Zero Product of Three Two Level Toeplitz Operators

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Abstract

In this paper we investigate conditions for $T_{f_1}T_{f_2}T_{f_3} - T_{f_1 f_2 f_3} = 0$ where T_{f_1}, T_{f_2}, and T_{f_3} are bi-level Toeplitz operators on the Hardy space of bidisk and f_1, f_2, f_3 are bounded and measurable complex valued functions on bidisk. We also provide that $T_{f_1}T_{f_2}T_{f_3}$ identical to zero matrix if and only if at least one of f_i is identically zero for $1 \le i \le 3$.

Keywords: Toeplitz operators, Hardy space, bidisk, bi-level Toeplitz matrices

1. Introduction

Denote $\mathbb{D}$ be the open unit disk in the complex plane ($\mathbb{C}$) and denote $\mathbb{T}$ be the boundary of it. Then the two subsets of $\mathbb{C}^2$, $\mathbb{D}^2$ and $\mathbb{T}^2$, are simply cartesian product of two copies of $\mathbb{D}$ and $\mathbb{T}$, respectively. Let $H^2(\mathbb{T})$ be the usual Hardy space of analytic functions and P be the orthogonal projection on $H^2(\mathbb{T})$.

A finite Toeplitz matrix is a matrix which has constant diagonals, namely entries along its diagonals are constant. For any $n \times n$ Toeplitz matrix, T_f where $f(z) = \sum_{k=-n+1}^{n-1} \hat{f}_k(z) z^k$ represents the generating function and $\hat{f}_k(z)$ represents Fourier coefficients of $f(z)$

$$T_f = \begin{bmatrix} \hat{f}_0 & \hat{f}_{-1} & \hat{f}_{-2} & \cdots & \hat{f}_{-n+1} \\ \hat{f}_1 & \hat{f}_0 & \hat{f}_{-1} & \cdots & \hat{f}_{-n+2} \\ \vdots & \ddots & \ddots & \ddots & \\ \hat{f}_{-n+1} & \cdots & \cdots & \hat{f}_1 & \hat{f}_0 \end{bmatrix}$$

where the function $\hat{f}(t) : \mathbb{T} \to \mathbb{C}$ and $\mathbb{T} = \{t \in \mathbb{C} : |t| = 1\}$. This structure is very interesting in itself for all the rich theoretical properties which it involves, but at the same time it is important for the dramatic impact that it has in applications.

Definition 1 $L^2(\mathbb{T})$ denotes the Hilbert space of square integrable lebesgue measurable complex valued function on $\mathbb{T}$. For instance, we can define the pointwise operations and inner product as follows:

Let $f_1, f_2 \in L^2(\mathbb{T})$. Then

$$< f_1, f_2 >= \frac{1}{2\pi} \int_{-\pi}^{\pi} f_1(z)\overline{f_2(z)} dz, z = e^{i\theta}.$$

Definition 2 $L^\infty(\mathbb{T})$ demotes the Banach space of essentially bounded Lebesgue measurable complex valued functions on $\mathbb{T}$. For instance, the pointwise operations and essential norm is defined as follows:

Let $f_1 \in L^\infty(\mathbb{T})$. Then

$$||f_1||_\infty = esssup_{|z|=1} |f_1(z)|.$$

Toeplitz matrices has applications in many different fields. To name a few, time series analysis, filter design, signal processing, etc. Because of its special structure, Toeplitz matrices were analysed by many mathematician. In particular, its algebra. More can be read in (Deepmala, 2014), (Mishra, 2014) and (Mishra, 2017). For instance, the product of two Toeplitz operators may not be Toeplitz again. It is easy to check that when a Toeplitz operator is the zero operator, then its corresponding symbol is zero and vice versa. It is also known that: the product of two such operators is identical to zero if and only if the symbol of one of them is zero (Brown & Halmos, 1963/1964). This result was later generalized for products of many Toeplitz operators, (Aleman & Vukotic, 2009). In (Ding, Sun & Zheng, 2012), Theorem 1.5 gives us necessary and sufficient condition for the product of bi-level Toeplitz operators Hardy space of the bidisk. In this paper we consider the following: Can we find a necessary and sufficient conditions to make the product of three bi-level infinite Toeplitz matrices is a Toeplitz matrix.? This paper is organized as follows: In section 1, we state and give the proof of some known results. In section 2, we give two main results: (1) We find necessary and sufficient conditions for when the product of three bi-level Toeplitz operators is again Toeplitz (2) We prove that, $T_{f_1}T_{f_2}T_{f_3}$ identical to zero matrix if and only if at least one of f_i is identically zero for $1 \leq i \leq 3$.

We would like to recall some known results regarding the zero product of one level Toeplitz operators.

Lemma 1 Suppose T_{f_1} and T_{f_2} are classical Toeplitz operators where $f_1(z)$ and $f_2(z)$ are bounded and measurable complex valued functions on the unit circle. Let $A = T_{f_1}T_{f_2}$ and let (a_{ij}) represent the (i, j)-entry of the matrix of A. Let the Fourier expansions of $f_1(z)$ and $f_2(z)$ be $f_1(z) = \sum_{k=-\infty}^{\infty} (\hat{f_1})_k(z)z^k$ and $f_2(z) = \sum_{k=-\infty}^{\infty} (\hat{f_2})_k(z)z^k$, where $(\hat{f_1})_k(z)$ and $(\hat{f_2})_k(z)$ are the Fourier coefficients of $f_1(z)$ and $f_2(z)$. Then

$$a_{i+1,j+1} = a_{ij} + (\hat{f}_1)_{i+1}(\hat{f}_2)_{-j-1},$$

whenever $i, j \geq 0$.

Proof. We can $(i, j)-$ entry of the matrix as follows

$$a_{ij} = \sum_{k=0}^{\infty} (\hat{f}_1)_{i-k}(\hat{f}_2)_{k-j}. \tag{1}$$

Then by some simple manipulations of the sum, we have the following

$$\begin{aligned}
a_{i+1,j+1} &= \sum_{k=0}^{\infty} (\hat{f}_1)_{i+1-k}(\hat{f}_2)_{k-j-1} \\
&= (\hat{f}_1)_{i+1}(\hat{f}_2)_{-j-1} + \sum_{k=1}^{\infty} (\hat{f}_1)_{i+1-k}(\hat{f}_2)_{k-j-1} \\
&= (\hat{f}_1)_{i+1}(\hat{f}_2)_{-j-1} + \sum_{k=0}^{\infty} (\hat{f}_2)_{i-k}(\hat{f}_2)_{k-j} \\
&= (\hat{f}_1)_{i+1}(\hat{f}_2)_{-j-1} + a_{ij}.
\end{aligned}$$

The following result is by (Brown & Halmos, 1963/1964).

Theorem 1

Let $f_1(z)$ and $f_2(z)$ be bounded and measurable complex valued functions. Then $T_{f_1}T_{f_2} = T_{f_1 f_2}$ if and only if one of the following holds

(i) f_1 and f_2 are analytic

(ii) f_1 and f_2 are co-analytic

(iii) There exist constants c_1 and c_2 with $|c_1| + |c_2| \neq 0$ such that $c_1 f_1 + c_2 f_2$ is constant.

Proof.

Suppose $f_2(z)$ is analytic. Then

$$T_{f_1}T_{f_2}g = T_{f_1}(f_2 g) = P(f_1 f_2 g) = T_{f_1 f_2}g,$$

where P is an orthogonal projection on $H^2(\mathbb{T})$ and for all $g(z) \in H^2$, and thus $T_{f_1}T_{f_2} = T_{f_1 f_2}$. Next suppose $f_1(z)$ is co-analytic. Then $\bar{f}_1(z)$ is analytic, and thus

$$T_{f_1}T_{f_2} = (T_{\bar{f}_2}T_{\bar{f}_1})^* = T^*_{\bar{f}_1\bar{f}_2} = T_{f_1 f_2}.$$

The following result is by (Riesz & Riesz, 1916).

Theorem 2

Let F be analytic on $\mathbb{D}$ and L^1-bounded, i.e, $F \in H^1(\mathbb{D})$. Assume that $F \neq 0$ and set $f = \lim_{r\to 1^-} F_r$. Then $\log|f| \in L^1(\mathbb{T})$. In particular, f does not vanish on a set of positive measure.

The following result is by (Brown & Halmos, 1963/1964).

Theorem 3

Let $f_1(z)$ and $f_2(z)$ be bounded measurable complex valued functions on the unit disk. Then $T_{f_1}T_{f_2}$ is identical to zero operator if and only if $f_1(z) = 0$ or $f_2(z) = 0$.

Proof.

Suppose either $f_1(z) = 0$ or $f_2(z) = 0$, then we have either $T_{f_1} = 0$ or $T_{f_2} = 0$, where 0 is zero operator matrix. Then, it follows that $T_{f_1}T_{f_2} = 0$. Next suppose that $T_{f_1}T_{f_2} = 0$. Then, since 0 is a Toeplitz operator, it follows from Theorem 1 that either $f_1(z)$ is co-analytic or $f_2(z)$ is analytic and that $f_1(z)f_2(z) = 0$. By Theorem 2 , since a non-zero analytic function, cannot vanish on a set of positive measure, it follows that if $f_1(z)$ is co-analytic, then $f_1(z) = 0$, and if $f_1(z)$ is analytic, then $f_1(z) = 0$.

2. Main Results

Theorem 4 Suppose

$$f_1(z_1,z_2) = \sum_{(i,j)\in\mathbb{Z}\times\mathbb{Z}} \hat{f}_1(i,j)z_1^i z_2^j, \quad f_2(z_1,z_2) = \sum_{(i,j)\in\mathbb{Z}\times\mathbb{Z}} \hat{f}_2(i,j)z_1^i z_2^j$$

and

$$f_3(z_1,z_2) = \sum_{(i,j)\in\mathbb{Z}\times\mathbb{Z}} \hat{f}_3(i,j)z_1^i z_2^j$$

where $f_1(z_1,z_2), f_2(z_1,z_2)$ and $f_3(z_1,z_2)$ are $L^\infty(\mathbb{T}^2)$ functions, and $\hat{f}_1(i,j)$, $\hat{f}_2(i,j), \hat{f}_3(i,j)$ are the Fourier coefficients of $f_1(z_1,z_2)$, $f_2(z_1,z_2), f_3(z_1,z_2)$, respectively. Then we have

$T_{f_1}T_{f_2}T_{f_3} = T_{f_1f_2f_3}$ is Toeplitz if and only if one of the following conditions holds for almost all $a, b \in \mathbb{T}$.

(i) $f_1(z_1,a), f_2(z_1,a)$ and $f_3(z_1,a)$ are all analytic in variable z_1 on $\mathbb{T}$.

(ii) $f_1(z_1,a), f_2(z_1,a)$ and $f_3(z_1,a)$ are co-analytic in variable z_1 on $\mathbb{T}$.

(iii) $f_1(b,z_2), f_2(b,z_2)$ and $f_3(b,z_2)$ are all analytic in variable z_2 on $\mathbb{T}$.

(iv) $f_1(b,z_2), f_2(b,z_2)$ and $f_3(b,z_2)$ are all co-analytic in variable z_2 on $\mathbb{T}$.

(v) $f_1(z_1,a), f_3(z_1,a)$ are analytic in variable z_1 and $f_2(z_1,a)$ is co-analytic in variable z_1 on $\mathbb{T}$.

(vi) $f_1(b,z_2), f_3(b,z_2)$ are analytic in variable z_2 and $f_2(b,z_2)$ is co-analytic in variable z_2 on $\mathbb{T}$.

Proof. Suppose that $T_{f_1}T_{f_2}T_{f_3} = T_{f_1f_2f_3}$. Let $T_{f_1}T_{f_2} = T_h$. Then using (Ding, Sun and Zheng, 2012), in particular Theorem 1.5, T_h is Toepltiz and we have that f_1 and f_2 are both analytic in variable z_1 on $\mathbb{D}$ or f_1 and f_2 are both co-analytic in variable z_1 on $\mathbb{D}$ and f_1 and f_2 are both analytic in variable z_2 on $\mathbb{D}$ or f_1 and f_2 are both co-analytic in variable z_2 on $\mathbb{D}$.

To prove the converse,we will prove (i),(ii) and the proof of other cases are similar. Now suppose $f_1(z_1,a), f_2(z_1,a)$ and $f_3(z_1,a)$ are all analytic in variable z_1 on $\mathbb{T}$, then we will prove that $T_hT_{f_3}$ is Toeplitz. Let $i, j, l \in \mathbb{N}_0^2$. Write

$$(T_hT_{f_3})_{(i,j)} = \sum_{k\in\mathbb{N}_0^2} \hat{h}_{(i-k)}\hat{f}_{3(k-j)} \tag{2}$$

Since $k_1 - i_1 \geq 0$ and $k_2 - i_2 \geq 0$, then the equation (2) is changed to the following form:

$$(T_hT_{f_3})_{(i,j)} = \sum_{k\in((i_1,0)+\mathbb{N}_0^2\cap(j_2,0)+\mathbb{N}_0^2)} \hat{h}_{i-k}\hat{f}_{3(k-j)} \tag{3}$$

Then we can write

$$(T_hT_{f_3})_{(i+l,j+l)} = \sum_{k\in((i_1+l_1,0)+\mathbb{N}_0^2\cap(i_2+l_2,0)+\mathbb{N}_0^2)} \hat{h}_{i-k+l}\hat{f}_{3(k-j-l)} \tag{4}$$

Substitute $\tilde{k}$ by $k-l$, the we get $i_1 \leq \tilde{k}_1$ and $i_2 \leq \tilde{k}_2$. So the equation (4) changed to the following form:

$$(T_hT_{f_3})_{(i+l,j+l)} = \sum_{\tilde{k}\in(i_1,i_2)+\mathbb{N}_0^2} \hat{h}_{i-\tilde{k}}\hat{f}_{3(\tilde{k}-j)} \tag{5}$$

Hence we showed that the equations (3) and (5) are equal.

Now suppose that $f_1(z_1, a)$, $f_2(z_1, a)$ and $f_3(z_1, a)$ are co-analytic in variable z_1 on $\mathbb{T}$. Then we can the following:

$$(T_h T_{f_3})_{(i,j)} = \sum_{k\in\mathbb{N}_0^2} \hat{h}_{i-k}\hat{f}_{3(k-j)} \tag{6}$$

Since $k_1 - i_1 \geq 0$ and $k_2 - j_2 \leq 0$, then the equation (6) is changed to the following form:

$$(T_h T_{f_3})_{(i,j)} = \sum_{k\in((i_1,0)+\mathbb{N}_0^2\cap(0,j_2)+\mathbb{N}_0^2)} \hat{h}_{i-k}\hat{f}_{3(k-j)} \tag{7}$$

Next we can write

$$(T_h T_{f_3})_{(i+l,j+l)} = \sum_{k\in((j_1+l_1,0)+\mathbb{N}_0^2\cap(0,j_2+l_2)+\mathbb{N}_0^2)} \hat{h}_{i-k+l}\hat{f}_{3(k-j-l)} \tag{8}$$

Letting $\tilde{k} = k - l$, we get $i_1 \leq \tilde{k}_1$, $\tilde{k}_2 \leq j_2$, $\tilde{k}_2 \leq -l_2$ and $-l_1 \leq \tilde{k}_1$. This implies that $\tilde{k} \in ((i_1, -l_2) + \mathbb{N}_0^2 \cap (-i_2, j_2) + \mathbb{N}_0^2) \cap ((i_1, 0) + \mathbb{N}_0^2 \cap (0, j_2) + \mathbb{N}_0^2)$. Then the equation (8) is changed to the following form:

$$(T_h T_{f_3})_{(i+l,j+l)} = \sum_{\tilde{k}\in\Lambda} \hat{h}_{i-\tilde{k}}\hat{f}_{3(\tilde{k}-j)}. \tag{9}$$

Now we can conclude that equations (7) and (9) are equal.

Theorem 5 $T_{f_1}T_{f_2}T_{f_3}$ is identically zero matrix if and only if at least one of $f_i = 0$ for $1 \leq i \leq 3$.

Proof. First suppose one of f_i is zero. Then it is clear that $T_{f_i} = 0$, where 0 is the zero matrix. For the converse, now suppose $T_{f_1}T_{f_2}T_{f_3} = 0$. Then, since 0 is a Toeplitz operator, it follows from Theorem 4 that one of the six cases hold and that $f_1 f_2 f_3 = 0$. In the case where f_1 is co-analytic in z_1 and z_2 or f_2 and f_3 is analytic in z_1 and z_2, then by Theorem of (F. and M. Riesz, 2016), a non-zero analytic function cannot vanish on a set of positive measure, this implies that if f_1 is co-analytic in z_1 and z_2, then $f_2 f_3 = 0$ and if f_3 is analytic in z_1 and z_2, then $f_1 f_2 = 0$. In the case where one of the f_i is co-analytic in z_1 and the others are analytic in z_2, we have that $f_1 f_2 f_3 = 0$ a.e on $\mathbb{T}^2$. Let $A \times B \subseteq \mathbb{T} \times \mathbb{T}$ be the zero set of f_1 and A have positive measure in $\mathbb{T}$. Since f_1 is co-analytic in z_1 and $f_1 \in L_\infty(\mathbb{T}^2)$, then for each fixed $z_2 \in \mathbb{T}$, we have that $f_1 \in H^2(\mathbb{T})$. Thus $f_1 = 0$ a.e for $z_1 \in \mathbb{T}$ and $z_2 \in B$, i.e, $A = B$ and hence $f_1 = 0$ on $\mathbb{T} \times B$. If the set B has full measure in $\mathbb{T}$, then $f_1 = 0$ on $\mathbb{T}^2$. However, assume that B is not of full measure in $\mathbb{T}$, then $f_2 = 0$ on $\mathbb{T}^2 - \mathbb{T} \times B = \mathbb{T} \times (\mathbb{T} - B)$ and $\mathbb{T} - B$ has positive measure. But f_2 is analytic in z_2 and for fixed $z_1 \in \mathbb{T}$, we have $f_1 \in H^2(\mathbb{T})$. Note that this implies $f_2 = 0$ a.e for $z_2 \in \mathbb{T}$. Thus $\mathbb{T} - B = \mathbb{T}$, i.e, $f_2 = 0$ on $\mathbb{T}^2$. Same argument can repeated for any f_i, where $1 \leq i \leq 3$.

3. Conclusion

In this manuscript, we consider the following question: Can we find a necessary and sufficient conditions to make the product of three bi-level infinite Toeplitz matrices again a Toeplitz matrix.? This question is just a natural extension of a well known a result by (Brown & Halmos, 1963/1964) in the case one level Toepltiz operators. In this paper, we extended the result by (Brown & Halmos, 1963/1964) to the case when three bi-level Toeplitz operator are given. Then we provide two main results: (1) we find necessary and sufficient conditions for when the product of three bi-level Toeplitz operators is again Toeplitz (2) we prove that, $T_{f_1}T_{f_2}T_{f_3}$ identical to zero matrix if and only if at least one of f_i is identically zero for $1 \leq i \leq 3$.

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