Existence, Uniqueness and $\mathscr{C}-$Differentiability of Solutions in a Non-linear Model of Cancerous Tumor

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Abstract

In this paper, we prove the existence and uniqueness of the weak solution of a system of nonlinear equations involved in the mathematical modeling of cancer tumor growth with a non homogeneous divergence condition. We also present a new concept of generalized differentiation of non linear operators : $\mathscr{C}-$differentiability. Through this notion, we also prove the uniqueness and the $\mathscr{C}-$differentiability of the solution when the system is perturbed by a certain number of parameters. Two results have been established. In the first one, differentiability is according to Fréchet. The proof is given uses the theorem of reciprocal functions in Banach spaces. First of all, we give the proof of strict differentiability of a direct mapping, according to Fréchet. In the second result, differentiability is understood in a weaker sense than that of Fréchet. For the proof we use Hadamard's theorem of small perturbations of Banach isomorphism of spaces as well as the notion of strict differentiability.

Keywords: cancer, existence, uniqueness and $\mathscr{C}-$differentiability, weak solution, perturbed system, isomorphism

1. Introduction

Cancer (Marcotte, 2008) is a serious genetic disease that results in an imbalance between cell division and death, leading to cells disequilibrium. The balance between these two processes regulates the number of cells in the tissues, and the breakdown of this equilibrium leads to the development of clusters of cancer cells (called tumors) irrespective of the normal functioning of the body. The cancer cell is a wanton cell that multiplies itself in an uncontrolled and excessive manner within a normal tissue of the body. This anarchic proliferation gives rise to increasingly large tumors that grow up and then destroy the surrounding organs. The cancer cells can also swarm away from a body to form a new tumor, or circulate in a free form. By destroying its environment, the cancer can become a real danger to the survival of human being. The fight against this disease is an important field of medical research. The need to adapt various types and forms of cancers as well as the understanding of complex phenomena involved in its growth has led to the development of many mathematical models (Patrick, 2013)(Tracqui, 1995) in recent decades. Mathematical modeling of cancer evolution is a rapidly developing field. Their interest lies in their ability to gather large quantity of information accumulated by biologists.

Indeed, it is important to understand that the mathematical complexity of a model is not a sufficient criterion to judge its relevance. Thus, the nature of this phenomenon (the cancer cells have a fluid movement) motivated us to use the non-stationary compressible Navier-Stokes model, which can describe the disease. These equations do not address the tumor environment and its interactions directly, but present measurable magnitudes such as the volume density denoted by $\rho = \rho(x,t)$ and the density of the outer forces denoted by $\xi_e = \xi_e(x,t)$, which models environmental factors. Furthermore, the cells are considered to be transported by a velocity field $\nu = \nu(x,t)$, with the corresponding pressure $\pi = \pi(x,t)$.

The choice of the Navier Stokes system as working equations permits to tackle problems like unknown coupling, nonlinearity, and time dependence. The nonlinear nature of the convection term $(\nu \cdot \nabla \nu)$ that appears in these equations is the source of difficulties in solving this problem. To overcome this difficulty, we use the method consisting of low estimates and convergences in regular spaces like $L^2((0,T); L^1(I, \mathbb{R}^{dim}))$. Note that in this paper the goal is to obtain the existence, uniqueness and $\mathscr{C}-$differentiability of a nonlinear dynamic system solution with $(\nu,\rho) \equiv \Re_\varepsilon(V)$, where $V = (\nu_0\rho_0, \xi_e)$, in which ν_0, ρ_0 and ξ_e are respectively the velocity, initial density and function that models the membrane surrounding the tumor and $\Re_\varepsilon$ the satisfactory operator

$$\left\|\Re_\varepsilon(V) - \Re_\varepsilon(\overline{V})\right\| \leq \frac{\eta}{\|\Psi^{-1}\|}\left\|V - \overline{V}\right\|, \tag{1.1}$$

where Ψ is a continuous invertible operator and η a positive constant. Our approach is therefore to perturb the system, using measurable functions and operators, twice continuously differentiable in Banach spaces in order to obtain the proof of the differentiability of the solution (v, ρ). We end the introduction with a brief description of the content of the document. In section 2, we formulate the problem and give some preliminary notations that will be used in the following. Section 3 introduces the definition of a weak solution by rewriting the system equations in a particular frame, then we present a result of the existence and a criterion of weak uniqueness of the weak solution for the case of constant viscosity coefficients. It generalizes classical criteria like (Solonnikov's, 1978) criterion for the case of the initial-value problem through an additional property of the weak solution. Then we give approximate estimates (see Varga, 1971; Aubin, 1972) used during the process of passing limit in the solution. Finally, section 4 is about uniqueness and generalized differentiability of the solution when the system is perturbed by a number of parameters.

2. Formulation of the Problem

Cancer is an important area of research in medicine, but also the suject of applied mathematics research. Mathematics is used in particular to model the growth of cancerous tumors, with the main goal of optimizing treatment by increasing antitumor efficacy and decreasing toxicity on healthy cells. In this paper, we present a model of differential equations modeling the tumor across a given area. We consider a non-homogeneous region (variable density) as a function of time $I_t = I \times (0, T)$ occupied by the tumor, where I is a lipchitz bounded open set of $\mathbb{R}^3$ and let ∂I be its border. Let $x \in I$, the size of the tumor and $t \in (0, T)$, the time parameter. At the initial time $t = 0$, the tumor has a size x_0 in the I domain. The non-stationary model is then described by the following differential equations

$$\begin{cases} \dfrac{\partial \rho v_i}{\partial t} + \displaystyle\sum_{j=1}^{3} \frac{\partial}{\partial x_j}(\rho v_i \otimes v_i) = \rho \xi_e + L_{\lambda,\mu}(v)\,, & \forall (x,t) \in I \times (0,T), \quad i = 1,2,3 \\ \dfrac{\partial \rho}{\partial t} + \displaystyle\sum_{j=1}^{3} \frac{\partial \rho v_j}{\partial x_j} = 0\,,\ \rho \geq 0, & \forall (x,t) \in I \times (0,T), \\ \displaystyle\sum_{j=1}^{3} \frac{\partial v_j}{\partial x_j} = 0, & \forall x \in I \times (0,T), \end{cases} \tag{2.1}$$

where ξ_e denotes the density of the external forces and $L_{\lambda,\mu}(v)$ an operator formally defined by

$$L_{\lambda,\mu}(v) \overset{\text{def}}{:=} \mu div(\nabla v) + (\lambda + \mu)\nabla(\nabla v) - \frac{\partial \pi}{\partial x_i}, \quad i = 1,2,3$$

with λ and μ respectively representing the volumetric and dynamic viscosity coefficients supposed to be constant. The system is supplemented by initial conditions on density and proliferation rate

$$\rho\Big|_{t=0} = \rho_0(x), \qquad v\Big|_{t=0} = v_0(x), \qquad \rho v\Big|_{t=0} = q_0(x), \quad \forall (x,t) \in I \times \{0\}. \tag{2.2}$$

It is assumed that on the ∂I border of the domain I, the velocity checks the boundary conditions

$$\begin{cases} v\Big|_{\partial I} = 0, & \forall (x,t) \in \partial I \times (0,T), \\ \lim\limits_{|x| \to \infty} (v, \rho) = (0,0), & \forall\, t \in (0,T). \end{cases} \tag{2.3}$$

It should be mentioned that $\rho v \otimes v \in \mathbb{R}^3$ in $(2.1)_1$ is a tensor product of ρv and v, and that

$$\sum_{i,j=1}^{3} \frac{\partial}{\partial x_j}(\rho v_i \otimes v_i) = \sum_{i,j=1}^{3} \frac{\partial}{\partial x_j}(\rho v_i) v_i + \sum_{i,j=1}^{3} \rho v_j \frac{\partial v_i}{\partial x_j}. \tag{2.4}$$

Before announcing the results, it is necessary to define the domains in which we work. In this sub section, we introduce the notation that will be used throughout this document.

2.1 General Framework and Preliminaries

Let's give here some notations. The following function spaces provide a standard framework for obtaining the unique results of overall and differentiability of the solution of system (2.1) − (2.3).

The underlying domain. Let $I \subset \mathbb{R}^3$, a delimited domain ∂I its sufficiently smooth border. For $T > 0$, the interval $(0, T)$ defines the considered time interval and $I_t = I \times (0, T)$ a space-time domain with boundary $\partial I_t = \partial I \times (0, T)$.

Standard operators. $x = (x_1, x_2, x_3)$ is the space variable in $\mathbb{R}^3$. For $x, y \in \mathbb{R}^3$, $x \cdot y = \sum_{i=1}^3 x_i y_i$. ∇ is the gradient and Δ is the laplacian. When $\mathscr{G}(x) = (\mathscr{G}_1, \mathscr{G}_2, \mathscr{G}_3)$ is an $\mathbb{R}^3$-valued function,

$$\nabla \cdot \mathscr{G} = \sum_{i=1}^{3} \frac{\partial \mathscr{G}_i}{\partial x_i}, \quad |\nabla \mathscr{G}|^2 = \sum_{i,j=1}^{3} \Big|\frac{\partial \mathscr{G}_i}{\partial x_j}\Big|^2, \quad \|\mathscr{G}\|_{L^p(I;\mathbb{R}^3)} = \Big(\sum_{i=1}^{3} \|\mathscr{G}\|^p_{L^p(I;\mathbb{R}^3)} \Big)^{1/p},$$

$$\|\nabla \mathscr{G}\|_{L^p(I;\mathbb{R}^3)} = \Big(\sum_{i,j=1}^{3} \Big\|\frac{\partial \mathscr{G}_i}{\partial x_j}\Big\|^p_{L^p(I;\mathbb{R}^3)} \Big)^{1/p}.$$

Standard Lebesgue spaces. Let m be a non-negative integer. We denote by $H^m(I;\mathbb{R}^3)$ the usual Sobolev space $W^{m,2}(I;\mathbb{R}^3)$ as defined in (Lions and Magenes, 1972).

We note by $\mathcal{D}(I)$, the space of infinitely differentiable functions with compact support. Its closure in the norm $W^{m,p}(I;\mathbb{R}^3)$ $(1 < p < s < +\infty)$ is noted by $W_0^{m,p}(I;\mathbb{R}^3)$. An alternate characteristic in the case where $m = 1$ and $p = 2$ is

$$H_0^1(I;\mathbb{R}^3) = \{v \in W^{1,2}(I;\mathbb{R}^3) : \gamma_0 v = 0 \text{ on } \partial I\},$$

where γ_0 is the v trace operator. We also note by $L^p(I)^3 = L^p(I;\mathbb{R}^3)$, the lebesgue space on I with the norm $\|.\|_p$ and by $\|.\|_{\mathbb{E}}$ the norm associated to a space $\mathbb{E}$. If $\mathbb{E}$ is a Banach space, $L^p(0,T;\mathbb{E})$ is the Banach space composed of functions, measurable on $(0,T)$ which values in $\mathbb{E}$. For details concerning these spaces, see (Adams, 1945) or (Girault ,1986).
Let introduce the solenoidal spaces. We consider zero divergence spaces introduced for the problem (2.1) – (2.3).

$$\begin{aligned} \mathbb{K}^1_{div} &:= \Big\{ v \in L^2(I;\mathbb{R}^3) : \sum_{j=1}^{3} \frac{\partial v_j}{\partial x_j} = 0,\ v \cdot \mathbf{n}\big|_{\partial I} = 0 \Big\}, \\ \mathbb{K}^0_{div} &:= \Big\{ v \in H_0^1(I;\mathbb{R}^3) : \sum_{j=1}^{3} \frac{\partial v_j}{\partial x_j} = 0 \Big\}, \\ C^\infty_{0,\sigma}(I;\mathbb{R}^3) &:= \Big\{ v \in \mathcal{D}(I;\mathbb{R}^3) : \sum_{j=1}^{3} \frac{\partial v_j}{\partial x_j} = 0 \Big\}, \end{aligned}$$

where $\mathbb{K}^1_{div}$ and $\mathbb{K}^0_{div}$ are the respective closure of $C^\infty_{0,\sigma}(I;\mathbb{R}^3)$ in $L^2(I;\mathbb{R}^3)$ and $H_0^1(I;\mathbb{R}^3)$.
Let us define the Stokes operator $\mathbf{A} : \mathbf{D}[\mathbf{A}] \cap \mathbb{K}^1_{div} \longrightarrow \mathbb{K}^1_{div}$ by

$$\mathbf{A} := -\mathbb{P}\Delta,\ \mathbf{D}[\mathbf{A}] = W^{2,2}(I;\mathbb{R}^3) \cap \mathbb{K}^1_{div}, \tag{2.5}$$

where $\mathbb{P} : L^2(I;\mathbb{R}^3) \longrightarrow \mathbb{K}^1_{div}$ is the orthogonal projection. Note also that, we have

$$\int_I \mathbf{A}v \cdot \phi dx = \int_I \nabla v \cdot \nabla \phi dx, \quad \forall v \in \mathbf{D}[\mathbf{A}],\ \phi \in \mathbb{K}^0_{div}. \tag{2.6}$$

It should also be noted that $\mathbf{A}^{-1} : \mathbb{K}^1_{div} \longrightarrow \mathbb{K}^1_{div}$ is a self-adjoint compact operator on $\mathbb{K}^1_{div}$ and by the classical spectral theorems, there exists a sequence $\ell_j > 0$ and a sequence function $\{\phi_j(x)\}_{j=1}^\infty \in \mathbf{D}[\mathbf{A}]$ such as $\mathbf{A}\phi_j = \ell_j \phi_j$ (for the existence and regularity of these functions see for example (Layzhenskaya, 1969) and (Temam, 1977).

Let us now give the definition of a weak solution for the system (2.1) – (2.3).

3. Weak Solution : Existence and Uniqueness

First, let's give the definition of the weak solution.

Definition 1 *Let I be a bounded domain in $\mathbb{R}^3$ with smooth boundary, and assume that the data $v_0(x)$, $\rho_0(x)$, $\xi_e(x,t)$ satisfy the regularity conditions $v_0 \in \mathbb{K}^1_{div}$, $\rho_0 \in W^{1,2}(I;\mathbb{R})$, $\xi_e(x,t) \in L^1((0,T); L^{\frac{2s}{s-1}}(I;\mathbb{R}^3))$, $(2 \le s < \infty)$. Then (v,ρ) is a solution of the problem* (2.1) – (2.3) *on* $(0,T)$ *corresponding to the initial conditions v_0 and ρ_0 if the following conditions are met :*
i) v and ρ satisfy

$$\begin{cases} v(x,t) > 0 \text{ and } v(x,t) \in L^2((0,T);\mathbb{K}^0_{div}) \cap L^\infty((0,T);\ \mathbb{K}^1_{div}), \\ \rho(x,t) > 0 \text{ and } \rho(x,t) \in L^\infty((0,T); W^{1,2}(I;\mathbb{R})), \\ \rho v \in C((0,T); L_w^{\frac{2s}{s+1}}(I;\mathbb{R}^3)), \end{cases} \tag{3.1}$$

where $C((0,T);X_w)$ is the space of continuous functions of $(0,T)$ with values in a closed ball of X equipped with the weak topology of the separable Banach space X.

ii) For all $\mathcal{G} \in C^1((0,T);\mathbb{K}^0_{div})$

$$-\int_0^T\int_I(\rho v\cdot\frac{\partial\mathcal{G}}{\partial t})dxdt-\sum_{j=1}^{3}\int_0^T\int_I\rho v_j\cdot\frac{\partial\mathcal{G}}{\partial x_j}v\,dxdt+(\lambda+2\mu)\sum_{j=1}^{3}\int_0^T\int_I\frac{\partial v}{\partial x_j}\cdot\frac{\partial\mathcal{G}}{\partial x_j}dxdt$$
$$=-\underbrace{\int_I(v(x,T)\mathcal{G}(x,T))dx}_{=0}+\int_I(v_0\cdot\mathcal{G}(x,0))dx+\int_0^T\int_I(\rho\xi_e\cdot\mathcal{G})dxdt. \tag{3.2}$$

iii) For all $\mathcal{Y} \in C^1((0,T);W^{1,2}(I;\mathbb{R}))$

$$-\int_0^T\int_I\rho\cdot\frac{\partial\mathcal{Y}}{\partial t}dxdt-\sum_{j=1}^{3}\int_0^T\int_I\rho v_j\cdot\frac{\partial\mathcal{Y}}{\partial x_j}dxdt=\int_I\rho_0(x)\mathcal{Y}(x,0)dx. \tag{3.3}$$

iv) Moreover, the following initial conditions hold in the weak sense ,i.e., for every $\upsilon \in \mathbb{K}^0_{div}$, we have $(v(.,t),\upsilon)\xrightarrow[t\to+\infty]{}(v_0,\upsilon)$ and for every $\Upsilon \in W^{1,2}(I;\mathbb{R})$, we have $(\rho(.,t),\Upsilon)\xrightarrow[t\to+\infty]{}(\rho_0,\Upsilon)$.

The next section, we discuss the existence and uniqueness of weak solution results for the system (2.1) − (2.3).

3.1 Existence

In this section, we are interested in a result of existence of a weak solution for the model (2.1) − (2.3) modeling the tumor in a three-dimensional I domain with volumetric viscosity and dynamic coefficients supposed to be constant that satisfy the following conditions :

$$\mu \geq 0, \quad 2\mu+\lambda>0. \tag{3.4}$$

The unknowns are the volume density $\rho(x,t)$, the tumor cell velocity field $v(x,t)$ and the pressure π that appears under the effect of tumor cell movements. The first result of this document is the following lemma on the existence of weak solutions to the (2.1) system, subject to (2.2) − (2.3).

Theorem 2 *Let $I \subset \mathbb{R}^3$ be a Lipschitz bounded domain with a regular border ∂I. Let $\rho_0 \in W^{1,2}(I;\mathbb{R})$ and $\xi_e \in L^1((0,T);L^{\frac{2s}{s-1}}(I;\mathbb{R}^3))$, $2 \leq s < \infty$, Furthermore, suppose $\beta > 0$ such that $\beta \leq \rho$. Then, for a given $T > 0$, there exists a unique weak solution $(v\,,\rho)$ of the problem (2.1) − (2.3) such that*

$$\begin{cases} v \in L^2((0,T);\mathbb{K}^0_{div}) \cap L^\infty((0,T);\,\mathbb{K}^1_{div}),\\ \rho \in L^\infty((0,T);W^{1,2}(I;\mathbb{R})),\\ \dfrac{\partial v}{\partial t} \in L^{2-\gamma}((0,T);\mathbb{K}'^0_{div}), \quad \gamma \in (0,1),\\ \dfrac{\partial\rho}{\partial t} \in L^2((0,T);L^2(I;\mathbb{R})). \end{cases}$$

Proof. For proof, we establish the Galerkin approximation to the (2.1) − (2.3) system. We first present the approximation scheme, then we estimate a priori the approximate solution, and finally we perform the process of passing to the limit to approximate solutions.

Step 1 (Construction of Approximate Solutions)

We construct approximate solutions using a semi-discrete Galerkin scheme as in (Dautary, 1992) and (Nakagiri, 1998). To implement it, we take a basic functional subset of $\mathbb{K}^1_{div}$ as follows

$$K_m := \Big\{\sum_{k=1}^{m}\ell_k\phi_k\,,\ \ell_1\leq\ell_2\leq\cdots\leq\ell_m\xrightarrow[m\to+\infty]{}\infty,\ \phi_k\in C^1(\bar{I})\Big\}. \tag{3.5}$$

We denote by $\{\phi_k(x)\}_{k=1}^m$ linearly independent generalized eigenfunctions corresponding to each distinct unstable eigenvalue ℓ_k of the operator $\mathbf{A} = -\mathbb{P}\Delta$ defined on $\mathbf{D}[\mathbf{A}] \cap \mathbb{K}^1_{div} \longrightarrow \mathbb{K}^1_{div}$, and $\mathbb{P}$ the orthogonal projection of $L^2(I;\mathbb{R}^3)$ on $\mathbb{K}^1_{div}$. consider the eigenvalue problem

$$\mathbb{P}\Delta\phi_j+\mathfrak{K}_j\phi_j=0, \quad \forall(x,t)\in I\times(0,T) \quad \text{and} \quad \phi_j=0, \quad x\in\partial I. \tag{3.6}$$

It is well known that $\{\phi_j(x)\}_{j=1}^\infty$ forms a complete orthogonal system in the space $\mathbb{K}^0_{div}$. For a detailed analysis of the convergence of expansions of eigenfunctions and the regularity of eigenfunctions, see (Ladyzhenskaya, 1969). Suppose that ρ_0, and ξ_e satisfy the assumptions of theoreme 2. By regularizing the initial density, we choose ρ_{0m} so that

$$\begin{cases} \rho_{0m} \in C^1(\bar{I}), \\ 0 < \underline{\rho_0} \le \rho_{0m} \le \overline{\rho_0}, \\ v_{0m} \in \mathbb{K}_m \quad v_{0m} \longrightarrow v_0, \text{ in } \mathbb{K}^1_{div}. \end{cases} \tag{3.7}$$

We define approximate solutions for the formulation (2.1) − (2.3) as follows : We say that (v_m, ρ_m) is an approximate solution if $(v_m, \rho_m) \in C^1((0,T); K_m) \times C^1((0,T); \bar{I})$ such as for the entire domain I

$$\begin{cases} \dfrac{\partial \rho_m}{\partial t} + \displaystyle\sum_{j=1}^{3} v_j \dfrac{\partial \rho_m}{\partial x_j} = 0, \quad \forall (x,t) \in I \times (0,T), \\ \rho_m|_{t=0} = \rho_{0m}(x), \quad \forall (x,t) \in I \times \{0\}, \end{cases} \tag{3.8}$$

and

$$\begin{cases} \displaystyle\int_I \Big(\rho_m \frac{\partial v_m}{\partial t} + \nabla : \rho_m v_m \otimes v_m\Big).\phi_k \, dx + (2\mu+\lambda)\int_I \nabla v_m.\nabla \phi_k dx = \int_I \rho_m \xi_{em} \phi_k dx, \\ v_m(x,t) = \displaystyle\sum_{k=1}^{m} \ell_{mk}(t)\, \phi_k(x), \quad v_m\Big|_{t=0} = v_0(x), \quad \forall (x,t) \in I \times \{0\}, \end{cases} \tag{3.9}$$

where ϕ_k , $k = 1, \cdots, m$ is a base of K_m, $m \in \mathbb{N}$ and $\ell_{mk}(t)$ is defined through the following system :

$$\begin{cases} \displaystyle\sum_{k,p=1}^{m} \int_I \rho_m \phi_k \phi_p dx \, \frac{\partial \ell_{mk}}{\partial t} + \sum_{k,p=1}^{m} \int_I \rho v \cdot \frac{\partial \phi_k}{\partial x_k} \phi_p \, dxdt + \\ +(2\mu+\lambda) \displaystyle\sum_{k,p=1}^{m} \int_I \frac{\partial \phi_k}{\partial x_k} \frac{\partial \phi_p}{\partial x_p} dxdt = \int_I \rho_m \xi_{em} \phi_p dx, \\ \ell_{mk}(0) = \langle v_0(x), \phi_k(x) \rangle_I, \quad \forall (x,t) \in I \times \{0\}, \quad k = 1, \cdots, m. \end{cases} \tag{3.10}$$

In order to resolve system (3.8), we use the classical method of characteristics to construct a solution. We thus have the following result.

Lemma 3 *Let $v \in C((0,T); C(\bar{I}))$ so that $div(v) = 0$ for all $(x,t) \in (0;T) \times \bar{I}$, $v(x,t) = 0$, for all $(x,t) \in (0,T) \times \partial I$ and $\rho(x) \in C^1(\bar{I})$, $\underline{\rho}_0 \le \rho_{0m} \le \bar{\rho}_0$ for all $(x,t) \in (0,T) \times \bar{I}$. Then (3.8) has a unique solution $\rho \in C^1((0,T) \times \bar{I})$. Moreover, for every $(x,t) \in (0,T) \times \bar{I}$, $\beta \le \rho \le \bar{\rho}$.*

Proof. The proof is standard and we can refer to (Kim, 1987) with a small change in the volumic density.
To simplify the mathematical formulations of the system (3.10), we introduce the following notations

$$(a^m_{kp})_{1 \le k,p \le m} = \int_I \rho_m \phi_k \phi_p dx \in C^1(0,T), \tag{3.11}$$

$$(b^m_{kp})_{1 \le k,p \le m} = \int_I \rho_m v_k \frac{\partial \phi_k}{\partial x_k} \phi_p dx \in C(0,T), \tag{3.12}$$

$$(c^m_{kq})_{1 \le k,p \le m} = (2\mu+\lambda) \int_I \frac{\partial \phi_k}{\partial x_k} \frac{\partial \phi_p}{\partial x_p} dx \in C(0,T), \tag{3.13}$$

$$(d^m_{p})_{1 \le k,p \le m} = \int_I \rho_m \xi_{em} \phi_p dx \in C(0,T). \tag{3.14}$$

So we can rewrite the above system of differential equations in matrix form as follows

$$A_m(t)\ell'_m(t) + (B_m(t) + C_m(t))\ell_m(t) = D_m(t), \tag{3.15}$$

where $A_m(t) = (a^m_{kp})_{m\times m}$, $B_m(t) = (b^m_{kp})_{m\times m}$, $C_m(t) = (c^m_{kp})_{m\times m}$, and $D_m(t) = (d^m_p)_{m\times m}$.
Since $v \in C((0,T); C(\bar{I}))$ and $\rho \in C^1((0,T) \times \bar{I})$, it is clear that the matrices defined above belong to $C(0,T)$. The matrix $(a^m_{kp})_{m\times m}$ is symmetric positive definite , thanks to the orthogonality of $(\phi_k)_{k=1,..,m}$ in $\mathbb{K}^0_{div}$. In particular, the matrix $(a^m_{kp})_{1\le k,p \le m}$ is non-singular. Then (3.15) can be written as

$$\begin{cases} \ell'_m(t) + A_m^{-1}(t)(B_m(t) + C_m(t))\ell_m(t) = A_m^{-1}(t) D_m(t), \\ \ell_{mk}(0) = \langle v_0(x), \phi_k(x) \rangle, \quad \forall (x,t) \in I \times \{0\}, \quad k = 1, ..., m. \end{cases} \tag{3.16}$$

Since $A_m(t), B_m(t), C_m(t), D_m(t) \in C(0,T)$, as well as $A_m^{-1}(t)$, the resolution of the initial value problem of the above system follows from the classical theory of ordinary differential equations, so we are assured of the existence and uniqueness of the solution of (3.10) and therefore, the one of the problem (3.9).

Step 2 (Parametric Sensitivity of Solutions). Let's give a prior estimate of the solutions of (3.8) − (3.9) with variable density and constant viscosity. For tumor-related reasons, we consider that the proliferation rate fields disappear on ∂I (see the condition (2.3)), and for this reason, we only consider the limit data from Dirichlet for v.

Lemma 4 *(estimated solution with low velocity assumption). Let I be a bounded domain in $\mathbb{R}^3$ with smooth boundary, and assume that the data $v_m(0)$, $\rho_m(0)$, ξ_{em} satisfy the regularity conditions $v_m(0) = v_{0m} \in \mathbb{K}^1_{div}$, $\rho_m(0) = \rho_{0m} \in L^2(I;\mathbb{R})$, $\xi_{em} \in L^1((0,T); L^{\frac{2s}{s-1}}(I;\mathbb{R}^3))$.*
For $T > 0$ (fixed), suppose there is a constant $\beta > 0$ such that $\forall (x,t) \in I \times (0,T)$

$$0 < \beta \leq \rho_m \leq \bar{\rho}, \quad p.\,p \text{ in } I. \tag{3.17}$$

Then, there is a solution $(v\,,\rho)$ of the system (3.8) − (3.9) satisfying the initial conditions (2.2) − (2.3) and the following inequality :

$$\|v_m\|^2_{\mathbb{K}^1_{div}} \leq \beta^{-1}\Big[\Big(\|\rho_{0m}v_{0m}\|^2_{L^{\frac{2s}{s+1}}(I;\mathbb{R}^3)} + \|\xi_{em}\|^2_{L^2((0,T);L^{\frac{2s}{s-1}}(I;\mathbb{R}^3))}\Big)\exp(T)\Big]. \tag{3.18}$$

If ρ_m has the additional regularity $\rho_{0m} \in W^{1,2}(I;\mathbb{R})$, $\rho_m \in L^\infty((0,T); W^{1,2}(I;\mathbb{R}))$, then

$$\|\rho_m\|_{W^{1,2}(I;\mathbb{R})} \leq |\rho_{0m}| exp\Big(\int_0^t \|\nabla v_m\|_{L^p(I;\mathbb{R}^3)} ds\Big). \tag{3.19}$$

Proof. By combining $(2.1)_1$ and $(2.1)_2$ and multiplying by v_m, we then obtain, by integration on the volume I, the following variational formulation

$$\int_I \rho_m\Big(\frac{\partial v_m^2}{\partial t}\Big)dx + \int_I (\nabla : \rho_m v_m \otimes v_m)v_m dx + \int_I \nabla\pi(\rho_m)\, v_m dx - \int_I \rho_m \xi_{em} v_m dx$$

$$= \mu\int_I (\Delta v_m)v_m dx + (\lambda + \mu)\int_I \nabla div(v_m)v_m dx. \tag{3.20}$$

Applying the derivation theorem, the first term on the left gives the following estimate

$$\int_I \rho_m\Big(\frac{\partial v_m}{\partial t}\Big)v_m dx = \frac{1}{2}\frac{d}{dt}\int_I \rho_m |v_m|^2 dx, \quad \forall\, t \in (0,T). \tag{3.21}$$

The slow mode reaction-diffusion equations allows as to write that the integral on the I volume of the term $\nabla : \rho_m v_m \otimes v_m$ is zero. Indeed we have

$$\int_I (\nabla : \rho_m v_m \otimes v_m)v_m dx = \frac{1}{2}\int_{\partial I} \rho_m v_{mj} v_{mi} v_{mi}\,\vec{n}\, ds - \frac{1}{2}\int_I \sum_{i,j=1}^{3} \rho_m \frac{\partial v_{mj}}{\partial x_j} v_{mi} v_{mi} dx.$$

Since $v_m = 0$ on ∂I, the boundary terms disappear. Further, thanks to the hypothesis of small speeds we have $\displaystyle\sum_{i,j=1}^{3} \rho_m \frac{\partial v_{mj}}{\partial x_j} = 0$, and so we find

$$\int_I \nabla : \rho_m v_m \otimes v_m dx = 0, \qquad \forall\, t \in (0,T). \tag{3.22}$$

(i) Estimate of $\mu\int_I (\Delta v_m)v_m dx$

Integrating by parts (Green's formula), we get

$$\mu\int_I (\Delta v_m)v_m dx = \mu\int_{\partial I} y_0 v_m(\nabla v_m \cdot \vec{n})ds - \mu\int_I tr(\nabla v_m \cdot \nabla^T v_m)dx,$$

(where y_0 is the unique continuous linear map defined from $W^{1,2}_0(I;\mathbb{R}^3) \to L^2(I;\mathbb{R}^3)$ such that $y_0 v_m = 0$, where $\vec{n}$ is the normal at the border of I, denoted ∂I and ds its surface element). It follows that

$$\mu\int_I (\Delta v_m)v_m dx = -\mu\sum_{i,j=1}^{3}\int_I \frac{\partial v_{mi}\partial v_{mj}}{\partial x_i \partial x_j} dx \leqq -\mu\int_I \Big\|\frac{Dv_m}{Dt}\Big\|^2 dx. \tag{3.23}$$

(ii) Estimate of $(\lambda+\mu)\int_I \nabla div(v_m)v_m\,dx$

$$(\lambda+\mu)\int_I \nabla div(v_m)v_m dx = (\lambda+\mu)\Big(\int_I \nabla(v div(v_m))dx - \int_I \Delta v_m^2 dx\Big),$$

$$(\lambda+\mu)\int_I \nabla div(v_m)v_m dx \leqq -(\lambda+\mu)\int_I \Big\|\frac{Dv_m}{Dt}\Big\|^2 dx. \tag{3.24}$$

Finally, the force provided by the membrane : $\forall\, t\in(0,T)$,

$$\int_I \rho_m\xi_{em}v_m dx \leq \big\|\rho_m v_m\big\|_{L_w^{\frac{2s}{s+1}}(I;\mathbb{R}^3)}\big\|\xi_{em}\big\|_{L^{\frac{2s}{s-1}}(I;\mathbb{R}^3)}. \tag{3.25}$$

Considering estimates (3.21) − (3.25), the equality (3.20) becomes :

$$\frac{1}{2}\frac{d}{dt}\int_I \rho_m\big|v_m\big|^2 dx \leq -(2\mu+\lambda)\int_I \Big\|\frac{Dv_m}{Dt}\Big\|^2 dx + \big\|\rho_m v_m\big\|_{L_w^{\frac{2s}{s+1}}(I;\mathbb{R}^3)}\big\|\xi_{em}\big\|_{L^{\frac{2s}{s-1}}(I;\mathbb{R}^3)} \tag{3.26}$$

$$\frac{1}{2}\frac{d}{dt}\int_I \rho_m\big|v_m\big|^2 dx \leq \big\|\rho_m v_m\big\|_{L_w^{\frac{2s}{s+1}}(I;\mathbb{R}^3)}\big\|\xi_{em}\big\|_{L^{\frac{2s}{s-1}}(I;\mathbb{R}^3)} \tag{3.27}$$

Applying Young's inequality, the estimate (3.27) becomes

$$\frac{1}{2}\frac{d}{dt}\int_I \rho_m\big|v_m\big|^2 dx \leq \frac{1}{2}\big\|\rho_m v_m\big\|^2_{L_w^{\frac{2s}{s+1}}(I;\mathbb{R}^3)} + \frac{1}{2}\big\|\xi_{em}\big\|^2_{L^{\frac{2s}{s-1}}(I;\mathbb{R}^3)} \tag{3.28}$$

If we integrate the inequality (3.28) on $(0,t)$ we get

$$\int_I \rho_m\big|v_m\big|^2 dx \leq \int_0^t \big\|\rho_{sm}v_{sm}\big\|^2_{L_w^{\frac{2s}{s+1}}(I;\mathbb{R}^3)} ds + \big\|\rho_{0m}v_{0m}\big\|^2_{L^{\frac{2s}{s+1}}(I;\mathbb{R}^3)} + \big\|\xi_{em}\big\|^2_{L^1((0,T);L^{\frac{2s}{s-1}}(I;\mathbb{R}^3)} \tag{3.29}$$

Applying the inequality of the Gronwall Lemma (Lions, 1972), the inequality (3.29) becomes, for $T>0$, fixed

$$\big\|v_m\big\|^2_{\mathbb{K}^1_{div}} \leq \beta^{-1}\Big(\big\|\rho_{0m}v_{0m}\big\|^2_{L^{\frac{2s}{s+1}}(I;\mathbb{R}^3)} + \big\|\xi_{em}\big\|^2_{L^1((0,T);L^{\frac{2s}{s-1}}(I;\mathbb{R}^3)}\Big)\exp(T). \tag{3.30}$$

On the other hand, it is easy to see that $(2.1)_2$ can be in the form

$$\frac{\partial}{\partial t}(\rho_m)^2 + \sum_{i,j=1}^{3}\frac{\partial}{\partial x_j}(\rho_m)^2\cdot v_{mi} = -2(\nabla\rho_m\cdot v_m)\rho_m. \tag{3.31}$$

By integrating on the I domain, we obtain

$$\frac{1}{2}\frac{d}{dt}\big|\rho_m\big|^2 \leq 2\int_I \big|\nabla\rho_m\big|^2\big|\nabla v_m\big|dx. \tag{3.32}$$

Integrating on $(0\,,t)$ and using Gronwall lemma inequality

$$\big\|\rho_m\big\|_{W^{1,2}(I;\mathbb{R})} \leq \big|\rho_{0m}\big|exp\Big(\varpi_1\int_0^t \|\nabla v_m\|_{L^p(I;\mathbb{R}^3)}ds\Big),\ 2\leq p<\infty.$$

■

Theorem 5 *Let I be a bounded domain in $\mathbb{R}^3$ with smooth boundary, and assume that the data $v_m(0)$, $\rho_m(0)$, ξ_{em} satisfy the regularity conditions*

$$v_m(0) = v_{0m}\in\mathbb{K}^1_{div},\ \rho_m(0) = \rho_{0m}\in L^2(I;\mathbb{R}),\ \xi_{em}\in L^1((0,T);L^{\frac{2s}{s-1}}(I;\mathbb{R}^3)).$$

So for $2\leq p<s<\infty$ and $2\mu+\lambda>0$, there is a solution (v_m,ρ_m) of the system (3.8) − (3.9) satisfying the initial conditions (2.1) − (2.3) and the following inequality :

$$(2\mu+\lambda)\big\|\Delta v_m^2\big\|_{L^2((0,T);L^p(I;\mathbb{R}^3))} \leq \big\|\rho_m v_m\big\|^2_{L_w^{\frac{2s}{s+1}}(I;\mathbb{R}^3)} + \big\|\xi_e\big\|^2_{L^1((0,T);L^{\frac{2s}{s-1}}(I;\mathbb{R}^3))}$$

$$+\frac{1}{2}\left\|\rho_{0m}v_{0m}\right\|^2_{L^{\frac{2s}{s+1}}(I;\mathbb{R}^3)} \tag{3.33}$$

Proof. According to (3.26), we have the following inequality

$$\frac{1}{2}\frac{d}{dt}\int_I \rho_m\left|v_m\right|^2 dx \leq -(2\mu+\lambda)\int_I\left\|\frac{Dv}{Dt}\right\|^2 dx + \int_I \rho_m\xi_{em}v_m dx.$$

By integrating on $(0\,,T)$, we get the estimate

$$\frac{1}{2}\int_I \rho_m\left|v_m(.,T)\right|^2 dx + \left(2\mu+\lambda\right)\int_0^T \left\|div(\nabla v_m^2)\right\|_{L^p(I;\mathbb{R}^3)}dt$$

$$\leq \int_0^T\Big(\int_I \rho_m\xi_{em}v dx\Big)dt + \int_I(\rho_{0m}v_{0m})^2 dx, \tag{3.34}$$

$$\left(2\mu+\lambda\right)\int_0^T \left\|\Delta v_m^2\right\|_{L^p(I;\mathbb{R}^3)}dx \leq \int_0^T \left\|\rho_m v_m\right\|_{L_w^{\frac{2s}{s+1}}(I;\mathbb{R}^3)}\left\|\xi_e\right\|_{L^{\frac{2s}{s-1}}(I;\mathbb{R}^3)}dt$$

$$+\frac{1}{2}\left\|\rho_{0m}v_{0m}\right\|^2_{L^{\frac{2s}{s+1}}(I;\mathbb{R}^3)}, \tag{3.35}$$

$$\left(2\mu+\lambda\right)\int_0^T \left\|\Delta v_m^2\right\|_{L^p(I;\mathbb{R}^3)}dx \leq \left\|\rho_m v_m\right\|^2_{L_w^{\frac{2s}{s+1}}(I;\mathbb{R}^3)} + \left\|\xi_e\right\|^2_{L^1((0,T);L^{\frac{2s}{s-1}}(I;\mathbb{R}^3))}$$

$$+\frac{1}{2}\left\|\rho_{0m}v_{0m}\right\|^2_{L^{\frac{2s}{s+1}}(I;\mathbb{R}^3)}. \tag{3.36}$$

Which completes the proof of theorem 5. ■

Step 3 (Process of passing limit in the approximate solution)

For the rest, we use the compactness results introduced by (Aubin-Lions, 1969) and (Simon, 1987). Indeed, considering the fact that v_m is bounded in a compact of $C((0,T);\mathbb{K}^1_{div})$ and ρ_m is also bounded in a compact of $C((0,T);W^{1,2}(I;\mathbb{R}))$, we can define, taking in to account previous estimates, sequences extracted from (ρ_m) and (v_m) so that when $m\longrightarrow\infty$

$$\begin{cases} v_m \xrightarrow{weak-*} v, \text{ in } L^\infty((0,T)\,;\,\mathbb{K}^1_{div}),\\ v_m \xrightarrow{weak} v, \quad \text{in } L^2((0,T)\,;\,\mathbb{K}^0_{div}),\\ \rho_m \xrightarrow{weak-*} \rho, \text{ in } L^\infty((0,T);W^{1,2}(I;\mathbb{R})). \end{cases} \tag{3.37}$$

Using the fact that $\partial\rho_m$ is bounded in $L^\infty((0,T);W^{-1,2}(I;\mathbb{R}))$, we deduce that $\rho_m v_m \longrightarrow \rho v$ in $C_0^\infty(I;\mathbb{R}^3)$ and weak in $L^2((0,T);L^2(I;\mathbb{R}^3))$, and $(\partial\rho_m/\partial t)\longrightarrow\partial\rho/\partial t$ weak-* in $L^\infty((0,T);W^{-1,2}(I;\mathbb{R}))$, therefore $(v_m\partial\rho_m/\partial t)$ and $(\partial\rho_m v_m/\partial t)$ are bounded in $L^2((0,T);W^{-1,2}(I;\mathbb{R}))$. Thus, we can easily obtain the following convergences

$$\begin{cases} \displaystyle\int_0^T\int_I(\rho_m\frac{\partial v_m}{\partial t}\cdot\phi_j)dxdt \xrightarrow[m\to\infty]{} \int_0^T\int_I\rho\frac{\partial v}{\partial t}\cdot\phi_j dxdt,\\ \displaystyle\int_0^T\int_I(\nabla v_m\cdot\nabla\phi_j)dxdt \xrightarrow[m\to\infty]{} \int_0^T\int_I\nabla v\cdot\nabla\phi_j dxdt. \end{cases} \tag{3.38}$$

Then

$$\int_0^T\int_I(\nabla:\rho_m v_m\otimes v_m)\phi_j dxdt \xrightarrow[m\to\infty]{} \int_0^T\int_I\nabla:\rho v\otimes v\,\phi_j dxdt. \tag{3.39}$$

So, from (3.8) – (3.10), we get

$$-\int_0^T\int_I(\rho_m v_m\cdot\frac{\partial\phi_j}{\partial t})dxdt - \sum_{j=1}^3\int_0^T\int_I\rho_m v_{mj}\cdot\frac{\partial\phi_j}{\partial x_j}v_m\,dxdt + (\lambda+2\mu)\sum_{j=1}^3\int_0^T\int_I\frac{\partial v_m}{\partial x_j}\cdot\frac{\partial\phi_j}{\partial x_j}dxdt$$

$$=\int_I(v_m(0)\cdot\phi_j(x,0))dx + \int_0^T\int_I(\rho_m\xi_{em}\cdot\phi_j)dxdt.$$

On the other hand, ϕ_j is the unit orthogonal basis. So we can easily get that for any function $\phi_j \in \mathbb{K}^0_{div}$, $\phi_j(.,T) = 0$, there holds

$$-\int_0^T \int_I (\rho v \cdot \frac{\partial \phi_j}{\partial t}) dxdt - \sum_{j=1}^3 \int_0^T \int_I \rho v_j \cdot \frac{\partial \phi_j}{\partial x_j} v_m \, dxdt + (\lambda + 2\mu) \sum_{j=1}^3 \int_0^T \int_I \frac{\partial v}{\partial x_j} \cdot \frac{\partial \phi_j}{\partial x_j} dxdt$$

$$= \int_I (v(0) \cdot \phi_j(x,0)) dx + \int_0^T \int_I (\rho \xi_e \cdot \phi_j) dxdt.$$

This completes the proof of the existence of the weak solution. ■

3.2 Uniqueness of the Solution of the Problem (2.1) − (2.3)

This section is dedicated to the proof of the uniqueness of solutions to the system (2.1) − (2.3). Let us first consider the case where the initial density has a positive lower limit. We have the following theorem :

Theorem 6 *Let* (v, ρ, π) *and* $(\widetilde{v}, \widetilde{\rho}, \widetilde{\pi})$ *be two weak solutions of the problem* (2.1) − (2.3) *on* $I \times (0,T)$ *for a time* $T > 0$ *with the same initial data* (v_0, ρ_0, ξ_e) *with regularity* $v_0 \in \mathbb{K}^1_{div}$, $\rho_0 \in W^{1,2}(I;\mathbb{R})$, $\xi_e \in L^1((0,T); L^{\frac{2s}{s-1}}(I;\mathbb{R}^3))$. *If moreover* (v,ρ) *,*$(\widetilde{v}, \widetilde{\rho})$ *satisfy the regularity of definition 1, then* $v = \widetilde{v}$ *and* $\rho = \widetilde{\rho}$.

Proof. To explain the ideas clearly, we present a formal argument. Let's start by using the following equations

$$\begin{cases} \rho \dfrac{\partial v}{\partial t} + \nabla : (\rho v \otimes v) = \rho \xi_e + div(2\mu \mathbf{D}[v] + \lambda \, div(v) - \pi(\rho)), \quad \forall (x,t) \in I \times (0,T), \\ \dfrac{\partial \rho}{\partial t} + \displaystyle\sum_{i=1}^3 \rho \frac{\partial v_i}{\partial x_i} = 0, \quad \forall (x,t) \in I \times (0,T), \\ \displaystyle\sum_{j=1}^3 \frac{\partial v_j}{\partial x_j} = 0, \quad \forall (x,t) \in I \times (0,T), \\ \rho\big|_{t=0} = \rho_0(x), \quad v\big|_{t=0} = v_0(x). \end{cases} \tag{3.40}$$

$$\begin{cases} \widetilde{\rho} \dfrac{\partial \widetilde{v}}{\partial t} + \nabla : (\widetilde{\rho v} \otimes \widetilde{v}) = \widetilde{\rho} \xi_e + div(2\mu \mathbf{D}[\widetilde{v}] + \lambda \, div(\widetilde{v}) - \widetilde{\pi}(\widetilde{\rho})), \quad \forall (x,t) \in I \times (0,T), \\ \dfrac{\partial \widetilde{\rho}}{\partial t} + \displaystyle\sum_{i=1}^3 \widetilde{\rho} \frac{\partial \widetilde{v}_i}{\partial x_i} = 0, \quad \forall (x,t) \in I \times (0,T), \\ \displaystyle\sum_{j=1}^3 \frac{\partial \widetilde{v}_j}{\partial x_j} = 0, \quad \forall (x,t) \in I \times (0,T), \\ \widetilde{\rho}\big|_{t=0} = \widetilde{\rho}_0(x), \quad \widetilde{v}\big|_{t=0} = \widetilde{v}_0(x). \end{cases} \tag{3.41}$$

By summing the two systems (3.40) and (3.41) we get the following system :

$$\begin{cases} \dfrac{\partial(\rho - \widetilde{\rho})}{\partial t} + \nabla : (v - \widetilde{v} \otimes \widetilde{\rho}) + \nabla v(\rho - \widetilde{\rho}) = 0, \quad (x,t) \in I \times (0,T), \\ \rho \dfrac{\partial(v - \widetilde{v})}{\partial t} + \nabla : \rho v \otimes \nabla(v - \widetilde{v}) - (\lambda + 2\mu)\Delta(v - \widetilde{v}) + \nabla(\pi - \widetilde{\pi}) \\ \qquad = (\rho - \widetilde{\rho})(\xi_e - \partial_t \widetilde{v} - \nabla : v \otimes \widetilde{v}) - \nabla : \widetilde{\rho}(v - \widetilde{v}) \otimes \widetilde{v}, \\ \displaystyle\sum_{j=1}^3 \frac{\partial(v - \widetilde{v})_j}{\partial x_j} = 0, \\ \rho - \widetilde{\rho}\big|_{t=0} = (\rho_0 - \widetilde{\rho}_0)(x), \quad v - \widetilde{v}\big|_{t=0} = (v_0 - \widetilde{v}_0)(x). \end{cases} \tag{3.42}$$

Multiplying equation $(3.42)_1$ by $(\rho - \tilde{\rho})$ and integrating on I we get

$$\frac{d}{dt} \int_I \left|\rho - \widetilde{\rho}\right|^2 dx + \int_I \nabla v \left|\rho - \widetilde{\rho}\right|^2 dx \leq L\left|v - \widetilde{v}\right|\left|\nabla \widetilde{\rho}\right|\left|\rho - \widetilde{\rho}\right|, \tag{3.43}$$

$$\frac{d}{dt} \int_I \left|\rho - \widetilde{\rho}\right|^2 dx \leq L\left\|\nabla(v - \widetilde{v})\right\|_{L^p(I;\mathbb{R}^3)} \left\|\nabla \widetilde{\rho}\right\|_{W^{1,2}(I;\mathbb{R})} \left\|\rho - \widetilde{\rho}\right\|_{W^{1,2}(I;\mathbb{R})}, \tag{3.44}$$

$$\frac{d}{dt} \left\|\rho - \widetilde{\rho}\right\|^2_{W^{1,2}(I;\mathbb{R})} dx \leq (\lambda + 2\mu)\left\|\nabla(v - \widetilde{v})\right\|^2_{L^p(I;\mathbb{R}^3)} + \left\|\nabla \widetilde{\rho}\right\|^2_{W^{1,2}(I;\mathbb{R})} \left\|\rho - \widetilde{\rho}\right\|^2_{W^{1,2}(I;\mathbb{R})}. \tag{3.45}$$

On the other hand, by multiplying equation $(3.42)_2$ by $2(v - \widetilde{v})$ and integrating on I
we have the next estimate :

$$\frac{d}{dt}\int_I \rho\left|v-\widetilde{v}\right|^2 dx + (2\lambda+4\mu)\int_I \left|\nabla(v-\widetilde{v})\right|^2 dx$$

$$\leq 2\int_I \left|\rho-\widetilde{\rho}\right|\left|\xi_e - \partial_t\widetilde{v} - \nabla : v\otimes\widetilde{v}\right|\left|v-\widetilde{v}\right| + 2\rho\left|v-\widetilde{v}\right|^2\left|\nabla\widetilde{v}\right|dx. \tag{3.46}$$

Applying the inequality of Hölder and Sobolev we have the following estimate :

$$\frac{d}{dt}\left\|\sqrt{\rho}(v-\widetilde{v})\right\|^2_{L^2(I;\mathbb{R}^3)} + (2\lambda+4\mu)\left\|\nabla(v-\widetilde{v})\right\|^2_{L^p(I;\mathbb{R}^3)} \leq \left\|\nabla\widetilde{v}\right\|_{L^p(I;\mathbb{R}^3)}\left\|\sqrt{\rho}(v-\widetilde{v})\right\|^2_{L^2(I;\mathbb{R}^3)}$$

$$+\left\|\rho-\widetilde{\rho}\right\|^2_{W^{1,2}(I;\mathbb{R})}\left\|\xi_e - \partial_t\widetilde{v} - \nabla : v\otimes\widetilde{v}\right\|^2_{L^{\frac{2s}{s-1}}(I;\mathbb{R}^3)} \tag{3.47}$$

Summing the equations (3.46) and (3.47) we get :

$$\frac{d}{dt}\left\|\sqrt{\rho}(v-\widetilde{v})\right\|^2_{L^2(I;\mathbb{R}^3)} + (2\mu+\lambda)\left\|\nabla(v-\widetilde{v})\right\|^2_{L^p(I;\mathbb{R}^3)} + \frac{d}{dt}\left\|\rho-\widetilde{\rho}\right\|^2_{W^{1,2}(I;\mathbb{R})}$$

$$\leq \left\|\rho-\widetilde{\rho}\right\|^2_{W^{1,2}(I;\mathbb{R})}\left\|\xi_e - \partial_t\widetilde{v} - \nabla : v\otimes\widetilde{v}\right\|^2_{L^{\frac{2s}{s-1}}(I;\mathbb{R}^3)}$$

$$+\left\|\nabla\widetilde{v}\right\|_{L^p(I;\mathbb{R}^3)}\left\|\sqrt{\rho}(v-\widetilde{v})\right\|^2_{L^2(I;\mathbb{R}^3)} + \left\|\nabla\widetilde{\rho}\right\|^2_{W^{1,2}(I;\mathbb{R})}\left\|\rho-\widetilde{\rho}\right\|^2_{W^{1,2}(I;\mathbb{R})}$$

$$\frac{d}{dt}\Big(\left\|\sqrt{\rho}(v-\widetilde{v})\right\|^2_{L^2(I;\mathbb{R}^3)} + \left\|\rho-\widetilde{\rho}\right\|^2_{W^{1,2}(I;\mathbb{R})}\Big) + (2\mu+\lambda)\left\|\nabla(v-\widetilde{v})\right\|^2_{L^2(I;\mathbb{R}^3)}$$

$$\leq \Big(\left\|\rho-\widetilde{\rho}\right\|^2_{W^{1,2}(I;\mathbb{R})} + \left\|\sqrt{\rho}(v-\widetilde{v})\right\|^2_{L^2(I;\mathbb{R}^3)}\Big)\Big(\left\|\xi_e - \partial_t\widetilde{v} - \nabla : v\otimes\widetilde{v}\right\|^2_{L^{\frac{2s}{s-1}}(I;\mathbb{R}^3)}$$

$$+\left\|\nabla\widetilde{v}\right\|_{L^p(I;\mathbb{R}^3)} + \left\|\nabla\widetilde{\rho}\right\|^2_{W^{1,2}(I;\mathbb{R})}\Big). \tag{3.48}$$

Thus, the application of the Gronwall inequality completes the proof of theoreme 6.

■

4. Theorems of Existence, Uniqueness and $\mathscr{C}$−Differentiability of the Solution of the Perturbed Problem

4.1 Linearization of the 3D Dynamic System

We consider a bounded domain I with the same initial conditions. In this paragraph, we construct a linear functional perturbation that linearizes the equation $(2.1)_1$. However, let's look at the term $(v\cdot\nabla v)$ that appears in the equation $(2.1)_1$. It is at the root of the difficulties encountered in solving this problem. We will therefore linearize the system by substituting this term with the following perturbation :

$$\mathfrak{F}(\mathcal{H},\varphi) \overset{\text{def}}{:=} \mathcal{H}_p(x,t) + \partial_v\varphi(x,t,v,w), \tag{4.1}$$

whese φ is a measurable function with respect to (x,t), twice continuously differentiable with respect to $(v\,,w)\in\mathbb{R}^3\times\mathbb{R}^9$, and $\mathcal{H}_p = \mathcal{P}\vartheta$ a continuous integral operator (see Silvia, 2014) which, at any function ϑ, matches $\mathcal{H}_p$. That is written in expanded form :

$$\mathcal{H}_{\mathcal{P}}\vartheta(.,t) \overset{\text{def}}{:=} \int_0^T\int_I \mathcal{P}(x-y,t-t')\partial_v\vartheta(y,t',v,w)dydt, \quad t>t',\ \ \forall x,y\in I, \tag{4.2}$$

where the $\mathcal{P}(x-y,t-t')$ operator is a linear and continuous application in $I\times(0,T)$. Using the new functions introduced, the initial value problem (2.1) − (2.3) is reformulated as follows

$$\begin{cases} \rho(\dfrac{\partial v_i}{\partial t}) + \displaystyle\int_0^T\int_I \mathcal{P}(x-y,t-t')\partial_v\vartheta(y,t',v,w)dydt + \partial_v\varphi(x,t,v,w) - L_{\lambda,\mu}(v) = \rho\xi_e \\ \dfrac{\partial\rho}{\partial t} + \displaystyle\sum_{j=1}^3\dfrac{\partial\rho v_j}{\partial x_j} = 0, \qquad \forall(x,t)\in I\times(0,T), \\ \displaystyle\sum_{j=1}^3\dfrac{\partial v_j}{\partial x_j} = 0, \qquad \forall\in I\times(0,T), \\ \rho\Big|_{t=0} = \rho_0(x), \quad v\Big|_{t=0} = v_0(x), \quad \forall(x,t)\in I\times\{0\}, \\ v\Big|_{\partial I} = 0, \qquad \forall(x,t)\in\partial I\times(0,T), \\ \displaystyle\lim_{|x|\to\infty}(v,\rho) = (0,0), \qquad \forall\,t\in(0,T). \end{cases} \tag{4.3}$$

Note that this system is a simpler version of the (2.1) system since the term $(v \cdot \nabla v)$ has been replaced by $\mathfrak{F}(\mathcal{H}, \varphi)$. This approach introduced new variables v, w which are considered respectively as an argument of the $v(x,t)$ field and its divergence. We will then make hypotheses about the functions $\varphi(x,t,v,w)$ and $\vartheta(y,t',v,w)$ defined on $I \times (0,T) \times \mathbb{R}^3 \times \mathbb{R}^9 \longrightarrow \mathbb{R}^9$.

4.1.1 Assumptions and Definition

(H-1) : Let $\beta, \widetilde{\beta} > 0$ and $T > 0$ (fixed). For every $(v,w) \in \mathbb{R}^3 \times \mathbb{R}^9$, the functions $(x,t,v,w) \longmapsto \varphi(x,t,v,w)$ and $(y,t',v,w) \longmapsto \vartheta(y,t',v,w)$ are measurable and verify the following conditions

$$\left|\varphi(x,t,v,w)\right| \leq \beta^{-1}(|v|^2 + |w|^2)exp(T), \tag{4.4}$$

$$\left|\vartheta(y,t',v,w)\right| \leq \widetilde{\beta}^{-1}(|v|^2 + |w|^2)exp(T). \tag{4.5}$$

(H-2) : For almost every $(x,t),(y,t') \in I \times (0,T)$, the functions $(x,t,v,w) \longmapsto \varphi(x,t,v,w)$ and $(y,t',v,w) \longmapsto \vartheta(y,t',v,w)$ are twice continuously differentiable with respect to couple (v,w). Moreover : $\forall (v,w) \in \mathbb{R}^3 \times \mathbb{R}^9$

$$\left|\Delta_v \varphi\right| + \left|\Delta_w \varphi\right| \leq 4\beta^{-1} exp(T), \tag{4.6}$$

$$\left|\Delta_v \vartheta\right| + \left|\Delta_w \vartheta\right| \leq 4\widetilde{\beta}^{-1} exp(T). \tag{4.7}$$

(H-3) : Let $\mathcal{A}_\epsilon$ and $\mathcal{B}_\epsilon$ be two nonlinear F-differentiable and G-differentiable operators. We note by $\mathcal{A}_\epsilon^{''}$ and $\mathcal{B}_\epsilon^{''}$, the respective second differential of $\mathcal{A}_\epsilon$ and $\mathcal{B}_\epsilon$ defined as follows

$$\begin{aligned} \mathcal{A}_\epsilon^{''} : \mathbb{X}_v^0 = L^\infty((0,T);\mathbb{K}_{div}^1) &\longrightarrow \mathcal{L}_2(\mathbb{X}_v^0, L^2((0,T);L^2(I\,;\mathbb{R}^3))) \\ v(x,t) &\longmapsto \mathcal{A}_\epsilon^{''}(v)(x,t) \end{aligned}$$

$$\begin{aligned} \mathcal{B}_\epsilon^{''} : L^\infty((0,T);\mathbb{K}_{div}^1) &\longrightarrow \mathcal{L}_2(\mathbb{X}_v^0, L^2((0,T);L^2(I\,;\mathbb{R}^3))) \\ v(x,t) &\longmapsto \mathcal{B}_\epsilon^{''}(v)(x,t) \end{aligned}$$

Let the increases h and g be difined on $L^2((0,T);H^1(I\,;\mathbb{R}^3))$. We also note by $d[\mathcal{A}_\epsilon^{'}(v)g,h]$ and $d[\mathcal{B}_\epsilon^{'}(v)g,h]$ (for these notations see Trenoguine, 1985), the second derivative of $\mathcal{A}_\epsilon(v)$ and $\mathcal{B}_\epsilon(v)$ in v with $\mathcal{A}_\epsilon^{'}(v)g = d\mathcal{A}_\epsilon(v,g)$.
For an increase of h, independent of g, we have

$$\mathcal{A}_\epsilon^{'}(v+g)h - \mathcal{A}_\epsilon^{'}(v)h = \sum_{i=1}^{3} \partial_{w_i}^2 \varphi \frac{\partial^2 gh}{\partial x_i \partial t} + \partial_v^2 \varphi gh + R_\epsilon(v, gh), \tag{4.8}$$

$$\mathcal{B}_\epsilon^{'}(v+g)h - \mathcal{B}_\epsilon^{'}(v)h = \sum_{i=1}^{3} \partial_{w_i}^2 \vartheta \frac{\partial^2 gh}{\partial x_i \partial t} + \partial_v^2 \vartheta gh + R_\epsilon(v, gh). \tag{4.9}$$

For $h = g$ we deduce the following formulas

$$d[\mathcal{A}_\epsilon^{'}(v)g,h]_{h=g} = \sum_{i=1}^{3} \partial_{w_i}^2 \varphi \frac{\partial^2 g^2}{\partial x_i \partial t} + \partial_v^2 \varphi g^2, \tag{4.10}$$

$$d[\mathcal{B}_\epsilon^{'}(v)g,h]_{h=g} = \sum_{i=1}^{3} \partial_{w_i}^2 \vartheta \frac{\partial^2 g^2}{\partial x_i \partial t} + \partial_v^2 \vartheta g^2. \tag{4.11}$$

Let us now give the definition of the generalized solution of the perturbed problem (4.3).

Definition 7 *Let $v_0 \in \mathbb{K}_{div}^1$, $\rho_0 \in W^{1,2}(I\,;\mathbb{R})$. Generalized solution of the problem* (4.3) *is a couple of functions $(v,\rho) \in L^\infty((0,T);\mathbb{K}_{div}^1) \times L^\infty((0,T);W^{1,2}(I\,;\mathbb{R}))$ such as*

1. *integral equality is verified :*

$$\begin{aligned} &\int_0^T \langle \rho \partial_t v\,, \mathcal{G} \rangle dt - \mu \int_0^T \langle \nabla v, \nabla \mathcal{G} \rangle dt - (\lambda + \mu) \int_0^T \langle \nabla v, div(\mathcal{G}) \rangle dt \\ &+ \int_0^T \int_I \Big[\partial_v \varphi(x,t,v,\nabla v) + \int_0^T \int_I \mathcal{P}(x-y,t-t') \partial_v \vartheta(y,t',v,\nabla v) dy dt \Big] \mathcal{G} dx dt = \int_0^T \langle \rho \xi_e, \mathcal{G} \rangle dt, \end{aligned} \tag{4.12}$$

for any $\mathcal{G} \in C\Big((0,T);\mathbb{K}_{div}^1\Big)$.

2. *The function $\rho(x,t)$ admits a generalized derivative with respect to (x,t). Moreover we have*

$$-\int_0^T \langle \rho, \frac{\partial \mathcal{Y}}{\partial t}\rangle dt - \sum_{j=1}^{3}\int_0^T \langle \rho v_j, \frac{\partial \mathcal{Y}}{\partial x_j}\rangle dt = \langle \rho_0(x), \mathcal{Y}(x,0)\rangle, \tag{4.13}$$

for any $\mathcal{Y} \in C\Big((0,T); W^{1,2}(I;\mathbb{R})\Big)$.

4.2 Generalized Differentiability of Non Linear Operators

In this section we propose a new concept of generalized differentiability. This concept encompasses the standard notions of Fr*é*chet differentiability, strict differentiability, according to Gateaux and Lipschitz continuity. We now begin with our first definition of generalized differentiability.

Definition 8 *(strong $\mathscr{C}-$differentiability) Let $\mathbb{E}$ and $\mathbb{H}$ be two normed spaces, U an open set in $\mathbb{E}$. The operator $F' : U \longrightarrow \mathbb{H}$ is said to be strongly $\mathscr{C}-$differentiable at the point $v_0 \in U$, where $\mathscr{C}$ is the system of all compacts in $\mathbb{E}$, if :*

$$F'(v+g)h - F'(v)h = d[F'(v_0)g, h]_{h=g} + R(v, gh), \tag{a}$$

where $d[F'(v_0)g,h]_{h=g} \; : \; \mathbb{E} \to \mathcal{L}_2(\mathbb{E},\mathbb{H})$ is a linear and continuous operator, and $R(v,gh)$ satisfies the following condition:

$$\forall\, \epsilon > 0,\ \forall\, S \in \mathscr{C},\ \exists\, \delta > 0$$

$$(D_\epsilon) : \left\{\left(\begin{array}{c} v \in U,\ g,\ h \in S,\ \tau \in]-1,1[,\ \|v - v_0\| < \delta, \\ |v| < \delta,\ v + \tau g h \in U \end{array} \right) \Longrightarrow \|R(x, \tau g h)\| \le \epsilon\, |\tau| \right\}, \tag{b}$$

Proposition 9 *The operator F' is $\mathscr{C}-$continuous, respectively $\mathscr{C}-$differentiable on U, if and only if $F' : U \longrightarrow \mathbb{H}$ is differentiable according to Gateaux and the operator $d[F'(v_0)g,h]_{h=g}$ from U to $\mathbb{H}$ is continuous on U for each fixed $h \in \mathbb{E}$.*

Proposition 10 *Let $I \subset \mathbb{R}^3$ be a bounded Lipschitz domain with a regular border ∂I. For all $(x,t) \in I \times (0,T)$, let's define the operator $\mathcal{A}_\epsilon$ by* $\mathcal{A}_\epsilon : L^\infty((0,T);\mathbb{K}^1_{div}) \longrightarrow L^2((0,T);L^2(I;\mathbb{R}^3))$

$$v \longmapsto \varphi(x,t,v,\nabla v).$$

If φ and $\mathcal{A}_\epsilon$ satisfy the assumptions (H-1)–(H-3), then, $d\mathcal{A}_\epsilon(v,g)$ is $\mathscr{C}-$continuous and $\mathscr{C}-$differentiable on $W(0,T)$. On the other hand, $\mathcal{A}^{''}_\epsilon(v)g^2$ is continuous and is defined by the following operator

$$d[\mathcal{A}^{'}_\epsilon(v)g,h]_{h=g} = \mathcal{A}^{''}_\epsilon(v)g^2,$$

for $g \in L^2((0,T);H^1(I;\mathbb{R}^3))$, fixed.

Proof. Suppose (H-1)–(H-3) are checked, and that the operator $\mathcal{A}_\epsilon(v)$ checks the equality (a), we have for all small enough g and h

$$\mathcal{A}'_\epsilon(v+g)h - \mathcal{A}'_\epsilon(v)h = d[\mathcal{A}'_\epsilon(v)g,h]_{h=g} + R_\epsilon(v,gh).$$

For $\tau \in]-1,1[\,,\ \tau \neq 0,\quad \mathcal{A}'_\epsilon(v+\tau g)h - \mathcal{A}'_\epsilon(v)h = d[\mathcal{A}'_\epsilon(v)\tau g,h]_{h=g} + R_\epsilon(v,\tau gh).$

Let's introduce the following reflexive space

$$W(0,T) =: \left\{ \mathcal{A}_\epsilon(v) \in L^2((0,T);L^2(I;\mathbb{R}^3)) : \left| \begin{array}{l} \mathcal{A}'_\epsilon(v) \in \mathcal{L}(\mathbb{X}^0_v, L^2((0,T);L^2(I;\mathbb{R}^3))) \\ \mathcal{A}''_\epsilon(v) \in \mathcal{L}_2(\mathbb{X}^0_v, L^2((0,T);L^2(I;\mathbb{R}^3))) \end{array} \right. \right\}.$$

Let's divide the last equality by τ and taking the norm on $W(0,T)$, we have

$$\begin{aligned}\left\|\frac{\mathcal{A}^{'}_\epsilon(v+\tau g)h - \mathcal{A}^{'}_\epsilon(v)h}{\tau} - \mathcal{A}^{''}_\epsilon(v)h^2\right\|^2_{W(0,T)} &= \left\|\frac{\mathcal{A}^{'}_\epsilon(v+\tau g)h}{\tau} - \mathcal{A}^{''}_\epsilon(v)h^2 - \frac{\mathcal{A}^{'}_\epsilon(v)h}{\tau}\right\|^2_{W(0,T)} \\ &= \left\|\frac{\varphi(x,t,v+\tau g,\nabla v + \tau\nabla g)h}{\tau} - \partial^2_v\varphi h^2 \right. \\ &\quad \left. - \sum_{i=1}^{3}\partial^2_{w_i}\varphi\frac{\partial^2 h^2}{\partial x_i \partial t} - \frac{\varphi(x,t,v,\nabla v)h}{\tau}\right\|^2_{W(0,T)}\end{aligned} \tag{4.14}$$

In order to make writing easier, we often omit the variables x and t.

$$\begin{aligned}\left\|\frac{\mathcal{A}'_\epsilon(v+\tau g)h-\mathcal{A}'_\epsilon(v)h}{\tau}-\mathcal{A}''_\epsilon(v)h^2\right\|^2_{W(0,T)} &= \left\|\frac{\varphi(v+\tau g,\nabla v+\tau g)h}{\tau}-\frac{\varphi(v,\nabla v+\tau\nabla g)h}{\tau}\right.\\ &-\partial^2_w\varphi h^2+\frac{\varphi(v,\nabla v+\tau\nabla g)h}{\tau}-\frac{\varphi(v,\nabla v)h}{\tau}\\ &\left.-\sum_{i=1}^{3}\partial^2_{w_i}\varphi\frac{\partial^2 h^2}{\partial x_i\partial t}\right\|^2_{W(0,T)}\end{aligned} \tag{4.15}$$

$$\begin{aligned}\left\|\frac{\mathcal{A}'_\epsilon(v+\tau g)h-\mathcal{A}'_\epsilon(v)h}{\tau}-\mathcal{A}''_\epsilon(v)h^2\right\|^2_{W(0,T)} &\le \int_0^T\int_I\left(\left|\frac{\varphi(x,t,v+\tau g,\nabla v+\tau\nabla g)h}{\tau}\right.\right.\\ &\left.\left.-\frac{\varphi(x,t,v,\nabla v+\tau\nabla g)h}{\tau}-\partial^2_v\varphi h^2\right|^2\right)dxdt\\ &+\int_0^T\int_I\left(\left|\frac{\varphi(x,t,v,\nabla v+\tau\nabla g)h}{\tau}\right.\right.\\ &\left.\left.-\frac{\varphi(x,t,v,\nabla v)h}{\tau}-\sum_{i=1}^{3}\partial^2_{w_i}\varphi\frac{\partial^2 h^2}{\partial x_i\partial t}\right|^2\right)dxdt\end{aligned} \tag{4.16}$$

Using the Lagrange formula (Trenoguine, 1985) for a certain $\theta\in[0\,;1]$, inequality (4.16) becomes

$$\begin{aligned}\left\|\frac{\mathcal{A}'_\epsilon(v+\tau g)h-\mathcal{A}'_\epsilon(v)h}{\tau}-\mathcal{A}''_\epsilon(v)h^2\right\|^2_{W(0,T)} &\le \int_0^T\int_I\left(\left|\int_0^1\mathcal{A}''_\varepsilon(v+\theta\tau g,\nabla v+\tau\nabla g)h^2\right.\right.\\ &\left.\left.-\partial^2_v\varphi h^2\right|^2 d\theta\right)dxdt+\int_0^T\int_I\left(\left|\int_0^1-\sum_{i=1}^{3}\partial^2_{w_i}\varphi\frac{\partial^2 h^2}{\partial x_i\partial t}\right.\right.\\ &\left.\left.+\mathcal{A}''_\epsilon(v,\nabla v+\theta\tau\nabla g)h^2\right|^2 d\theta\right)dxdt\end{aligned} \tag{4.17}$$

$$\begin{aligned}\left\|\frac{\mathcal{A}'_\epsilon(v+\tau g)h-\mathcal{A}'_\epsilon(v)h}{\tau}-\mathcal{A}''_\epsilon(v)h^2\right\|^2_{W(0,T)} &\le \int_0^T\int_I\left(\left|\int_0^1\left(\left(\partial^2_v\varphi(x,t,v+\theta\tau g,\nabla v+\tau\nabla g\right)h^2\right.\right.\right.\\ &\left.\left.\left.-\partial^2_v\varphi h^2\right)\times\sum_{i=1}^{3}\partial^2_{w_i}\varphi\frac{\partial^2 h^2}{\partial x_i\partial t}\right|^2 d\theta\right)dxdt\\ &+\int_0^T\int_I\left(\left|\int_0^1\sum_{i=1}^{3}\left(\left(\partial^2_{w_i}\varphi(x,t,\nabla v+\theta\tau\nabla g\right)h^2\right.\right.\right.\\ &\left.\left.\left.-\partial^2_w\varphi h^2\right)\times\sum_{i=1}^{3}\partial^2_{w_i}\varphi\frac{\partial^2 h^2}{\partial x_i\partial t}\right|^2 d\theta\right)dxdt\end{aligned} \tag{4.18}$$

$$\begin{aligned}\left\|\frac{\mathcal{A}'_\epsilon(v+\tau g)h-\mathcal{A}'_\epsilon(v)h}{\tau}-\mathcal{A}''_\epsilon(v)h^2\right\|^2_{W(0,T)} &\le \int_0^1\left(\int_0^T\int_I\left|\partial^2_v\varphi\left(x,t,v+\theta\tau g,\nabla v+\tau\nabla g\right)h^2\right.\right.\\ &\left.\left.-\partial^2_v\varphi h^2\right|^2\times\left|\sum_{i=1}^{3}\partial^2_{w_i}\varphi\frac{\partial^2 h^2}{\partial x_i\partial t}\right|^2 dxdt\right)d\theta\\ &+\int_0^1\left(\int_0^T\int_I\left|\partial^2_v\varphi\left(x,t,\nabla v+\theta\tau\nabla g\right)h^2\right.\right.\\ &\left.\left.-\partial^2_v\varphi h^2\right|^2\times\left|\sum_{i=1}^{3}\partial^2_{w_i}\varphi\frac{\partial^2 h^2}{\partial x_i\partial t}\right|^2 dxdt\right)d\theta\end{aligned} \tag{4.19}$$

Passing limit in (4.19) for $\tau\longrightarrow 0$, we get

$$\boxed{\lim_{\tau\to 0}\frac{\mathcal{A}'_\epsilon(v+\tau g)h-\mathcal{A}'_\epsilon(v)h}{\tau}=\mathcal{A}''_\varepsilon(v)h^2}$$

On this result, it has been shown that $\mathcal{A}'_\epsilon(v)$ is differentiable according to Gateaux. Therefore, according to the proposition 9, we can say that $\mathcal{A}'_\epsilon(v)h$ is $\mathscr{C}-$continous and $\mathscr{C}-$differentiable on $W(0,T)$. Furthermore, $\mathcal{A}''_\epsilon(v)h^2$ is a bilinear operator and $\mathcal{A}''_\epsilon(v)h^2 \in \mathcal{L}_2(L^2((0,T), H^1(I\,;\mathbb{R}^3)); W(0,T))$.
Which concludes the proof. ■

Remark 1 Similary, we can show that $\mathcal{B}'_\epsilon(v)h$ is $\mathscr{C}-$continuous and $\mathscr{C}-$differentiable on $W(0,T)$. Furthermore, $\mathcal{B}''_\epsilon(v)h^2$ is a bilinear operator and $\mathcal{B}''_\epsilon(v)h^2 \in \mathcal{L}_2(L^2((0,T); H^1(I\,;\mathbb{R}^3)); W(0,T))$.

4.3 Lipschitz Approximation Scheme

Proposition 11 *Let I be a bounded Lipschitz and open set in* $\mathbb{R}^3$ *and* $T > 0$*, fixed. Let's consider* $v_1(x,t),\ v_2(x,t) \in \mathbb{X}^0_v = L^\infty((0,T); \mathbb{K}^1_{div})$ *and* $\nabla v_1, \nabla v_2 \in L^2((0,T); L^p(I\,;\mathbb{R}^3))$ $(2 \leq p < \infty)$*. Suppose then that the operator* ∇ *satisfies at each point of* $I \times (0\,,T)$ *the following inequality*

$$\left\|\nabla v_1 - \nabla v_2\right\|_{L^p(I\,;\mathbb{R}^3)} \leq exp(T)\left\|v_1 - v_2\right\|_{\mathbb{K}^1_{div}} \tag{4.20}$$

Suppose g is small enough and $\|g\|^2_{H^1(I\,;\mathbb{R}^3)} \leq \varpi exp(T)$ *and finally* $\left\|\frac{\partial^2 g}{\partial x \partial t}\right\|^2 \leq \|g\|^2_{H^1(I\,;\mathbb{R}^3)}$*. Then, there exists a constant* $\beta > 0$ *such that* $\forall(x,t) \in I \times (0,T)$*, the operator* $\mathcal{A}'_\epsilon(v)$ *satisfies*

$$\left\|\mathcal{A}'_\epsilon(v_1) - \mathcal{A}'_\epsilon(v_2)\right\|_{W(0,T)} \leq M(\beta,T)\left\|v_1 - v_2\right\|_{L^\infty((0,T);\mathbb{K}^1_{div})} \tag{4.21}$$

where $M(\beta,T) = 2\beta^{-1}\sqrt{T\,\varpi}\,exp(3T/2)(1+\sqrt{3})(1+exp(T))$.

Proof Suppose $v_1, v_2 \in \mathbb{X}^0_v = L^\infty((0,T); \mathbb{K}^1_{div})$

$$\begin{aligned}
\left\|\mathcal{A}'_\epsilon(v_1) - \mathcal{A}'_\epsilon(v_2)\right\|_{W(0,T)} &= \Bigg\| \sum_{i=1}^{3} \partial_{w_i}\varphi(v_1, \nabla v_1)\frac{\partial^2 g}{\partial x_i \partial t} + \partial_v \varphi(v_1, \nabla v_1) g \\
&\quad - \sum_{i=1}^{3} \partial_{w_i}\varphi(v_2, \nabla v_2)\frac{\partial^2 g}{\partial x_i \partial t} - \partial_v\varphi(v_2, \nabla v_2) g \Bigg\|_{W(0,T)} \\
&= \Bigg[\int_0^T \Bigg\| \partial_v\varphi(v_1, \nabla v_1) g + \sum_{i=1}^{3} \partial_{w_i}\varphi(v_1, \nabla v_1)\frac{\partial^2 g}{\partial x_i \partial t} \\
&\quad - \sum_{i=1}^{3} \partial_{w_i}\varphi(v_2, \nabla v_2)\frac{\partial^2 g}{\partial x_i \partial t} - \partial_v\varphi(v_2, \nabla v_2) g \Bigg\|^2_{W(0,T)} dt \Bigg]^{\frac{1}{2}}
\end{aligned} \tag{4.22}$$

$$\begin{aligned}
\left\|\mathcal{A}'_\epsilon(v_1) - \mathcal{A}'_\epsilon(v_2)\right\|_{W(0,T)} &= \Bigg[\int_0^T \Bigg\| \partial_v\varphi(v_1, \nabla v_1) g + \sum_{i=1}^{3} \partial_{w_i}\varphi(v_1, \nabla v_1)\frac{\partial^2 g}{\partial x_i \partial t} \\
&\quad - \partial_v\varphi(v_2, \nabla v_1) g - \sum_{i=1}^{3} \partial_{w_i}\varphi(v_2, \nabla v_1)\frac{\partial^2 g}{\partial x_i \partial t} \\
&\quad + \partial_v\varphi(v_2, \nabla v_1) g + \sum_{i=1}^{3} \partial_{w_i}\varphi(v_2, \nabla v_1)\frac{\partial^2 g}{\partial x_i \partial t} \\
&\quad - \sum_{i=1}^{3} \partial_{w_i}\varphi(v_2, \nabla v_2)\frac{\partial^2 g}{\partial x_i \partial t} - \partial_v\varphi(v_2, \nabla v_2) g dt \Bigg\|^2_{W(0,T)} \Bigg]^{\frac{1}{2}}
\end{aligned} \tag{4.23}$$

Using Minskowski's inequality, (4.23) becomes

$$\left\|\mathcal{A}'_\epsilon(v_1) - \mathcal{A}'_\epsilon(v_2)\right\|_{W(0,T)} \leq M_1 + M_2 + M_3 + M_4. \tag{4.24}$$

where

$$\begin{aligned}
M_1 &= \left[\int_0^T \left\|\partial_v\varphi(v_1,\nabla v_1)g - \partial_v\varphi(v_2,\nabla v_1)g\right\|^2_{W(0,T)} dt\right]^{\frac{1}{2}},\\
M_2 &= \left[\int_0^T \Big\|\sum_{i=1}^3 \partial_{w_i}\varphi(v_1,\nabla v_1)\frac{\partial^2 g}{\partial x_i\partial t} - \sum_{i=1}^3 \partial_{w_i}\varphi(v_2,\nabla v_1)\frac{\partial^2 g}{\partial x_i\partial t}\Big\|^2_{W(0,T)} dt\right]^{\frac{1}{2}},\\
M_3 &= \left[\int_0^T \left\|\partial_v\varphi(v_2,\nabla v_1)g - \partial_v\varphi(v_2,\nabla v_2)g\right\|^2_{W(0,T)} dt\right]^{\frac{1}{2}},\\
M_4 &= \left[\int_0^T \Big\|\sum_{i=1}^3 \partial_{w_i}\varphi(v_2,\nabla v_1)\frac{\partial^2 g}{\partial x_i\partial t} - \sum_{i=1}^3 \partial_{w_i}\varphi(v_2,\nabla v_2)\frac{\partial^2 g}{\partial x_i\partial t}\Big\|^2_{W(0,T)} dt\right]^{\frac{1}{2}}.
\end{aligned}$$

i) Estimate of $M_1 = \left[\int_0^T \left\|\partial_v\varphi(v_1,\nabla v_1)g - \partial_v\varphi(v_2,\nabla v_1)g\right\|^2_{W(0,T)} dt\right]^{\frac{1}{2}}$

$$\begin{aligned}
M_1^2 &\le \int_0^T \|g\|^2_{H^1(I;\mathbb{R}^3)}\left\|\partial_v\varphi(v_1,\nabla v_1) - \partial_v\varphi(v_2,\nabla v_1)\right\|^2_{W(0,T)} dt\\
&\le 4\beta^{-2}exp(2T)\int_0^T \|g\|^2_{H^1(I;\mathbb{R}^3)}\left\||v_1|_{\mathbb{K}^1_{div}} - |v_2|_{\mathbb{K}^1_{div}}\right\|^2_{L^2(I;\mathbb{R}^3)} dt.
\end{aligned}$$

Using Minkowski's inequality and inequality (4.5), we get

$$\begin{aligned}
M_1^2 &\le 4\beta^{-2}exp(2T)\int_0^T \|v_1 - v_2\|^2_{\mathbb{K}^1_{div}}\|g\|^2_{H^1(I;\mathbb{R}^3)} dt\\
&\le 4\beta^{-2}exp(2T)\|v_1 - v_2\|^2_{L^\infty((0,T);\mathbb{K}^1_{div})}\int_0^T \|g\|^2_{H^1(I;\mathbb{R}^3)} dt.
\end{aligned}$$

Let's now consider $\int_0^T ||g||^2_{H^1(I;\mathbb{R}^3)}dt$. Since g is small enough and $||g||^2_{H^1(I;\mathbb{R}^3)} \le \varpi exp(T)$ we deduce that

$$M_1 \le 2\beta^{-1}\sqrt{T\varpi}\,exp(3T/2)\|v_1 - v_2\|_{L^\infty((0,T);\mathbb{K}^1_{div})}. \tag{4.25}$$

ii) Estimate of $M_2 = \left[\int_0^T \Big\|\sum_{i=1}^3 \partial_w\varphi(v_1,\nabla v_1)\frac{\partial^2 g}{\partial x\partial t} - \sum_{i=1}^3 \partial_w\varphi(v_2,\nabla v_1)\frac{\partial^2 g}{\partial x\partial t}\Big\|^2_{W(0,T)} dt\right]^{\frac{1}{2}}$

$$M_2^2 \le \int_0^T \sum_{i=1}^3 \left\|\partial_{w_i}\varphi(v_1,\nabla v_1) - \partial_{w_i}\varphi(v_2,\nabla v_1)\right\|^2_{W(0,T)}\Big\|\frac{\partial^2 g}{\partial x_i\partial t}\Big\|^2_{L^2(I;\mathbb{R}^3)} dt$$

$$\begin{aligned}
M_2^2 &\le 12\beta^{-2}exp(2T)\int_0^T \left\||v_1|_{\mathbb{K}^1_{div}} - |v_2|_{\mathbb{K}^1_{div}}\right\|^2_{L^2(I;\mathbb{R}^3)}\Big\|\frac{\partial^2 g}{\partial x\partial t}\Big\|^2_{L^2(I;\mathbb{R}^3)} dt\\
M_2^2 &\le 12\beta^{-2}exp(2T)\int_0^T \|v_1 - v_2\|^2_{\mathbb{K}^1_{div}}\Big\|\frac{\partial^2 g}{\partial x\partial t}\Big\|^2_{L^2(I;\mathbb{R}^3)} dt\\
&\le 12\beta^{-2}exp(2T)\|v_1 - v_2\|^2_{L^2((0,T);\mathbb{K}^1_{div})}\int_0^T \Big\|\frac{\partial^2 g}{\partial x\partial t}\Big\|^2_{L^2(I;\mathbb{R}^3)} dt.
\end{aligned}$$

$$M_2 \le 2\sqrt{3T\varpi}\,\beta^{-1}\,exp(3T/2)\|v_1 - v_2\|_{L^2((0,T);\mathbb{K}^1_{div})}. \tag{4.26}$$

The other members M_3 and M_4 are evaluated in the same way.

$$M_3 \le 2\beta^{-1}\sqrt{T\varpi}\,exp(5T/2)\|v_1 - v_2\|_{L^2((0,T);\mathbb{K}^1_{div})}. \tag{4.27}$$

$$M_4 \le 2\sqrt{3T\varpi}\beta^{-1}\,exp(5T/2)\|v_1 - v_2\|_{L^2((0,T);\mathbb{K}^1_{div})}. \tag{4.28}$$

Now, using (4.25), (4.26), (4.27), (4.28) in (4.24), we get

$$\begin{aligned}\left\|\mathcal{A}_\epsilon^{'}(v_1)-\mathcal{A}_\epsilon^{'}(v_2)\right\|_{W(0,T)} &\le 2\beta^{-1}\ \sqrt{T\,\varpi}\,exp(3T/2)\left\|v_1-v_2\right\|_{L^2((0,T);\mathbb{K}^1_{div})}\\ &\quad +2\beta^{-1}\ \sqrt{3T\,\varpi}\,exp(3T/2)\left\|v_1-v_2\right\|_{L^2((0,T);\mathbb{K}^1_{div})}\\ &\quad +2\beta^{-1}\ \sqrt{T\,\varpi}\,exp(5T/2)\left\|v_1-v_2\right\|_{L^2((0,T);\mathbb{K}^1_{div})}\\ &\quad +2\beta^{-1}\ \sqrt{3T\,\varpi}\,exp(5T/2)\left\|v_1-v_2\right\|_{L^2((0,T);\mathbb{K}^1_{div})}\\ &\le M(\beta,T)\left\|v_1-v_2\right\|_{L^2((0,T);\mathbb{K}^1_{div})}.\end{aligned}$$

where $M(\beta,T)=2\beta^{-1}\ \sqrt{T\,\varpi}\,exp(3T/2)(1+\ \sqrt{3})(1+exp(T))$ is a constant that depends only on β and T. We have proved that $\mathcal{A}_\epsilon^{'}(v)$ is a Lipschitz operator and therefore continuous and satisfies (4.20). ■

Remark 2 Note that, practically, we do not need to know the exact v solution to show the proposition 11. It suffices to establish that v is sufficiently regular. It is also sufficient that g is sufficiently small in norm and that the condition (4.20) be checked for all $t\in(0,T)$. Similary, considering that $\mathcal{B}_\epsilon^{'}(v)=\vartheta(y,t',v,\nabla v)$, we can also show that $\mathcal{B}_\epsilon^{'}(v)$ is Lipschitz with constant $N(\widetilde{\beta},T)=2\widetilde{\beta}^{-1}\ \sqrt{T\,\varpi}\,exp(3T/2)(1+\ \sqrt{3})(1+exp(T))$.

Proposition 12 *Let I be a bounded Lipschitz set of $\mathbb{R}^3$ and let $T>0$ be fixed.*
Let's consider $v_1(x,t)$, $v_2(x,t)\in\mathbb{X}^0_v=L^2((0,T);\mathbb{K}^1_{div})$ and $\nabla v_1,\nabla v_2\in L^2((0,T);L^p(I\,;\mathbb{R}^3))$. Suppose the increase g is small enough and $\left\|g^2\right\|^2_{H^1(I\,;\mathbb{R}^3)}\le\varpi exp(2T)$ and finally $\left\|\frac{\partial^2 g^2}{\partial x\partial t}\right\|^2_{L^2(I\,;\mathbb{R}^3)}\le\left\|g^2\right\|^2_{H^1(I\,;\mathbb{R}^3)}$. Then there is a constant $\widetilde{\beta}>0$ such that $\forall(x,t)\in I\times(0,T)$, we have :

$$\left\|\mathcal{B}_\epsilon^{''}(v_1)g^2-\mathcal{B}_\epsilon^{''}(v_2)g^2\right\|_{W(0,T)}\le P(\widetilde{\beta},T) \tag{4.29}$$

where $P(\widetilde{\beta},T)=4\widetilde{\beta}^{-1}\ \sqrt{T\,\varpi}\,exp(2T)(1+\ \sqrt{3})$.

Proof Suppose $v_1,v_2\in L^\infty((0,T);\mathbb{K}^1_{div})$

$$\begin{aligned}\left\|\mathcal{B}_\epsilon^{''}(v_1)g^2-\mathcal{B}_\epsilon^{''}(v_2)g^2\right\|_{W(0,T)} &=\Bigg\|\sum_{i=1}^{3}\partial^2_{w_i}\vartheta^2(v_1,\nabla v_1)\frac{\partial^2 g^2}{\partial x_i\partial t}+\partial^2_v\vartheta^2(v_1,\nabla v_1)g^2\\ &\quad-\sum_{i=1}^{3}\partial^2_{w_i}\vartheta^2(v_2,\nabla v_2)\frac{\partial^2 g^2}{\partial x_i\partial t}-\partial^2_v\vartheta^2(v_2,\nabla v_2)g^2\Bigg\|_{W(0,T)}\\ &=\Big[\int_0^T\Big\|\partial^2_v\vartheta^2(v_1,\nabla v_1)g^2+\sum_{i=1}^{3}\partial_{w_i}\vartheta^2(v_1,\nabla v_1)\frac{\partial^2 g^2}{\partial x_i\partial t}\\ &\quad-\sum_{i=1}^{3}\partial^2_{w_i}\vartheta^2(v_2,\nabla v_2)\frac{\partial^2 g^2}{\partial x_i\partial t}-\partial^2_v\vartheta^2(v_2,\nabla v_2)g^2\Big\|^2_{W(0,T)}dt\Big]^{\frac{1}{2}}\end{aligned} \tag{4.30}$$

$$\begin{aligned}\left\|\mathcal{B}_\epsilon^{''}(v_1)g^2-\mathcal{B}_\epsilon^{''}(v_2)g^2\right\|_{W(0,T)} &=\Bigg[\int_0^T\Big\|\partial^2_v\vartheta^2(v_1,\nabla v_1)g^2+\sum_{i=1}^{3}\partial^2_{w_i}\vartheta^2(v_1,\nabla v_1)\frac{\partial^2 g^2}{\partial x_i\partial t}\\ &\quad-\partial^2_v\vartheta^2(v_2,\nabla v_1)g^2-\sum_{i=1}^{3}\partial^2_{w_i}\vartheta^2(v_2,\nabla v_1)\frac{\partial^2 g^2}{\partial x_i\partial t}\\ &\quad+\partial^2_v\vartheta^2(v_2,\nabla v_1)g^2+\sum_{i=1}^{3}\partial^2_{w_i}\vartheta^2(v_2,\nabla v_1)\frac{\partial^2 g^2}{\partial x_i\partial t}\\ &\quad-\sum_{i=1}^{3}\partial^2_{w_i}\vartheta^2(v_2,\nabla v_2)\frac{\partial^2 g^2}{\partial x_i\partial t}-\partial^2_v\vartheta^2(v_2,\nabla v_2)g^2dt\Big\|^2_{W(0,T)}\Bigg]^{\frac{1}{2}}\end{aligned} \tag{4.31}$$

Using Minkowski's inequality, we get

$$\left\|\mathcal{B}_\epsilon^{''}(v_1)g^2-\mathcal{B}_\epsilon^{''}(v_2)g^2\right\|_{W(0,T)}\le N_1+N_2+N_3+N_4 \tag{4.32}$$

where

$$\begin{aligned}
N_1 &= \left[\int_0^T \left\|\partial_v^2\vartheta^2(v_1,\nabla v_1)g^2 - \partial_v^2\vartheta^2(v_2,\nabla v_1)g^2\right\|^2_{W(0,T)} dt\right]^{\frac{1}{2}},\\
N_2 &= \left[\int_0^T \Big\|\sum_{i=1}^3 \partial_{w_i}^2\vartheta^2(v_1,\nabla v_1)\frac{\partial^2 g^2}{\partial x_i\partial t} - \sum_{i=1}^3 \partial_{w_i}^2\vartheta^2(v_2,\nabla v_1)\frac{\partial^2 g^2}{\partial x_i\partial t}\Big\|^2_{W(0,T)} dt\right]^{\frac{1}{2}},\\
N_3 &= \left[\int_0^T \left\|\partial_v^2\vartheta^2(v_2,\nabla v_1)g^2 - \partial_v\vartheta^2(v_2,\nabla v_2)g^2\right\|^2_{W(0,T)} dt\right]^{\frac{1}{2}},\\
N_4 &= \left[\int_0^T \Big\|\sum_{i=1}^3 \partial_{w_i}^2\vartheta^2(v_2,\nabla v_1)\frac{\partial^2 g^2}{\partial x_i\partial t} - \sum_{i=1}^3 \partial_{w_i}^2\vartheta^2(v_2,\nabla v_2)\frac{\partial^2 g^2}{\partial x_i\partial t}\Big\|^2_{W(0,T)} dt\right]^{\frac{1}{2}}.
\end{aligned}$$

i) Estimate of $N_1 = \left[\int_0^T \left\|\partial_v^2\vartheta^2(v_1,\nabla v_1)g^2 - \partial_v\vartheta^2(v_2,\nabla v_1)g^2\right\|^2_{W(0,T)} dt\right]^{\frac{1}{2}}$

$$\begin{aligned}
N_1^2 &\le \int_0^T \left\|\partial_v^2\vartheta^2(v_1,\nabla v_1) - \partial_v^2\vartheta^2(v_2,\nabla v_1)\right\|^2_{L^2(I;\mathbb{R}^3)}\left\|g^2\right\|^2_{H^1(I;\mathbb{R}^3)} dt\\
&\le 4\widetilde{\beta}^{-2}exp(2T)\int_0^T \left\|g^2\right\|^2_{H^1(I;\mathbb{R}^3)} dt\\
N_1 &\le 2\widetilde{\beta}^{-1}\sqrt{T\,\varpi}\,exp(2T)
\end{aligned} \tag{4.33}$$

ii) Estimate of $N_2 = \left[\int_0^T \Big\|\sum_{i=1}^3 \partial_{w_i}^2\vartheta^2(v_1,\nabla v_1)\frac{\partial^2 g^2}{\partial x_i\partial t} - \sum_{i=1}^3 \partial_{w_i}^2\vartheta^2(v_2,\nabla v_1)\frac{\partial^2 g^2}{\partial x_i\partial t}\Big\|^2_{W(0,T)} dt\right]^{\frac{1}{2}}$

$$N_2^2 \le \int_0^T \sum_{i=1}^3 \left\|\partial_{w_i}^2\vartheta^2(v_1,\nabla v_1) - \partial_{w_i}^2\vartheta^2(v_2,\nabla v_1)\right\|^2_{W(0,T)}\left\|\frac{\partial^2 g^2}{\partial x_i\partial t}\right\|^2_{L^2(I;\mathbb{R}^3)} dt$$

Using Minkowski's inequality

$$\begin{aligned}
N_2^2 &\le 12\widetilde{\beta}^{-2}exp(2T)\int_0^T \left\|\frac{\partial^2 g^2}{\partial x\partial t}\right\|^2_{L^2(I;\mathbb{R}^3)} dt\\
N_2 &\le 2\sqrt{3\,\varpi\,T}\,\widetilde{\beta}^{-1}\,exp(2T)
\end{aligned} \tag{4.34}$$

The other members N_3 and N_4 are evaluated in the same way.

$$N_3 \le 2\widetilde{\beta}^{-1}\sqrt{T\,\varpi}\,exp(2T). \tag{4.35}$$

$$N_4 \le 2\widetilde{\beta}^{-1}\sqrt{3\,\varpi\,T}\,exp(2T). \tag{4.36}$$

By using now (4.33), (4.34), (4.35), (4.36) in (4.32), we get

$$\left\|\mathcal{B}_\epsilon^{''}(v_1)g^2 - \mathcal{B}_\epsilon^{''}(v_2)g^2\right\|_{W(0,T)} \le P(\widetilde{\beta},T)$$

where $P(\widetilde{\beta},T) = 4\widetilde{\beta}^{-1}\sqrt{T\,\varpi}\,exp(2T)(1+\sqrt{3})$.
Q.E.D ■

Remark 3 Similarly, it is easy to prove that the operator $\mathcal{A}_\epsilon^{''}(v)g^2$ verifies the following inequality

$$\left\|\mathcal{A}_\epsilon^{''}(v_1)g^2 - \mathcal{A}_\epsilon^{''}(v_2)g^2\right\|_{W(0,T)} \le 4\beta^{-1}\sqrt{T\,\varpi}\,exp(2T)(1+\sqrt{3}). \tag{4.37}$$

Proposition 13 *(strict differentiability). Let I be an open bounded Lipschitz set of* $\mathbb{R}^3$*.*
Let $l(x,t)$, $g(x,t) \in L^2((0,T);\ H^1(I;\mathbb{R}^3))$*. Since* $W(0,T)$ *is a reflexive space, then there is a subsequence* $(v_n)_{n\in\mathbb{N}}$ *which*

converges strongly in $L^\infty((0,T);\, L^2(I\,;\mathbb{R}^3))$*, such that* $v_n \longrightarrow v_\infty$ *and* $\tau^n g_n \longrightarrow 0$ *almost everywhere in* $I \times (0,T)$ *for* $\tau^n \in]-1;1[$*. Moreover, if*

$$\left|\left\langle \frac{1}{\tau^n} R_\epsilon(\left\|\tau^n g_n^2\right\|), l(x,t)\right\rangle\right| \longrightarrow 0, \tag{4.38}$$

then $\mathcal{A}_\epsilon^{'}(v)$ *is strictly differentiable on* $W(0,T)$*.*

Proof. Let's fix $m \in \mathbb{N}$. Suppose that for $n \longrightarrow \infty$, $v_n \longrightarrow v_\infty$ in $L^\infty((0,T);\, L^2(I\,;\mathbb{R}^3))$. The operators $\mathcal{A}_\epsilon^{'m}(v_n)$ and $\mathcal{A}_\epsilon^{''m}(v_\infty)$ satisfy (4.8). Then let's put

$$\begin{aligned} R_\epsilon^m(\|h_n^2\|) &= \mathcal{A}_\epsilon^{'m}(v_n + g_n)g_n - \mathcal{A}_\epsilon^{'m}(v_n)g_n - \mathcal{A}_\epsilon^{''m}(v_\infty)g_n^2 \\ R_\epsilon^m(\|\tau^n g_n^2\|) &= \mathcal{A}_\epsilon^{'m}(v_n + \tau^n g_n)g_n - \mathcal{A}_\epsilon^{'m}(v_n)g_n - \mathcal{A}_\epsilon^{''m}(v_\infty)\tau^n g_n^2, \quad \forall \tau^n \in]-1;1[. \end{aligned}$$

$$\begin{aligned} \left|\left\langle \frac{1}{\tau^n} R_\epsilon^m\left(\|\tau^n g_n^2\|\right), l(x,t)\right\rangle\right| &= \left|\int_0^T \int_I \frac{1}{\tau^n} R_\epsilon^m\left(\left\|\tau^n g_n^2\right\|\right) \times l(x,t)dxdt\right| \\ &= \left|\int_0^T \int_I \frac{1}{\tau^n}\Big[\partial_v\varphi(x,t,v_n + \tau^n h_n, \nabla v_n + \tau^n \nabla v_n)g_n\right. \\ &\left.-\partial_v\varphi(x,t,v_n,\nabla v_n)g_n - \partial_v^2\varphi(x,t,v_\infty)\tau^n g_n^2\Big] l(x,t)dxdt\right| \end{aligned} \tag{4.39}$$

Since the operator $\mathcal{A}_\epsilon^{'}$ is $\mathscr{C}-$differentiable (therefore Gateaux differentiable), the use of the Lagrange formula (Trenoguine, 1985) for some $\theta \in [0\,;1]$ gives

$$\begin{aligned} \left|\left\langle \frac{1}{\tau^n} R_\epsilon^m\left(\|\tau^n g_n^2\|\right), l(x,t)\right\rangle\right| &= \left|\int_0^T \int_I \frac{1}{\tau^n}\Big[\int_0^1 \left(\partial_v^2\varphi(x,t,v_n + \theta\tau^n g_n)\tau^n g_n^2 - \partial_v^2\varphi(x,t,v_\infty)\tau^n g_n^2 d\theta\right)\Big] l(x,t)dxdt\right| \\ &= \left|\int_0^T \int_I \frac{1}{\tau^n}\Big[\int_0^1 \partial_v^2\varphi(x,t,v_n + \theta\tau^n g_n) - \partial_v^2\varphi(x,t,v_\infty)d\theta\Big]\tau^n g_n^2\, l(x,t)dxdt\right| \end{aligned} \tag{4.40}$$

Using the Cauchy Schwarz inequality, we deduce that

$$\left|\left\langle \frac{1}{\tau^n} R_\epsilon^m\left(\|\tau^n g_n^2\|\right), l(x,t)\right\rangle\right| \le C\int_0^T \int_I\left(\left|\int_0^1 \partial_v^2\varphi(x,t,v_n + \theta\tau^n g_n) - \partial_v^2\varphi(x,t,v_\infty)d\theta\right|^2 dxdt\right)^{\frac{1}{2}} \left\|l(x,t)\right\|_{L^2((0,T);H^1(I\,;\mathbb{R}^3))}$$

On the other hand, as $\|v_n - v_\infty\| \longrightarrow 0$ and $\tau^n g_n \longrightarrow 0$ almost everywhere in $I \times (0,T)$, so

$$\partial_v^2\varphi(x,t,v_n + \theta\tau^n g_n) - \partial_v^2\varphi(x,t,v_\infty) \longrightarrow 0.$$

■

Remark 4 Following the same steps, we arrive at the same strict differentiability result for the operator $\mathcal{B}_\epsilon^{'}(v)$. On the other hand, as $\mathcal{P}$ is a linear continuous mapping, from Proposition 11, we deduce that the operator $\mathcal{P}[\mathcal{B}'_\epsilon(v)]$ is also Lipschitz. As the operator $\mathcal{A}'_\epsilon(v)$ is lipschitz we show from (4.21) and (4.30) that

$$\left\|\mathcal{A}'_\epsilon(v_1) - \mathcal{A}'_\epsilon(v_2) + \mathcal{P}(\mathcal{B}'_\epsilon(v_1)) - \mathcal{P}(\mathcal{B}'_\epsilon(v_2))\right\|_{W(0,T)} \le Q\left\|v_1 - v_2\right\|_{\mathbb{K}_{div}^1} \tag{4.41}$$

$$\left\|\mathcal{A}''_\epsilon(v_1) - \mathcal{A}''_\epsilon(v_2) + \mathcal{P}(\mathcal{B}''_\epsilon(v_1)) - \mathcal{P}(\mathcal{B}''_\epsilon(v_2))\right\|_{W(0,T)} \le \chi(T)\max(\beta^{-1},\widetilde{\beta}^{-1}) \tag{4.42}$$

where $Q = 2\chi(T)cosh(T)\max(\beta^{-1},\widetilde{\beta}^{-1})$ and $\chi(T) = 4\sqrt{T\,\varpi}\,exp(2T)(1+\sqrt{3})$.

Theorem 14 *(uniqueness of the solution of the perturbed system). Let* $v_0 \in \mathbb{K}_{div}^1$*,* $\rho_0 \in W^{1,2}(I;\mathbb{R})$*. There is a number* $\zeta > 0$ *such that* $0 < max(\beta^{-1},\widetilde{\beta}^{-1}) < \zeta$ *and suppose that the assumptions (H-1), (H-2) and (H-3) about the functions* φ *and* ϑ *are satisfied. Then the problem* (4.3) *admits a unique solution* $(v,\rho) = \mathfrak{R}_\epsilon(q_0,\xi_e)$ *for all* $\xi_e \in \mathbb{X}_{\xi_e}^0$*. Moreover :*

$$\mathfrak{R}_\epsilon : L^{\frac{2s}{s+1}}(I;\mathbb{R}^3) \times \mathbb{X}_{\xi_e}^0 \longrightarrow L^1((0,T);L^{\frac{2s}{s-1}}(I;\mathbb{R}^3))$$

$$(v_0\rho_0\,,\xi_e) \longmapsto \mathfrak{R}_\epsilon(q_0\,,\xi_e)$$

is $\mathscr{C}-$*continuous and* $\mathscr{C}-$*differentiable. On the other hand, the operator* $\mathfrak{R}_\epsilon$ *is highly differentiable on* $L^{\frac{2s}{s+1}}(I;\mathbb{R}^3) \times \mathbb{X}_{\xi_e}^0$ *as an application on space* $(L^2((0,T);\mathbb{K}_{div}^1);\sigma)$ *where* σ *is the weak topology in* $L^2((0,T);\mathbb{K}_{div}^1)$*.*

Proof. Let $\mathcal{V}$ be the functional space defined by

$$\mathcal{V}(0,T) := \Big\{ v \in \mathbb{X}^0_v,\ \ \exists \xi_e \in \mathbb{X}^0_{\xi_e},\ \ \exists\, q_0 \in L^{\frac{2s}{s+1}}(I;\mathbb{R}^3)\ \ , \Theta_0 \equiv (q_0,\xi_e) \Big\},$$

where

$$\|v\|^2_{\mathcal{V}(0,T)} = \|q_0\|^2_{L^{\frac{2s}{s+1}}(I;\mathbb{R}^3)} + \|\xi_e\|^2_{\mathbb{X}^0_{\xi_e}} \tag{4.43}$$

with $L_{\lambda,\mu}(v)$ formally defined by

$$\Theta_0 :\ \mathcal{V}(0,T) \longrightarrow L^{\frac{2s}{s+1}}(I;\mathbb{R}^3) \times L^1((0,T); L^{\frac{2s}{s-1}}(I;\mathbb{R}^3))$$
$$v \longmapsto \frac{1}{\rho} L_{\lambda,\mu}(v) + \frac{\partial v}{\partial t} \equiv (q_0,\xi_e)$$

According to (Ladyzhenskaya, 1970) , it follows that the operator $L_{\lambda,\mu}(v)$ is a continuous and bijective isomorphism and $\forall \xi_e \in \mathbb{X}^0_{\xi_e}$, $q_0 \in L^{\frac{2s}{s+1}}(I;\mathbb{R}^3)$, there is a single couple (v,ρ) of the perturbed problem (4.3).

Moreover, if $\int_0^T \int_I \mathcal{P}(x-y,t-t')\partial_v\vartheta(y,t',v,w)dydt + \partial_v\varphi(x,t,v,w) = 0$, then the equality (4.43) is satisfied. We deduce that $L_{\lambda,\mu}(v)$ is linear, continuous and admits an inverse which is also continuous. On this basis, if moreover, the inequalities (4.21) and (4.42) are satisfed, so $L_{\lambda,\mu}(v)$ is continuous and invertible. Since $\mathcal{A}'_\epsilon(v) + \mathcal{P}(\mathcal{B}'_\epsilon(v))$ is Lipschitz, then using the Hadamard theorem, we can write that for all $v_0 \in \mathcal{V}(0,T)$, the operator $\mathfrak{R}(V) \equiv \Big(q_0\,, L_{\lambda,\mu}(v) + d[\mathcal{A}'_\epsilon(v_0)g,h]_{h=g} + \int_0^T \int_I \mathcal{P}d[\mathcal{B}'_\epsilon(v_0)g,h]_{h=g}dxdt\Big)$ defined from $\mathcal{V}(0,T)$ in $L^{\frac{2s}{s+1}}(I;\mathbb{R}^3)\times\mathbb{X}^0_{\xi_e}$ admits a continuous inverse having the following form

$$\mathfrak{R}(v)^{-1} \equiv \Big(q_0\,, L_{\lambda,\mu}(v) + \mathcal{A}'_\epsilon(v)g + \int_0^T \int_I \mathcal{P}\mathcal{B}'_\epsilon(v)gdxdt\Big), \tag{4.44}$$

from $L^{\frac{2s}{s+1}}(I;\mathbb{R}^3) \times L^1((0,T); L^{\frac{2s}{s-1}}(I;\mathbb{R}^3))$ in $\mathcal{V}(0,T)$.
$\mathfrak{R}(v)^{-1}$ has an inverse Lipschitz function, so there is a unique $(v,\rho) \equiv \mathfrak{R}(V)$, solution to the (4.3) problem. However, under Proposition 11, $\mathfrak{R}^{-1}$ is a strongly $\mathscr{C}-$differentiable function. Thus, by using the theorems of strong $\mathscr{C}-$differentiability, we conclude that $\mathfrak{R}_\epsilon$ is $\mathscr{C}-$continuous and $\mathscr{C}-$differentiable. ■

In this paper, we presented a non-stationary, time-dependent mathematical model that models a given tumor. The model is based on partial differential equations with some initial parameters and conditions. An estimate of the rate v of cell development and propagation is given. The addition of linear members to the first system allowed us to find a domain in which we could solve this problem. However, the results obtained can be used in the theory of optimal control, to establish the necessary optimality conditions.

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