Riemannian Lie Subalgebroid

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Received: September 25, 2018 Accepted: October 19, 2018 Online Published: November 27, 2018

doi:10.5539/jmr.v10n6p80 URL: https://doi.org/10.5539/jmr.v10n6p80

Abstract

This paper talks about Riemannian Lie subalgebroid. We investigate the induced Levi-civita connection on Riemannian Lie subalgebroid, and give a construction of the second fondamental form like in case of Riemannian submanifold. We also give the Gauss formula in the case of Riemannian Lie subalgebroid. In the case of the Lie subalgebroid induced by a Leaf of a characteristic foliation, we obtain that the leaf carries more curvature than the manifold as shown by Boucetta (2011).

Keywords: Lie algebroid, Riemannian submanifold

1. Introduction

The notion of Lie groupoids and Lie algebroid are nowadays central resarch subjects in differential geometry. Lie algebroid was first introduced by Pradines and appeared as an infinitesimal counterpart of Lie groupoid. It is a generalization of the notion of Lie algebra and fiber bundle. A. Wenstein shows the crucial role of Lie algebroid in the study of Poisson manifold and Lagrangian mechanics. This motivated many studies on Lie algebroid, among others integrability by M. Crainic and R. L. Fernandes (2003), covariantes derivatives by Fernandes (2002) and Riemannian metric by M. Boucitta (2011), $\cdots$

The authors introduced in [1] the notion of geodesically complete Lie algebroid, and the notion of Riemannian distance. They also give a Hopf Rinow type theorem, caracterise the base connected manifold and characteristic leaf of this Lie algebroid.

This paper deals with Riemannian Lie subalgebroid. We will introduce the notion of Riemannian Lie algebroid as a generalisation of Riemannian submanifold. Hence we will give the second fondamental form of Lie subalgebroid and rewrite the Gauss's formulas type. We will investigate the special case of Riemannian Lie subalgebroid induced by a leaf of a characteristic foliation of a Riemannian manifold.

The paper is organised as follow. After this introduction, the second section gives some preliminary. In the third section, we introduce the notion of Lie subalgebroid of a Lie algebroid, induced by a submanifold of the base manifold. The fourth section deals with Riemannian Lie subalgebroid, and the second fondamental form in the case of Riemannian Lie subalgebroid will be introduced. We also give the Gauss's type formulas. In the last section we investigate a special case of Riemannian Lie subalgebroid induced by a leaf of a characteristic foliation.

2. Some Basic Facts on Lie Algebroids

Most of notions introduced in this section come from Boucetta (2003) and from J.-P. Dufour and N. T. Zung' s book (2005).

2.1 Lie Algebroid

A Lie algebroid is a vector bundle $p : A \to M$ such that :

- the sectons space $\Gamma(A)$ carry a Lie structure $[,]$;
- there is a bundle map $\sharp : A \to TM$ named anchor;
- For all $a, b \in \Gamma(A)$ and $f \in C^\infty(M)$, then

$$[a, fb] = f[a, b] + \sharp(a)(f)b \tag{1}$$

Note that a Lie algebroid is said to be transitive if the anchor is surjective.

The anchor $\sharp$ also satisfies:

$$\sharp[a,b] = [\sharp(a),\sharp(b)]$$

where $a,b \in \Gamma(A)$ and the bracket in the right is the natural Lie bracket of vector bundle. We also have:

$$[fa,b] \quad = \quad f[a,b] - \sharp(b)(f)a \tag{2}$$

and

$$[fa,gb] \quad = \quad fg[a,b] + f(\sharp(a)(g))b - g(\sharp(b)(f))a \tag{3}$$

for any $a,b \in \Gamma(A)$ and $f,g \in C^\infty(M)$. We also have a local splitting of a lie algebroid. given by R. Fernandes in (2002).

Theorem 2.1(2002)(local splitting) *Let $x_0 \in M$ be a point where $\sharp_{x_0}$ has rank q. There exists a system of cordinates $(x_1,\cdots,x_q,y_1,\cdots,y_{n-q})$ valid in a neighborhood U of x_0 and a basis of sections $\{a_1,\cdots,a_r\}$ of A over U, such that*

$$\begin{aligned}\sharp(a_i) &= \partial_{x_i} \quad (i=1,\cdots,q),\\ \sharp(a_i) &= \sum_j b^{ij}\partial_{y_j} \quad (i=q+1,\cdots,r),\end{aligned}$$

where $b^{ij} \in C^\infty(U)$ are smooth functions depending only on the $y's$ and vanishing at x_0 : $b^{ij} = b^{ij}(y^s), b^{ij}(x_0) = 0$. Moreover, for any $i,j = 1,\cdots,r$,

$$[a_i,a_j] = \sum_u C^u_{ij}a_u$$

where $C^u_{ij} \in C^\infty(U)$ vanish if $u \leq q$ and satisfy $\sum_{u>q}\frac{\partial C^u_{ij}}{\partial x_s}b^{ut} = 0$.
$n = \dim M$, *and* $r = \dim A$.

2.2 Linear A-connection

The notion of connection on Lie algebroids was first introduced in the contexte of Poisson geometry by Vaisman (1994) and R. Fernandes (2002). It appears as natural extension of the usual connection on fiber bundle (covariant derivative).

Remark 2.1 *The notion of A-connection is a generalization of the notion of the usual linear connection on a vector bundle. Lot of classic notions associate with covariant derivative can be written in the case of Lie algebroid.*

To introduce the notion of parallel transport, Boucetta sets the following definition and then we can introduce the notion of linear A-connexion.

Definiton 2.1 *Let $p : A \to M$ be a Lie algebroid. A linear A-connection D is an A-connection on the vector bundle $A \to M$*

If $(x_1,\cdots,x_n)$ is a system of local coordinates in a neighborhood $U \subset M$ in which $\{a_1,\cdots,a_r\}$ is a base of sections of $\Gamma(A)$, then the Christoffel's symbols of the linear connection D can be defined by:

$$D_{a_i}a_j = \sum_{k=1}^r \Gamma^k_{ij}a_k.$$

The most interesting fact about this notion is that one can ask about his relationship with the natural covariant derivative. The answer given by Fernandes in (2002) is relative to the notion of compatibility with Lie algebroid structure.

Defintion 2.2 *A linear A-connection D is compatible with the Lie algebroid structure of A if there is a linear connection on TM (covariant derivative) ∇ such that*

$$\sharp D = \nabla\sharp$$

Proposition 2.1 (2002) *Every Lie algebroid admits a compatible linear connection.*

Remark 2.2 *There is another notion of compatibility between linear A-connection and Lie algebroid structure introduced by Boucetta (2011) which is less stronger than the above one. A linear A-connection D is strongly compatible with the*

Lie algebroid structure if, for any A-path α, the parallel transport τ_α preserves $Ker\sharp$. A linear A-connection D is weakly compatible with the Lie algebroid structure if, for any vertical A-path α, the parallel transport τ_α preserves $Ker\sharp$.

Proposition 2.2 (2011)

1. *A linear A-connection is strongly compatible with Lie algebroid structure if and only if, for any leaf L and any sections $\alpha \in \Gamma(A_L)$ and $\beta \in \Gamma(Ker\sharp_L)$, $D_\alpha\beta \in \Gamma(Ker\sharp_L)$.*
2. *A linear A-connection D is weakly compatible with the Lie algebroid structure if and only if, for any leaf L and any sections $\alpha \in \Gamma(Ker\sharp_L)$ and $\beta \in \Gamma(Ker\sharp_L)$, $D_\alpha\beta \in \Gamma(Ker\sharp_L)$.*

2.3 Riemannian Metric on Lie Algebroid

Let $p : A \to M$ be a Lie algebroid.

Definition 2.3 *A Riemannian metric on a Lie algebroid $p : A \to M$ is the data, for any $x \in M$, of a scalar product g_x on the fiber A_x such that, for any local sections $a, b \in \Gamma(A)$, the function $g(a, b)$ is smooth.*

Like in the classic case of Riemannian manifold, one of the most interesting fact is the existence of a linear A-connection which has the same characteristics with the Levi-civita connection. The A-connection is

- metric, i.e. $\sharp(a)g(b, c) = g(D_a b, c) + g(b, D_a c)$,
- free torsion $D_a b - D_b a = [a, b]$

Also, the Linear A-connection D is characterized by the Koszul type formula: for all sections $a, b, c \in \Gamma(A)$

$$\begin{aligned} 2g(D_a b, c) &= \sharp(a)g(b, c) + \sharp(b)g(a, c) - \sharp(c)g(a, b) \\ &+ g([c, a], b) + g([c, b], a) + ([a, b], c) \end{aligned}$$

The Christoffel's symbols of the Levi-civita A-connexion are defined, in a local coordinates system $(x_1, \cdots, x_n)$ over a trivializing neighborhood U of M where $\Gamma(A)$ admits a local basis of sections $\{a_1, \cdots, a_r\}$, by:

$$\begin{aligned} \Gamma^k_{ij} &= \frac{1}{2}\sum_{l=1}^{r}\sum_{u=1}^{n} g^{kl}(b^u_i \partial x_u(g_{jl}) + b^u_j \partial x_u(g_{il}) - b^u_l \partial x_u(g_{ij})) \\ &+ \frac{1}{2}\sum_{l=1}^{r}\sum_{u=1}^{r} g^{kl}(C^u_{ij} g_{ul} + C^u_{li} g_{uj} + C^u_{lj} g_{ui}) \end{aligned}$$

where the structures functions $b^i_s, C^u_{st} \in C^\infty(U)$ are given by

$$\sharp(a_s) = \sum_{i=1}^{n} b^i_s \partial_{x_i} \quad (s = 1, ..., r)$$

and

$$[a_s, a_t] = \sum_{u=1}^{r} C^u_{st} a_u \quad (s, t = 1, ..., r),$$

$g_{ij} =< a_i, a_j >$ and (g^{ij}) denote the inverse matrix of (g_{ij}).

The author of [**?**] define the sectional curvature of two linearly independant vectors $a, b \in A_x$ by

$$K(a, b) = -\frac{g(R(a, b)a, b)}{g(a, a)g(b, b) - g(a, b)^2}. \tag{4}$$

where R is Riemannian curvature.

2.4 Notion of Lie Subalgebroid

The notion of Lie subalgebroid is studied by many authors. K. Mackenzie (1987) gives the following definition

Definition 2.4 *Let $p : A \to M$ be a Lie algebroid. A Lie subalgebroid of A is a Lie algebroid A' on M together with an injective morphism $A' \to A$ of Lie algebroid over M.*

Here we give a construction of this notion which generalised the classic notion of a submanifold.

Let $p : A \to M$ be a Lie algebroid with anchor map $\sharp$ and let N be a submanifold of M. Let's set

$$(A_N)_x = \{a_x \in A_x/\sharp_x a \in T_x N\}$$

and

$$A_N = \bigcup_{x \in N} (A_N)_x$$

Then $A_N \subset A$. Let $p_N : A_N \to N$ be the restriction of $p : A \to M$ to A_N.

If there is no risk of confusion, let's denote by $\sharp : A_N \to TN$ the restriction of $\sharp$ to A_N. Then we can defined a Lie bracket $[,]_N$ on $\Gamma(A_N)$, which is like a restriction of the Lie bracket of $\Gamma(A)$, by setting for all $a, b \in \Gamma(A_N)$ and $x \in N$:

$$[a, b]_N(x) = [a, b]_{|_M}(x)$$

This bracket is well defined and induced a Lie algebroid structure on $p_N : A_N \to N$.

Thus

$$\begin{aligned}
[a, fb]_N(x) &= [(a), fb]_{|_M}(x) \\
&= (f[a, b]_{|_M} + \sharp(a)(f)b)(x) \\
&= (f[a, b]_{|_M})(x) + (\sharp(a)(f)b)(x) \\
&= (f[a, b]_N)(x) + (\sharp(a)(f)b)(x) \\
&= (f[a, b]_N + \sharp(a)(f)b)(x)
\end{aligned}$$

then

$$[a, fb]_N = f[a, b]_N + \sharp(a)(f)b$$

Remark 2.3 *One of the best example of this Lie subalgebroid structure, it's the one induced by a leaf of a characteristic foliation L. This structure Lie subalgebroid is transitive. We will use this structure to study some particular Riemannian Lie subalgebroid.*

3. Riemannian Lie Subalgebroid

Let $p : A \to M$ be a Lie algebroid with anchor map $\sharp$ and let $\tilde{g}$ be a Riemannian metric on A. Let N be submanifold of M and $p_N : A_N \to N$ be an induced Lie subalgebroid. As in the Riemannian case, if $f : A_N \to A$ is an isometric immersion, then $g = f^*\tilde{g}$ is an induced metric on A_N.

Definition 3.1 (A_N, g) *is a Riemannian Lie algebroid called Riemannian Lie subalgebroid.*

Like in the classic case of Riemannian submanifold, one can ask about the induced Levi-Civita A_N-connection and it's relationship with the Levi-civita A-connection.

As an answer, we will give a similar connection construction with the classic one of Riemannian submanifold. For this construction for all $x \in N$ we denote by $(A_N)_x^\perp$ the orthogonal complementary of $(A_N)_x$ in respect with $\tilde{g}_x$. then we have :

$$A_x = (A_N)_x \oplus (A_N)_x^\perp$$

hence

$$A = A_N \oplus A_N^\perp$$

where $(A_N)_x^\perp$ can be defined by:

$$(A_N)_x^\perp = \{a \in A/\tilde{g}_x(a, b) = 0, \forall b \in (A_N)_x\}$$

Thus for all $a \in \Gamma(A)$ one has $a = a^\top + a^\perp$ with $a^\top \in \Gamma(A_N)$ and $a^\perp \in \Gamma(A_N)^\perp$.

Moreover if D is the Levi-civita A-connection, then for all $\alpha, \beta \in \Gamma(A_N)$ one has:

$$D_\alpha \beta = (D_\alpha \beta)^\top + (D_\alpha \beta)^\perp \tag{5}$$

Let's set $D^N_\alpha\beta = (D_\alpha\beta)^\top$.

Then we have the following proposition.

Proposition 3.1 *D^N is the Levi-civita A_N-connection associate to the Riemannian metric g.*

proof *Indeed D^N is free torsion: for all $\alpha,\beta \in \Gamma(A_N)$ one has*

$$\begin{aligned} D^N_\alpha\beta - D^N_\beta\alpha &= (D_\alpha\beta)^\top - (D_\beta\alpha)^\top \\ &= (D_\alpha\beta - D_\beta\alpha)^\top \\ &= ([\alpha,\beta])^\top \end{aligned}$$

since $\alpha,\beta \in \Gamma(A_N)$, one has $[\alpha,\beta] \in \Gamma(A_N)$ and

$$[\alpha,\beta]^\top = [\alpha,\beta] = [\alpha,\beta]_N$$

then we have

$$D^N_\alpha\beta - D^N_\beta\alpha = [\alpha,\beta]_N$$

D^N is metric: since D is metric, then for all $\alpha,\beta,\gamma \in \Gamma(A_N)$ one has:

$$\begin{aligned} \sharp(\alpha)g(\beta,\gamma) &= \sharp(\alpha)\tilde{g}(\beta,\gamma) \\ &= \tilde{g}(D_\alpha\beta,\gamma) + \tilde{g}(\beta, D_\alpha\gamma) \\ &= \tilde{g}((D_\alpha\beta)^\top + (D_\alpha\beta)^\perp,\gamma) + \tilde{g}(\beta,(D_\alpha\gamma)^\top + (D_\alpha\gamma)^\perp) \\ &= \tilde{g}(D^N_\alpha\beta,\gamma) + \tilde{g}(\beta, D^N_\alpha\gamma) \\ &= g(D^N_\alpha\beta,\gamma) + g(\beta, D^N_\alpha\gamma) \end{aligned}$$

Then

$$\sharp(\alpha)g(\beta,\gamma) = g(D^N_\alpha\beta,\gamma) + g(\beta, D^N_\alpha\gamma).$$

3.1 Second Fondamental Form

This notion is a generalization of the classical case of Riemannian submanifold. we will show that the A -second fondamental form satisfies all properties of the second fondamental form of a Riemannian submanifold.

From the equation (5), let's set $h(\alpha,\beta) = (D_\alpha\beta)^\perp$. Then we have the following definition.

Definition 3.2 *The operator*

$$\begin{aligned} h : \Gamma(A_N)\times\Gamma(A_N) &\to \Gamma(A_N)^\perp \\ (\alpha,\beta) &\mapsto h(\alpha,\beta) \end{aligned}$$

is called the A_N-second fondamental form associated to Riemannian Lie subalgebroid $p_N : A_N \to N$.

This operator h is $C^\infty(N)$-bilinear and symmetric.

3.2 Gauss and Kodazzi's Equations

Here we rewrite the Gauss and Kodazzi's formulas in the case of Riemannian Lie subalgebroid.

For all $\alpha \in \Gamma(A_N)$ and $\xi \in \Gamma(A_N)^\perp$ the relation (5) becomes :

$$D_\alpha\xi = (D_\alpha\xi)^\top + (D_\alpha\xi)^\perp \tag{6}$$

Now if we set $B_\xi\alpha = -(D_\alpha\xi)^\top$ and $L(\alpha,\xi) = (D_\alpha\xi)^\perp$ then we have the Gauss type formula:

$$\tilde{g}(\xi, h(\alpha,\beta)) = g(B_\xi\alpha,\beta) \quad \text{(Gauss formula)} \tag{7}$$

for all $\alpha,\beta \in \Gamma(A_N)$ and $\xi \in \Gamma(A_N)^\perp$.

Proposition 3.2 *The map*

$$\begin{array}{rcl} B : \Gamma(A_N) \times \Gamma(A_N)^{\perp} & \rightarrow & \Gamma(A_N) \\ (\alpha, \xi) & \mapsto & B_{\xi}\alpha \end{array}$$

is $C^{\infty}(N)$-bilinear and symmetric.

Proof *For all $\alpha \in \Gamma(A_N), \xi \in \Gamma(A_N)^{\perp}$ and $f_1, f_2 \in C^{\infty}(N)$ one has:*

$$\begin{array}{rcl} B_{f_1\xi}(f_2\alpha) & = & -(D_{f_2\alpha}(f_1\xi))^{\top} = -(f_2 D_{\alpha}(f_1\xi))^{\top} \\ & = & -f_2[f_1 D_{\alpha}\xi + \sharp(\alpha)(f_1)\xi]^{\top} \\ & = & -f_2 f_1 (D_{\alpha}\xi)^{\top} \\ & = & f_1 f_2 B_{\xi}\alpha \end{array}$$

One of the important fact in the study of Riemannian submanifold is the relationship between the Riemannian curvature of the manifold and the one on the submanifold. Similarly to this case we can investigate this relationship between Riemannian curvature associate to $\tilde{g}$ and the induced one associate to g.

The following proposition gives the Gauss equation in the case of Riemannian Lie subalgebroid.

Proposition 3.3 *If R and R^N are respectively the curvature associate to $\tilde{g}$ and g, then we have:*

$$g(R^N(\alpha,\beta)\gamma,\delta) = \tilde{g}(R(\alpha,\beta)\gamma,\delta) + \tilde{g}(h(\alpha,\gamma), h(\beta,\delta)) - \tilde{g}(h(\alpha,\delta), h(\beta,\gamma)) \tag{8}$$

Particulary, if $K^N(\alpha,\beta) = \dfrac{g(R^N(\alpha,\beta)\beta,\alpha)}{g(\alpha,\alpha)g(\beta,\beta) - g(\alpha,\beta)^2}$ and $K(\alpha,\beta) = \dfrac{\tilde{g}(R(\alpha,\beta)\beta,\alpha)}{\tilde{g}(\alpha,\alpha)\tilde{g}(\beta,\beta) - \tilde{g}(\alpha,\beta)^2}$ are respectively the sectional curvature on A_N and A, then the Gauss equation (8) become by sitting $Q(\alpha,\beta) = \tilde{g}(\alpha,\alpha)\tilde{g}(\beta,\beta) - \tilde{g}(\alpha,\beta)^2$:

$$K^N(\alpha,\beta) = K(\alpha,\beta) - \frac{\tilde{g}(h(\alpha,\alpha), h(\beta,\beta))}{Q(\alpha,\beta)} - \frac{||h(\alpha,\beta)||}{Q(\alpha,\beta)}. \tag{9}$$

Moreover if α and β are orthornormal, then the equation (9) becomes :

$$K^N(\alpha,\beta) = K(\alpha,\beta) - \tilde{g}(h(\alpha,\alpha), h(\beta,\beta)) - ||h(\alpha,\beta)|| \tag{10}$$

3.3 Totally Geodesic Lie Subalgebroid

Definition 3.3 *Let (A_N, g) be Riemannian Lie subalgebroid of a Riemannian Lie algebroid $(A, \overline{g})$. A_N is said to be totally geodesic if any A_N-geodesic is an A-geodesic.*

This class of Riemannian Lie subalgebroid is characterised by the following proposition.

Proposition 3.4 *Let $p_N : A_N \rightarrow N$ be a Riemannian Lie subalgebroid of the Riemannian Lie algebroid $p : A \rightarrow M$. Then, the following assertions are equivalent:*

1. *A_N is totally geodesic;*
2. *the second fondamental A-form is identically nul.*

Proof

1) $\Rightarrow$ 2. *Supposed that A_N is a totally Riemannian Lie subalgebroid. Let s be an A_N-geodesic, then $D^{N s} s = 0$ and s is an A-geodesic ($D^s s = 0$). Since $D^s s = D_s s + \frac{ds}{dt}$, one has:*

$$\begin{array}{rcl} D^s s & = & (D_s s)^{\top} + (D_s s)^{\perp} + \dfrac{ds}{dt} \\ & = & D_s^N s + (D_s s)^{\perp} + \dfrac{ds}{dt} \\ & = & (D_s s)^{\perp} \\ & = & h(s, s) \end{array}$$

hence $h(s, s) = 0$.

$2 \Rightarrow 1$). *Supposed $B_\xi \equiv 0$. Let s be an A_N-geodesic, then :*

$$D^s s = D^{Ns} s + \frac{ds}{dt}.$$

With the Gauss's formula (7), $h(s, s) = 0$ and s is an A-geodesic.

Corollary 3.1 *If (A_N, g) is a totally geodesic Riemannian Lie subalgebroid, then the Gauss type formula becomes: for all linearly independent vectors $\alpha, \beta \in (A_N)_x$*

$$g(R^N(\alpha,\beta)\gamma, \delta) = \tilde{g}(R(\alpha,\beta)\gamma, \delta) \tag{11}$$

3.4 Minimal, Parallel and Totally Umbilical Lie Subalgebroid

From the splitting theorem, consider $\{a_1, \cdots, a_r\}$ a local basis of the sections space $\Gamma(A)$. Supposed that the sections space $\Gamma(A_N)$ is dimension d ($d \le r$) and let $\{a_1, \cdots, a_d\}$ an induced orthonormal basis. As in the case of Riemannian submanifold, the mean curvature H associated to the second fondamental A-form is defined by:

$$H = \frac{1}{d}\sum_{i=1}^{d} h(a_i, a_i) \tag{12}$$

If $\{\xi_1, \cdots, \xi_{r-d}\}$ the g-orthogonal basis of $\{a_1, \cdots, a_d\}$, then with the Gauss formula (7) the mean curvature become:

$$H = \frac{1}{d}\sum_{k=1}^{r-d} Tr(B_{\xi_k}) \tag{13}$$

Definiton 3.4 *A Riemannian Lie subalgebroid de Lie $p_N : A_N \to N$ is called minimal, if the mean curvature H vanishes identically (i.e. $H \equiv 0$). And A_N is called quasi-minimal (pseudo-minimal) if $H \neq 0$ and $< H, H >= 0$ at each point of N.*

Corollary 3.2 *If for all $\xi \in \Gamma(A_N)^\perp$, $B_\xi \equiv 0$, then (A_N, g) is minimal.*

Proof *Supposed that for all $\xi \in \Gamma(A_N)^\perp$, $B_\xi \equiv 0$. Since $\{\xi_1, \cdots, \xi_{r-d}\}$ is the g-orthogonal basis of $\{a_1, \cdots, a_d\}$ and*

$$H = \frac{1}{d}\sum_{k=1}^{r-d} Tr(B_{\xi_k}) \tag{14}$$

we have $\xi_k \in \Gamma(A_N)^\perp$ and $B_{\xi_k} = 0$. Then $H \equiv 0$ and A_N is minimal.

Definition 3.5 *Let (A, g) be a Riemannian Lie algebroid and (A_N, g) a Riemannian Lie subalgebroid of (A, g).*

1. *A normal section $\xi \in \Gamma(A_N)^\perp$ is called parallel, if for any section $\alpha \in \Gamma(A_N)$, one has:*

$$D_\alpha \xi = 0.$$

2. *A Riemannian Lie subalgebroid is called parallel if the second fondamental A-form is parallel. i.e. $Dh = 0$*
3. *(A_N, g) is said to be totally umbilic if, for all normal section $\xi \in \Gamma(A_N)^\perp$, there exist a scalar λ such that:*

$$\forall \alpha \in \Gamma(A_N)^\top, \quad B_\xi(\alpha) = \lambda\alpha$$

Theorem 3.1 *Let $(A, \tilde{g})$ be a Riemannian Lie algebroid with constant sectional curvature c. If the Riemannian Lie subalgebroid (A_N, g) is minimal, then*

$$Ric(\alpha) \le c \tag{15}$$

for all unit A_N-section α. Moreover the equality hold if and only if A_N is totally geodesic. Where Ric is a Ricci curvature.

Proof *Let α be a unit A_N-section and $\{a_1, \cdots, a_d\}$ be a local orthonormal basis of the space of sections $\Gamma(A_N)$. Then we have:*

$$Ric(\alpha) = \frac{1}{d-1}\sum_{i=1}^{d} g(R^N(a_i, \alpha)\alpha, a_i)$$

with the Gauss equation (8), one has:

$$
\begin{aligned}
Ric(\alpha) &= \frac{1}{d-1}\sum_{i=1}^{d}[\tilde{g}(R(a_i,\alpha)\alpha,a_i)+\tilde{g}(h(a_i,a_i),h(\alpha,\alpha))-\tilde{g}(h(a_i,\alpha,h(a_i,\alpha))] \\
&= \frac{1}{d-1}\sum_{i=1}^{d}[c+\tilde{g}(h(a_i,a_i),h(\alpha,\alpha))-\|h(a_i,\alpha)\|^2] \\
&= c+\frac{1}{d-1}\tilde{g}(\sum_{i=1}^{d}h(a_i,a_i),h(\alpha,\alpha))-\frac{1}{d-1}\sum_{i=1}^{d}\|h(a_i,\alpha)\|^2 \\
&= c+\frac{d}{d-1}g(H,h(\alpha,\alpha))-\frac{1}{d-1}\sum_{i=1}^{d}\|h(a_i,\alpha)\|^2
\end{aligned}
$$

since A_N is minimal and $\frac{1}{d-1}\sum_{i=1}^{d}\|h(a_i,\alpha)\|^2 \geq 0$, one can conclude $Ric(\alpha) \leq c$.
Moreover $Ric(\alpha) = c$ if and only if $\frac{1}{d-1}\sum_{i=1}^{d}\|h(a_i,\alpha)\|^2 = 0$ which hold if and only if $h = 0$ and we have A_N is totally geodesic.

Corollary 3.3 *Let $(A,\tilde{g})$ be a Riemannian Lie algebroid with constant sectional curvature $\tilde{c}$. If the Riemannian Lie subalgebroid (A_N, g) has constant sectional curvature c, then:*

1. *$c \leq \tilde{c}$,*
2. *$c = \tilde{c}$ if and only if A_N is totally geodesic.*

Theorem 3.2 *Let $(A,\tilde{g})$ be a Riemannian Lie algebroid with constant sectional curvature $\tilde{c}$. The scalar curvature of the Riemannian Lie subalgebroid (A_N, g), S is given by :*

$$S = c+\frac{d}{d-1}\|H\|^2-\frac{1}{d(d-1)}\|h\|^2 \tag{16}$$

Proof *Let $\{a_1,\cdots,a_d\}$ be a local orthogonal base of the space of A_N-section $\Gamma(A_N)$; then one has the scalar curvature:*

$$
\begin{aligned}
S &= \sum_{j=1}^{d}Ric(a_j) \\
&= \sum_{j=1}^{d}[\frac{1}{d-1}\sum_{i=1}^{d}g(R(a_i,a_j)a_j,a_i)] \\
&= \frac{1}{d-1}\sum_{i,j}g(R(a_i,a_j)a_j,a_i) \\
&= \frac{1}{d-1}\sum_{i,j}[c+\tilde{g}(h(a_i,a_i),h(a_j,a_j))-\tilde{g}(h(a_i,a_j),h(a_i,a_j))] \\
&= c+\frac{d}{d-1}\|H\|^2-\frac{1}{d(d-1)}\|h\|^2.
\end{aligned}
$$

4. Characteristic Riemannian Lie Subalgebroid

One of the interesting case is the situation of N with a leaf of characteristic foliation. Here we investigate on the following case :

If $N = L$: In this case one has locally :

$$\forall x \in L, \quad A_x = (A_L)_x \oplus Ker\sharp_x$$

hence

$$(A_L)_x^{\perp} = Ker\sharp_x$$

Hence with the notion of compatibility of the A-connection D with the Lie algebroid structure (in the sens of Fernandes(2002)), one has the following theorem.

Theorem 4.1 *Let $(A, \tilde{g})$ be a Riemannian Lie algebroid. If L is a Leaf of the characteristic foliation, then:*

1. *For all $\xi \in \Gamma(A_L)^{\perp}$, one has $B_\xi = 0$. Hence (A_L) is totally geodesic.*
2. *The Gauss's equation become:*
$$g(R^L(\alpha,\beta)\gamma,\delta) = \tilde{g}(R(\alpha,\beta)\gamma,\delta) \;\; \forall \alpha,\beta,\gamma,\delta \in \Gamma(A_L). \tag{17}$$
Moreover $K^L = K$.
3. *(A_L, g) is minimal*

Proof

1. *For all $\alpha \in \Gamma(A_L)$ and $\xi \in \Gamma(A_L)^{\perp}$, one has:*
$$D_\alpha \xi = (D_\alpha \xi)^{\top} + (D_\xi \alpha) = -B_\xi \alpha + (D_\alpha \xi)^{\perp}$$
thus
$$\sharp(D_\alpha \xi) = \sharp(-B_\xi \alpha) + \sharp(D_\alpha \xi)^{\perp} = 0$$
since $(D_\alpha \xi) \in \Gamma(A_L)^{\perp}$ one has:
$$\sharp(-B_\xi \alpha) = 0 \quad \text{and} \quad B_\xi \alpha \in \Gamma(A_L)^{\perp}$$
from the definition of B_ξ one has $B_\xi \alpha \in \Gamma(A_L)$, and with the fact that $\tilde{g}$ is not degenerate, one has: $B_\xi = 0$. And A_L is totally geodesic.
2. *Since $B_\xi = 0$, then with the gauss formula we have :*
$$\tilde{g}(h(\alpha,\gamma), h(\beta,\delta)) = \tilde{g}(h(\alpha,\delta), h(\beta,\gamma)) = 0$$
3. *This is because that $\forall \xi \in \Gamma(A_L)^{\perp}, \quad B_\xi \equiv 0$.*

Corollary 4.1 *Let $(A, \tilde{g})$ be Riemannian Lie algebroid with constant sectional curvature $\tilde{c}$ and L a leaf of the characteristic foliation. If (A_L, g) has constant curvature c, then we have:*

1. *$\tilde{c} = c$*
2. *for any unit section $\alpha \in \Gamma(A_L)$, one has : $Ric(\alpha) = c$*

One of the particularity of the notion of A-connection is the existence of the $\mathcal{F}$-connection (see [**?**]). Hence we have the following proposition.

Proposition 4.1 *If D is an $\mathcal{F}$-connection, then we have:*
$$(D_\xi \alpha)^{\perp} = (D_\xi \alpha)^{\top} = 0$$
and
$$[\alpha, \xi] = D_\alpha \xi \in \Gamma(A_L)^{\perp}$$
for all $\alpha \in \Gamma(A_L)$ and $\xi \in \Gamma(A_L)^{\perp}$.

Proof *For $\alpha \in \Gamma(A_L)$ and $\xi \in \Gamma(A_L)^{\perp}$, one has:*
$$\begin{aligned} D_\xi = 0 \qquad &\Rightarrow \qquad D_\xi \alpha = (D_\xi \alpha)^{\top} + (D_\xi \alpha)^{\perp} = 0 \\ &\Rightarrow \qquad (D_\xi \alpha)^{\top} = -(D_\xi \alpha)^{\perp}. \end{aligned}$$

since $(D_\xi\alpha)^\top \in \Gamma(A_L)$ *and* $(D_\xi\alpha)^\perp \in \Gamma(A_L)^\perp$ *one has:*

$$(D_\xi\alpha)^\top = (D_\xi\alpha)^\perp = 0$$

Moreover, since $D_\xi = 0$, *one has :*

$$[\alpha, \xi] = D_\alpha\xi - D_\xi\alpha = D_\alpha\xi$$

thus

$$\sharp[\alpha, \xi] = [\sharp\alpha, \sharp\xi] = 0 \quad \Rightarrow \quad D_\alpha\xi \in \Gamma(A_L)^\perp$$

Proposition 4.2 *If D is an* $\mathcal{F}$*-connection, then the isotropic algebra Ker*$\sharp$ *is a Riemannian Lie algebra.*

Proof *A consequence of the definition of an* $\mathcal{F}$*-connection (As in the case of Poisson manifolds).*

If $L \subset N$

In this case, one has the following decomposition:

$$A_N = A_L \oplus (A_L)^{\perp_N}$$

where $(A_L)^{\perp_N}$ is the complementary g_N-orthogonale of A_L in A_N. Thus for all $\alpha, \beta \in \Gamma(A_L)$ one has:

$$D^N_\alpha\beta = D^L_\alpha\beta + h^1(\alpha, \beta)$$

Moreover :

$$\begin{aligned} D_\alpha\beta &= D^N_\alpha\beta + h(\alpha, \beta) \\ &= D^L_\alpha\beta + h^1(\alpha, \beta) + h(\alpha, \beta) \end{aligned}$$

In other words, since $L \subset N$ than $A_L \subset A_N$ and $(A_N)^\perp \subset (A_L)^\perp$. Thus

$$h^N(\alpha, \beta) = h^L(\alpha, \beta) + h^1(\alpha, \beta)$$

Hence we have the following proposition

Theorem 4.2 *If the submanifold N is entierelly contained in a leaf L of a characteristic foliation, then the Riemannian* A_N *Lie subalgebroid is totally geodesic. Moreover* A_N *have a reduction of codimension.*

Proof *This is a consequence of the fact that if L is a leaf of characteristic foliation, then* A_L *is totally geodesic.*

Corollary 4.2 *If the Lie algebroid* $p : A \to M$ *is transitive, then N is a totally geodesic Riemannian submanifold of the Riemannian manifold M.*

Proof *Let* γ *be a geodesic on M and s an A-path with base path* γ*. Then* α *is an A-geodesic (see [*?*]). From the above theorem,* α *is an* A_N*-geodesic and we can conclude.*

References

Ali, M. S., Hassirou, M., & Mahaman, B. Geodesically Complete Lie Algebroid.

Besse, A. L. (1987). Einstein Manifolds, Springer-Verlag, Berlin, Heidelberg New York. https://doi.org/10.1007/978-3-540-74311-8

Boucetta, M. (2011). Riemannian geometry of Lie algebroids. *Journal of the Egyptian Mathematical Society, 19*(1-2), 57-70. https://doi.org/10.1016/j.joems.2011.09.009

Da Silva, A. C., & Weinstein, A. (1999). Geometric models for noncommutative algebras (Vol. 10). American Mathematical Soc.

Crainic, M., & Fernandes, R. L. (2003). Integrability of Lie brackets. *Annals of Mathematics,* 575-620. https://doi.org/10.4007/annals.2003.157.575

Zung, N. T. (2005). Poisson Structures and Their Normal Forms: Progress in Mathematics. Springer.

Fernandes, R. L. (2002). Lie algebroids, holonomy and characteristic classes. *Advances in Mathematics, 170*(1), 119-179. https://doi.org/10.1006/aima.2001.2070

Fernandes, R. (2002).Connexion in Poisson geometry. I. Holonomy and invariants, *J. Diff. Geom. 54*(2), 303-365. https://doi.org/10.4310/jdg/1214341648

Gallot, S., Hulin, D., & LaFontaine, J. (2004). Riemannian Geometry. Universitext (1979). https://doi.org/10.1007/9783-64218855-8_2

HASSIROU, M. (2013). Introduction in Poisson manifolds with lightlike kaehler foliation. *Pioneer Jour. of Math. and Math. Sci, 7*(1), 37-50.

Lafontaine, J. (1996). Introduction aux Variétés Différentielles, Presse Universitaire de Grenoble.

Mackenzie, K. (1987). *Lie Groupoids and Lie Algebroids in Differential Geometry*, London Math. Soc. Lectures Notes Series. No. 124, Cambridge University Press. https://doi.org/10.1017/CBO9780511661839

Mackenzie, K. (2005). *General Theory of Lie Groupoids and Lie algebroids*, Cambridge University Press. https://doi.org/10.1017/CBO9781107325883

Vaisman, I. (1994). *Lecture on the Geometry of Poisson Manifolds*,Prog. Math., vol. 118, Birkhausser, Berlin. https://doi.org/10.1007/978-3-0348-8495-2