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Strict Positivity of Operators and Inflated Schur Products

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Abstract

In this paper we provide a characterization of strictly positive $n\times n$ matrices of operators and a factorization of their inverses. Consequently, we provide a test of strict positivity of matrices in $M_n(C)$. We give equivalent conditions for the inequality $I > T^*T + TT^*$. We prove a theorem involving inflated Schur products [4, P. 153] of positive matrices of operators with invertible elements in the main diagonal which extends the results [3, P. 479, Theorem 7.5.3 (b), (c)]. We also discuss strictly completely positive linear maps in the paper.

Keywords: positive operators, strictly positive operators, strictly completely positive, inflated Schur products.

AMS Subject Classification (2010): 47A63.

1. Introduction

Let $M_n(C)$ be the algebra of all complex $n\times n$ matrices and $B(H)$ be the algebra of all bounded linear operators on a Hilbert space H. For two self-adjoint operators S and T in $B(H)$, the order relation $S \ge T$ means that $\langle (S-T)h,h\rangle \ge 0$ for all h in H. If $T \ge 0$ and invertible, then T is called a strictly positive operator denoted by $T > 0$. Let A and B be C^*-algebras and let $L: A \to B$ be a bounded linear map. The map L is strictly positive if $L(a) > 0$ whenever $a > 0$ in A. The map L is strictly completely positive if $L\otimes I_n : A\otimes M_n \to B\otimes M_n$ defined by $L\otimes I_n(a\otimes b) = L(a)\otimes b$ is strictly positive for all n. Applying operator determinant of a positive 2×2 matrix [2, Problem 71], we give a characterization of a strictly positive 2×2 operator matrix [6, Lemma 2.1]. In Section 2, we extend 2×2 positive matrices to $n\times n$ positive matrices, give an equivalent condition for strictly positive $n\times n$ matrices of operators, and prove properties of strictly positive operators. In Section 3, we provide characterizations of the inequality $I > T^*T + TT^*$ where $T\in B(H)$ and have the famous estimate by Kittaneh 2005 [Studia Math 168]. In Section 4, applying the property that an inflated Schur product of a positive matrix of operators is positive [4], we provide properties of a Schur product of positive matrix of operators which extend the results [3, P. 479, Theorem 7.5.3 (b) (c)]. Finally, we prove that the inflated Schur product of a positive matrix of operators with invertible elements in the main diagonal is strictly completely positive.

2. Strictly Positive Operators

Applying [6, Lemma 2.1] we have a characterization of strictly positive $n\times n$ operator matrices as follows:

Proposition 2.1. Suppose that $x_{ij}\in B(H_j, H_i)$ where $H_i (i=1,2,...,n)$ is a Hilbert space. If x_{ii} is self-adjoint $(i=1,2,...,n)$ and x_{11} is invertible, then the $n\times n$ operator matrix

$$X_n = \begin{pmatrix} x_{11} & x_{12} & . & x_{1n} \\ x_{12}^* & x_{22} & . & x_{2n} \\ . & . & . & . \\ x_{1n}^* & x_{2n}^* & . & x_{nn} \end{pmatrix} : H_1\oplus H_2 \oplus ... \oplus H_n \to H_1\oplus H_2\oplus ...\oplus H_n$$

is strictly positive if and only if x_{11} and

$$\Omega_{n-1} = \begin{pmatrix} x_{22} & x_{23} & . & x_{2n} \\ x_{23}^* & x_{33} & . & . \\ . & . & . & . \\ x_{2n}^* & x_{3n}^* & . & x_{nn} \end{pmatrix} - (x_{12}x_{13}....x_{1n})^* (x_{11}^{-1})(x_{12}x_{13}....x_{1n})$$

are strictly positive. Moreover,

$$X_n^{-1} = \begin{pmatrix} 1 & -x_{11}^{-1}x_{12} & . & -x_{11}^{-1}x_{1n} \\ 0 & 1 & 0 & 0 \\ . & 0 & . & 0 \\ 0 & . & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{11}^{-1} & 0 \\ 0 & \Omega_{n-1}^{-1} \end{pmatrix} \begin{pmatrix} 1 & -x_{11}^{-1}x_{12} & . & -x_{11}^{-1}x_{1n} \\ 0 & 1 & 0 & 0 \\ . & 0 & . & 0 \\ 0 & . & 0 & 1 \end{pmatrix}^* .$$

Proof. From [6, Lemma 2.1] with $A = x_{11}$, $B = (x_{12}x_{13}....x_{1n})$, and $C = \begin{pmatrix} x_{22} & x_{23} & . & x_{2n} \\ x_{23}^* & x_{33} & . & . \\ . & . & . & x_{n-1n} \\ x_{2n}^* & . & x_{n-1n}^* & x_{nn} \end{pmatrix}$ we know that

$$X_n = \begin{pmatrix} x_{11} & 0 & . & 0 \\ x_{12}^* & x_{22} - x_{12}^* x_{11}^{-1} x_{12} & . & x_{2n} - x_{12}^* x_{11}^{-1} x_{1n} \\ . & . & . & . \\ x_{1n}^* & x_{2n}^* - x_{1n}^* x_{11}^{-1} x_{12} & . & x_{nn} - x_{1n}^* x_{11}^{-1} x_{1n} \end{pmatrix} \begin{pmatrix} 1 & x_{11}^{-1}x_{12} & . & x_{11}^{-1}x_{1n} \\ 0 & 1 & 0 & 0 \\ . & 0 & . & 0 \\ 0 & . & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & x_{11}^{-1}x_{12} & . & x_{11}^{-1}x_{1n} \\ 0 & 1 & 0 & 0 \\ . & 0 & . & 0 \\ 0 & . & 0 & 1 \end{pmatrix}^* \begin{pmatrix} x_{11} & 0 & . & 0 \\ 0 & x_{22} - x_{12}^* x_{11}^{-1} x_{12} & . & x_{2n} - x_{12}^* x_{11}^{-1} x_{1n} \\ . & . & . & . \\ 0 & x_{2n}^* - x_{1n}^* x_{11}^{-1} x_{12} & . & x_{nn} - x_{1n}^* x_{11}^{-1} x_{1n} \end{pmatrix} \begin{pmatrix} 1 & x_{11}^{-1}x_{12} & . & x_{11}^{-1}x_{1n} \\ 0 & 1 & 0 & 0 \\ . & 0 & . & 0 \\ 0 & . & 0 & 1 \end{pmatrix}$$ Since

$$\begin{pmatrix} 1 & x_{11}^{-1}x_{12} & . & x_{11}^{-1}x_{1n} \\ 0 & 1 & 0 & 0 \\ . & 0 & . & 0 \\ 0 & . & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -x_{11}^{-1}x_{12} & . & -x_{11}^{-1}x_{1n} \\ 0 & 1 & 0 & 0 \\ . & 0 & . & 0 \\ 0 & . & 0 & 1 \end{pmatrix}$$, we can see that X_n is strictly positive if and only if x_{11} and Ω_{n-1} are strictly positive.

Corollary 2.2. The operator matrix X_n in the Proposition 2.1 is positive if and only if x_{11} and Ω_{n-1} are positive.
Corollary 2.3, Corollary 2.4, and Corollary 2.5 in the followings are known for the case of positive sub-matrices, but for the case of strictly positive sub-matrices, we need to discuss more.

Corollary 2.3. Suppose that the main diagonal elements x_{ii} $(i = 1,2,...,n)$ of the $n \times n$ operator matrix X_n are self-adjoint and invertible . If X_n is strictly positive, then the matrices $x_{ii}(i = 1,2,...n)$, $\begin{pmatrix} x_{11} & x_{12} \\ x_{12}^* & x_{22} \end{pmatrix}$,

$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{12}^* & x_{22} & x_{23} \\ x_{13}^* & x_{23}^* & x_{33} \end{pmatrix}$, ….., $\begin{pmatrix} x_{11} & x_{12} & . & x_{1n-1} \\ x_{12}^* & x_{22} & . & x_{2n-1} \\ . & . & . & . \\ x_{1n-1}^* & . & x_{n-2n-1}^* & x_{n-1n-1} \end{pmatrix}$ are strictly positive.

Proof. Since

$$\begin{pmatrix} 0 & . & 0 & 1 \\ . & 0 & 1 & 0 \\ 0 & . & 0 & . \\ 1 & 0 & . & 0 \end{pmatrix} X_n \begin{pmatrix} 0 & . & 0 & 1 \\ . & 0 & 1 & 0 \\ 0 & . & 0 & . \\ 1 & 0 & . & 0 \end{pmatrix} = \begin{pmatrix} x_{nn} & x_{n-1n}{}^* & . & x_{1n}{}^* \\ x_{n-1n} & . & . & x_{1n-1}{}^* \\ . & . & x_{22} & . \\ x_{1n} & . & x_{12} & x_{11} \end{pmatrix} > 0$$, applying Proposition 2.1, we have that

x_{nn} and $\begin{pmatrix} x_{n-1n-1} & x_{n-2n-1}{}^* & . & x_{1n-1}{}^* \\ . & . & . & . \\ x_{2n-1} & . & x_{22} & x_{12}{}^* \\ x_{1n-1} & . & x_{12} & x_{11} \end{pmatrix}$ are strictly positive.

Hence

$\begin{pmatrix} x_{11} & x_{12} & . & x_{1n-1} \\ x_{12}{}^* & x_{22} & . & . \\ . & . & . & x_{n-2n-1} \\ x_{1n-1}{}^* & . & x_{n-2n-1}{}^* & x_{n-1n-1} \end{pmatrix}$ is strictly positive. Follow the same procedures, we have the Corollary.

Corollary 2.4. If $\begin{pmatrix} a_{11} & a_{12} & . & a_{1n} \\ \overline{a}_{12} & a_{22} & . & a_{2n} \\ . & . & . & . \\ \overline{a}_{1n} & . & \overline{a}_{n-1n} & a_{nn} \end{pmatrix}$ is strictly positive in $M_n(C)$, then $a_{ii}(i=1,2,...,n)$, $\begin{pmatrix} a_{11} & a_{12} \\ \overline{a}_{12} & a_{22} \end{pmatrix}$, …..,

$\begin{bmatrix} a_{11} & a_{12} & . & a_{1n-1} \\ \overline{a}_{12} & a_{22} & . & a_{2n-1} \\ . & . & . & . \\ \overline{a}_{1n-1} & . & \overline{a}_{n-2n-1} & a_{n-1n-1} \end{bmatrix}$ are strictly positive.

Proof. From [3, p.432, 7.1.10] we know that a_{ii} is nonzero and self-adjoint $(i=1,2,...,n)$.

Corollary 2.5. Suppose that the main diagonal elements x_{ii} $(i=1,2,...,n)$ of the $n \times n$ operator matrix X_n are self-adjoint and invertible. If X_n is strictly positive, then the matrices $\begin{pmatrix} x_{11} & x_{12} \\ x_{12}{}^* & x_{22} \end{pmatrix}$, $\begin{pmatrix} x_{11} & x_{13} \\ x_{13}{}^* & x_{33} \end{pmatrix}$, …..,

$\begin{pmatrix} x_{11} & x_{1n} \\ x_{1n}{}^* & x_{nn} \end{pmatrix}$ are strictly positive.

Proof. From Corollary 2.3 and Proposition 2.1, we know that $\begin{bmatrix} x_{11} & x_{12} \\ x_{12}{}^* & x_{22} \end{bmatrix} > 0$ and $x_{22} - x_{12}{}^* x_{11}{}^{-1} x_{12} > 0$. If

$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{12}{}^* & x_{22} & x_{23} \\ x_{13}{}^* & x_{23}{}^* & x_{33} \end{pmatrix} > 0$, then $\begin{pmatrix} x_{22} & x_{23} \\ x_{23}{}^* & x_{33} \end{pmatrix} - (x_{12}x_{13})^* (x_{11}{}^{-1})(x_{12}x_{13}) > 0$.

Or $x_{33} - x_{13}^{*} x_{11}^{-1} x_{13} > (x_{23}^{*} - x_{13}^{*} x_{11}^{-1} x_{12})(x_{22} - x_{12}^{*} x_{11}^{-1} x_{12})^{-1}(x_{23} - x_{12}^{*} x_{11}^{-1} x_{13}) \geq 0$.

Hence $\begin{pmatrix} x_{11} & x_{13} \\ x_{13}^{*} & x_{33} \end{pmatrix} > 0$.

Applying the mathematical Induction, if $n = k$ is true. That is,

If $\begin{pmatrix} x_{11} & x_{12} & . & x_{1k} \\ x_{12}^{*} & x_{22} & . & x_{2k} \\ . & . & . & . \\ x_{1k}^{*} & . & x_{k-1k}^{*} & x_{kk} \end{pmatrix} > 0$, then $\begin{pmatrix} x_{11} & x_{1k} \\ x_{1k}^{*} & x_{kk} \end{pmatrix} > 0$.

Now, if $\begin{pmatrix} x_{11} & x_{12} & . & x_{1k+1} \\ x_{12}^{*} & x_{22} & . & x_{2k+1} \\ . & . & . & . \\ x_{1k+1}^{*} & . & x_{kk+1}^{*} & x_{k+1k+!} \end{pmatrix} > 0$, Applying Proposition 2.1, we have

$$\begin{pmatrix} x_{22} & x_{23} & . & x_{2k+1} \\ x_{23}^{*} & x_{33} & . & x_{3k+1} \\ . & . & . & . \\ x_{2k+1}^{*} & . & x_{kk+1}^{*} & x_{k+1k+1} \end{pmatrix} - \begin{pmatrix} x_{12}^{*} x_{11}^{-1} x_{12} & x_{12}^{*} x_{11}^{-1} x_{13} & . & x_{12}^{*} x_{11}^{-1} x_{1k+1} \\ x_{13}^{*} x_{11}^{-1} x_{12} & x_{13}^{*} x_{11}^{-1} x_{13} & . & . \\ . & . & . & x_{1k}^{*} x_{11}^{-1} x_{1k+1} \\ x_{1k+1}^{*} x_{11}^{-1} x_{12} & . & x_{1k+1}^{*} x_{11}^{-1} x_{1k} & x_{1k+1}^{*} x_{11}^{-1} x_{1k+1} \end{pmatrix} > 0.$$

By induction we obtain

$$\begin{pmatrix} x_{22} - x_{12}^{*} x_{11}^{-1} x_{12} & x_{2k+1} - x_{12}^{*} x_{11}^{-1} x_{1k+1} \\ x_{2k+1}^{*} - x_{1k+1}^{*} x_{11}^{-1} x_{12} & x_{k+1k+1} - x_{1k+1}^{*} x_{11}^{-1} x_{1k+1} \end{pmatrix} > 0.$$

Or

$$x_{k+1k+1} - x_{1k+1}^{*} x_{11}^{-1} x_{1k+1} > (x_{2k+1}^{*} - x_{1k+1}^{*} x_{11}^{-1} x_{12})(x_{22} - x_{12}^{*} x_{11}^{-1} x_{12})^{-1}(x_{2k+1} - x_{12}^{*} x_{11}^{-1} x_{1k+1}) \geq 0.$$

Hence

$$\begin{pmatrix} x_{11} & x_{1k+1} \\ x_{1k+1}^{*} & x_{k+1k+1} \end{pmatrix} > 0.$$

We prove an inequality of strictly positive operators without equal sign as follows:

Proposition 2.6. Let $A, B \in B(H)$, where H is a Hilbert space. Then

$A > B > 0$ if and only if $B^{-1} > A^{-1} > 0$.

Proof. Applying Proposition 2.1, we know that the following five inequalities are equivalent:

$A > B > 0$, $A > B^{\frac{1}{2}} I B^{\frac{1}{2}} > 0$, $\begin{pmatrix} I & B^{\frac{1}{2}} \\ B^{\frac{1}{2}} & A \end{pmatrix} > 0$, $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} I & B^{\frac{1}{2}} \\ B^{\frac{1}{2}} & A \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} A & B^{\frac{1}{2}} \\ B^{\frac{1}{2}} & I \end{pmatrix} > 0$, and $I > B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}}$.

Hence

$B^{-1} > A^{-1} > 0$.

Theorem 2.7. Let $X_i = \begin{pmatrix} x_{11} & x_{12} & . & x_{1i} \\ x_{12}^* & x_{22} & . & x_{2i} \\ . & . & . & . \\ x_{1i}^* & x_{2i}^* & . & x_{ii} \end{pmatrix}$ be strictly positive, $x_{ii} = x_{ii}^*$, and x_{ii} be invertible $(i = 1,2,.....n)$, then

$X_n^{-1} = A_n \begin{pmatrix} \Delta_1^{-1} & 0 & . & 0 \\ 0 & \Delta_2^{-1} & . & . \\ . & . & . & 0 \\ 0 & . & 0 & \Delta_n^{-1} \end{pmatrix} A_n^*$ where A_n is invertible and Δ_i^{-1} is the (i,i)-entry of X_i^{-1}.

Proof. If $i = 2$, then $X_2^{-1} = \begin{pmatrix} 1 & -\Delta_1^{-1}x_{12} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \Delta_1^{-1} & 0 \\ 0 & \Delta_2^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -x_{12}^*\Delta_1^{-1} & 1 \end{pmatrix}$ where $\Delta_1^{-1} = x_{11}^{-1}$,

$\Delta_2^{-1} = (x_{22} - x_{12}^* x_{11}^{-1} x_{12})^{-1} =$ the $(2,2)$-entry of X_2^{-1}.

If $i = 3$, then $X_3^{-1} = \begin{pmatrix} 1 & -\Delta_1^{-1}x_{12} & -\Delta_1^{-1}x_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\Delta_2^{-1}(x_{23} - x_{12}^* x_{11}^{-1} x_{13}) \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \Delta_1^{-1} & 0 & 0 \\ 0 & \Delta_2^{-1} & 0 \\ 0 & 0 & \Delta_3^{-1} \end{pmatrix}$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -(x_{23}^* - x_{13}^* x_{11}^{-1} x_{12})\Delta_2^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -x_{12}^*\Delta_1^{-1} & 1 & 0 \\ -x_{13}^*\Delta_1^{-1} & 0 & 1 \end{pmatrix}$ where

$\Delta_3^{-1} = [x_{33} - x_{13}^* x_{11}^{-1} x_{13} - (x_{23}^* - x_{13}^* x_{11}^{-1} x_{12})\Delta_2^{-1}(x_{23} - x_{12}^* x_{11}^{-1} x_{13})] =$ the $(3,3)$-entry of X_3^{-1},

and the inverse of

$\begin{pmatrix} 1 & -\Delta_1^{-1}x_{12} & -\Delta_1^{-1}x_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\Delta_2^{-1}(x_{23} - x_{12}^* x_{11}^{-1} x_{13}) \\ 0 & 0 & 1 \end{pmatrix}$ is

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \Delta_2^{-1}(x_{23} - x_{12}^* x_{11} x_{13}) \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \Delta_1^{-1}x_{12} & \Delta_1^{-1}x_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

In general, applying Proposition 2.1, we have

$$X_n^{-1} = \begin{pmatrix} 1 & -x_{11}^{-1}x_{12} & . & -x_{11}^{-1}x_{1n} \\ 0 & 1 & 0 & 0 \\ . & 0 & . & 0 \\ 0 & . & 0 & 1 \end{pmatrix}\begin{pmatrix} x_{11}^{-1} & 0 \\ 0 & \Omega_{n-1}^{-1} \end{pmatrix}\begin{pmatrix} 1 & -x_{11}^{-1}x_{12} & . & -x_{11}^{-1}x_{1n} \\ 0 & 1 & 0 & 0 \\ . & 0 & . & 0 \\ 0 & . & 0 & 1 \end{pmatrix}^*$$

$$= B_1\begin{pmatrix} x_{11}^{-1} & 0 \\ 0 & \Omega_{n-1}^{-1} \end{pmatrix} B_1 \text{ where } B_1 = \begin{bmatrix} 1 & -x_{11}^{-1}x_{12} & . & -x_{11}^{-1}x_{1n} \\ 0 & & & \\ . & & I_{n-1} & \\ 0 & & & \end{bmatrix},$$

and

$$\Omega_{n-1}^{-1} = \begin{pmatrix} 1-\Delta_2^{-1}(x_{23}-x_{12}^{*}x_{11}^{-1}x_{13}) & . & . & 1-\Delta_2^{-1}(x_{2n}-x_{12}^{*}x_{11}^{-1}x_{1n}) \\ 0 & & & \\ . & & I_{n-2} & \\ 0 & & & \end{pmatrix} \begin{bmatrix} \Delta_2^{-1} & 0 \\ 0 & \Omega_{n-2}^{-1} \end{bmatrix}$$

$$\begin{pmatrix} 1-\Delta_2^{-1}(x_{23}-x_{12}^{*}x_{11}^{-1}x_{13}) & . & . & 1-\Delta_2^{-1}(x_{2n}-x_{12}^{*}x_{11}^{-1}x_{1n}) \\ 0 & & & \\ . & & I_{n-2} & \\ 0 & & & \end{pmatrix}^{*}.$$

Let $B_2 = \begin{bmatrix} 1 & 0 & . & 0 \\ 0 & 1-\Delta_2^{-1}(x_{23}-x_{12}x_{11}^{-1}x_{13}) & . & 1-\Delta_2^{-1}(x_{2n}-x_{12}^{*}x_{11}^{-1}x_{1n}) \\ . & & & \\ 0 & & I_{n-2} & \end{bmatrix}$, then

$$X_n^{-1} = B_1B_2\begin{bmatrix} \Delta_1^{-1} & 0 & . \\ 0 & \Delta_2^{-1} & 0 \\ . & 0 & \Omega_{n-2}^{-1} \\ & & \end{bmatrix} B_2^{*}B_1^{*}.$$

Repeating the same steps by applying Proposition 2.1, we have

$$X_n^{-1} = B_1B_2...B_{n-2}\begin{bmatrix} I_{n-2} & & 0 \\ & 1 & -\Delta_{n-1}^{-1}t \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \Delta_1^{-1} & 0 & . & 0 \\ 0 & . & 0 & . \\ . & . & \Delta_{n-1}^{-1} & 0 \\ 0 & . & 0 & \Delta_n^{-1} \end{pmatrix} \begin{bmatrix} I_{n-2} & & 0 \\ & 1 & -\Delta_{n-1}^{-1}t \\ & 0 & 1 \end{bmatrix}^{*} B_{n-2}^{*}.....B_1^{*},$$

where $t =$ the $(1,2)$ -entry of Ω_2. Now let $A_n = B_1B_2...B_{n-1}$, we have the theorem.

Theorem 2.8. $X_n = \begin{pmatrix} x_{11} & x_{12} & . & x_{1n} \\ x_{12}^{*} & x_{22} & . & x_{2n} \\ . & . & . & . \\ x_{1n}^{*} & x_{2n}^{*} & . & x_{nn} \end{pmatrix}$ is strictly positive if and only if $\Delta_1^{-1}, \Delta_2^{-1}, ...\Delta_n^{-1}$ are strictly positive where Δ_i^{-1} is the (i,i) -entry of X_i^{-1} $(i = 1,2,.....n)$.

Proof. From Corollary 2.3, we know that $X_1, X_2, ..., X_n$ are strictly positive. By Proposition 2.7, we have the Theorem.

We give an example of Theorem 2.8 as follows:

Example 2.9. Let $T \in B(H)$ for some Hilbert space H, $E_{i,j}$ be the $n \times n$ matrix with 1 in the (i, j)-entry and 0 elsewhere, and I_n be the identity in $M_n(B(H))$.

If $\begin{pmatrix} I & T & 0 & 0 \\ T^* & I & . & 0 \\ 0 & . & . & T \\ 0 & 0 & T^* & I \end{pmatrix}_{n\times n} > 0$, then $\begin{pmatrix} I & T & 0 & 0 \\ T^* & I & . & 0 \\ 0 & . & . & T \\ 0 & 0 & T^* & I \end{pmatrix}_{n\times n}^{-1} = A_n^* \begin{bmatrix} \Delta_1^{-1} & 0 & . & 0 \\ 0 & \Delta_2^{-1} & . & . \\ . & . & . & 0 \\ 0 & . & 0 & \Delta_n^{-1} \end{bmatrix} A_n$ where

$\Delta_1^{-1} = I$, $\Delta_2^{-1} = (I - T^*T)^{-1}$, $\Delta_k^{-1} = (I - T^*\Delta_{k-1}^{-1}T)^{-1}$ are strictly positive $(k = 3,4,...,n)$ and $A_n^* = (I_n - T \otimes E_{1,2})(I_n - \Delta_2^{-1}T \otimes E_{2.3})....(I_n - \Delta_{n-1}^{-1}T \otimes E_{n-1,n})$.

Corollary 2.10. let $X_n = \begin{pmatrix} x_{11} & . & . & x_{1n} \\ \overline{x_{12}} & x_{22} & . & x_{2n} \\ . & . & . & . \\ \overline{x_{1n}} & \overline{x_{2n}} & . & x_{nn} \end{pmatrix}$ be strictly positive in $M_n(C)$ and x_{ii} $(i = 1,2,3,.....,n)$ are nonzero real numbers, then the determinant of $X_n = \Delta_1\Delta_2....\Delta_n$.

Proof. From proposition 2.7, we know that $X_n = A_n^{*-1} \begin{pmatrix} \Delta_1 & 0 & . & 0 \\ 0 & \Delta_2 & . & . \\ . & . & . & 0 \\ 0 & . & 0 & \Delta_n \end{pmatrix} A_n^{-1}$. It is not difficult to see that the determinant of A_n and A_n^* are equal to 1.

We have a test for strictly positive matrices in $M_n(C)$ as follows:

Corollary 2.11. Let $X_n = \begin{pmatrix} x_{11} & . & . & x_{1n} \\ \overline{x_{12}} & x_{22} & . & x_{2n} \\ . & . & . & . \\ \overline{x_{1n}} & \overline{x_{2n}} & . & x_{nn} \end{pmatrix} \in M_n(C)$. Then

$X_n > 0$ is strictly positive if and only if the determinant of $X_i > 0$. $(i = 1,2,.....n)$.

Proof. By Theorem 2.8 and Corollary 2.10, we have the Corollary.

Corollary 2.12. Let $X_n = \begin{pmatrix} x_{11} & . & . & x_{1n} \\ \overline{x_{12}} & x_{22} & . & x_{2n} \\ . & . & . & . \\ \overline{x_{1n}} & \overline{x_{2n}} & . & x_{nn} \end{pmatrix} \in M_n(C)$.

If $X_n > 0$, then $\Delta_1 = x_{11}$ and $\Delta_i = \frac{\det X_i}{\det X_{i-1}}$ for $i = 2,3,....,n$.

Proof. Applying Corollary 2.10, we have the Corollary.

Proposition 2.13. Let $X_n = \begin{bmatrix} x_{11} & . & . & x_{1n} \\ x_{12}^* & x_{22} & . & x_{2n} \\ . & . & . & . \\ x_{1n}^* & x_{2n}^* & . & x_{nn} \end{bmatrix}$ and $Y_n = \begin{bmatrix} y_{11} & . & . & y_{1n} \\ y_{12}^* & y_{22} & . & y_{2n} \\ . & . & . & . \\ y_{1n}^* & y_{2n}^* & . & y_{nn} \end{bmatrix}$ be strictly positive. If $X_n > Y_n$, then $\Delta_i^{-1}(X) < \Delta_i^{-1}(Y)$ where $\Delta_i^{-1}(X)$ and $\Delta_i^{-1}(Y)$ are the (i,i)-entry of X_i^{-1} and Y_i^{-1} respectively $(i = 1,2,...,n)$.

Proof. From Corollary 2.3, Proposition 2.6, and Theorem 2.7,we have the proposition.

Corollary 2.14. Let $X_n = \begin{pmatrix} x_{11} & . & . & x_{1n} \\ \overline{x_{12}} & x_{22} & . & x_{2n} \\ . & . & . & . \\ \overline{x_{1n}} & \overline{x_{2n}} & . & x_{nn} \end{pmatrix}$ and $Y_n = \begin{pmatrix} y_{11} & . & . & y_{1n} \\ \overline{y_{12}} & y_{22} & . & y_{2n} \\ . & . & . & . \\ \overline{y_{1n}} & \overline{y_{2n}} & . & y_{nn} \end{pmatrix}$ be strictly positive in $M_n(C)$.

If $X_n > Y_n$, then det $X_n >$ det Y_n.

Proof. Applying Corollary 2.10 and Proposition 2.13, we have the Corollary.

3. Operator Inequalities

Proposition 3.1. Suppose that x_{11}, x_{22}, x_{33} are self-adjoint and invertible. Then

$$\begin{pmatrix} x_{11} & x_{12} & 0 \\ x_{12}^* & x_{22} & x_{23} \\ 0 & x_{23}^* & x_{33} \end{pmatrix} > 0 \text{ if and only if } x_{22} > x_{12}^* x_{11}^{-1} x_{12} + x_{23} x_{33}^{-1} x_{23}^*.$$

Proof. Applying Proposition 2.1 we have $\begin{pmatrix} x_{22} & x_{23} \\ x_{23}^* & x_{33} \end{pmatrix} > \begin{pmatrix} x_{12}^* x_{11}^{-1} x_{12} & 0 \\ 0 & 0 \end{pmatrix}$.

Or

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_{22} - x_{12}^* x_{11}^{-1} x_{12} & x_{23} \\ x_{23}^* & x_{33} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} x_{33} & x_{23}^* \\ x_{23} & x_{22} - x_{12}^* x_{11}^{-1} x_{12} \end{pmatrix} > 0.$$

Hence $x_{22} - x_{12}^* x_{11}^{-1} x_{12} > x_{23} x_{33}^{-1} x_{23}^*$.

We give a characterization of the inequality $I > I + T^* + T$ as follows:

Corollary 3.2. Let $T \in B(H)$ for some Hilbert space H. Then the following three inequalities are equivalent:

$$\begin{pmatrix} I & T & 0 \\ T^* & I & T \\ 0 & T^* & I \end{pmatrix} > 0, \quad I > T^*T + TT^*, \text{ and } \left\| T^*T + TT^* \right\| < 1.$$

Proof. Applying Proposition 3.1 with $x_{11} = x_{22} = x_{33} = 1$ and $x_{12} = x_{23} = T$, we obtain the Corollary.

Corollary 3.3. Let $T \in B(H)$. If T is normal $(T^*T = TT^*)$, then

$\begin{pmatrix} I & T & 0 \\ T^* & I & T \\ 0 & T^* & I \end{pmatrix} > 0$ if and only if $\|T\| < \frac{1}{\sqrt{2}}$.

We know that $\min\{ k : \begin{pmatrix} k & T \\ T^* & k \end{pmatrix}$ is positive$\} = \|T\|$. We can find

$\min\{ k : \begin{pmatrix} k & T & 0 \\ T^* & k & T \\ 0 & T^* & k \end{pmatrix}$ is positive$\}$ as follows:

Corollary 3.4. let $T \in B(H)$, then $\min\{ k : \begin{pmatrix} k & T & 0 \\ T^* & k & T \\ 0 & T^* & k \end{pmatrix}$ is positive$\}$ $= \|T^*T + TT^*\|^{\frac{1}{2}}$.

Proof. If $\begin{pmatrix} k & T & 0 \\ T^* & k & T \\ 0 & T^* & k \end{pmatrix}$ is positive, from Corollary 3.2, we have

$\|T^*T + TT^*\| \le k^2$. Conversely, since $\|T^*T + TT^*\| I \ge T^*T + TT^*$,

we have $\|T^*T + TT^*\|^{\frac{1}{2}} I \ge \dfrac{T^*T + TT^*}{\|T^*T + TT^*\|^{\frac{1}{2}}}$ where $T \neq 0$.

Applying Proposition 3.1, we have $\begin{pmatrix} \|T^*T + TT^*\|^{\frac{1}{2}} I & T & 0 \\ T^* & \|T^*T + TT^*\|^{\frac{1}{2}} I & T \\ 0 & T^* & \|T^*T + TT^*\|^{\frac{1}{2}} I \end{pmatrix}$ to be positive.

Hence

$\|T^*T + TT^*\|^{\frac{1}{2}} \ge \min\{ k : \begin{pmatrix} k & T & 0 \\ T^* & k & T \\ 0 & T^* & k \end{pmatrix}$ is positive$\}$.

The following Corollary is the famous estimate by Kittaneh 2005 [Studia Math 168].

Corollary 3.5. Let $T \in B(H)$, then $2w(T) \ge \|T^*T + TT^*\|^{\frac{1}{2}}$.

Proof. By [5, Proposition 2.2], we have the Corollary.

4. Inflated Schur Product

In [4] the authors discuss the positivity of inflated Schur product. In this Section we mainly discuss the strict positivity of inflated Schur product. Moreover, we extend the results [3, P.479, Theorem 7.5.3 (b), (c)] to the case of inflated Schur product.

V. Paulsen, S. Power, and R. Smith [4] introduced the following definition:

Definition 4.1. Suppose that $x_{ij} \in B(H_j, H_i)$, where $H_i (i = 1,2,...,n)$ is a Hilbert space. Let $X_n = (x_{ij})$. A linear map $S_{X_n} : M_n(C) \to B(H_1 \oplus H_2 \oplus ... \oplus H_n)$ defined by $S_{X_n}((a_{ij})) = X_n \circ (a_{ij}) = (a_{ij} x_{ij})$ is called an inflated Schur product.

Theorem 4.2. Suppose that $x_{ij} \in B(H_j, H_i)$ where $H_i (i = 1,2,...,n)$ is a Hilbert space. If $x_{ii} (i = 1,2,...,n)$ is invertible and $X_n = \begin{pmatrix} x_{11} & x_{12} & . & x_{1n} \\ x_{12}^* & x_{22} & . & x_{2n} \\ . & . & . & . \\ x_{1n}^* & x_{2n}^* & . & x_{nn} \end{pmatrix} : H_1 \oplus H_2 \oplus ... \oplus H_n \to H_1 \oplus H_2 \oplus ... \oplus H_n$ is positive, then the inflated Schur product

$S_{X_n} : M_n(C) \to B(H_1 \oplus H_2 \oplus ... \oplus H_n)$ is strictly positive.

Proof. (i) Suppose that $n = 2$.

Since $\begin{pmatrix} x_{11} & x_{12} \\ x_{12}^* & x_{22} \end{pmatrix} \geq 0$ and $\begin{pmatrix} a_{11} & a_{12} \\ \overline{a}_{12} & a_{22} \end{pmatrix} > 0,$ applying Corollary 2.3, Proposition 2.1, and corollary 2.2, we have

$x_{11} > 0, x_{22} > 0,$ and $a_{22} x_{22} > \dfrac{|a_{12}|^2}{a_{11}} x_{22} \geq \dfrac{|a_{12}|^2}{a_{11}} x_{12}^* x_{11}^{-1} x_{12}$.

Hence

$\begin{pmatrix} a_{11} x_{11} & a_{12} x_{12} \\ \overline{a}_{12} x_{12}^* & a_{22} x_{22} \end{pmatrix}$ is strictly positive.

(ii) Suppose that $n = k$ is true.

Since $\begin{pmatrix} x_{11} & x_{12} & . & x_{1k+1} \\ x_{12}^* & x_{22} & . & x_{2k+1} \\ . & . & . & . \\ x_{1k+1}^* & x_{2k+1}^* & . & x_{k+1k+1} \end{pmatrix} \geq 0$ and $\begin{pmatrix} a_{11} & a_{12} & . & a_{1k+1} \\ \overline{a}_{12} & a_{22} & . & a_{2k+1} \\ . & . & . & . \\ \overline{a}_{1k+1} & \overline{a}_{2k+1} & . & a_{k+1k+1} \end{pmatrix} > 0$, from Corollary 2.2, Proposition 2.1, and Corollary 2.3, we have

$x_{ii} > 0 (i = 1,2,..., k+1), \begin{pmatrix} x_{22} & x_{23} & . & x_{2k+1} \\ x_{23}^* & x_{33} & . & x_{3k+1} \\ . & . & . & . \\ x_{2k+1}^* & x_{3k+1}^* & . & x_{k+1k+1} \end{pmatrix} \geq 0,$ and

$$\begin{pmatrix} a_{22} & a_{23} & \cdot & a_{2k+1} \\ \bar{a}_{23} & a_{33} & \cdot & a_{3k+1} \\ \cdot & \cdot & \cdot & \cdot \\ \bar{a}_{2k+1} & \bar{a}_{3k+1} & \cdot & a_{k+1k+1} \end{pmatrix} - \begin{pmatrix} \frac{|a_{12}|^2}{a_{11}} & \frac{\bar{a}_{12}a_{13}}{a_{11}} & \cdot & \frac{\bar{a}_{12}a_{1k+1}}{a_{11}} \\ \frac{\bar{a}_{13}a_{12}}{a_{11}} & \frac{|a_{13}|^2}{a_{11}} & \cdot & \frac{\bar{a}_{13}a_{1k+1}}{a_{11}} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\bar{a}_{1k+1}a_{12}}{a_{11}} & \frac{\bar{a}_{1k+1}a_{13}}{a_{11}} & \cdot & \frac{|a_{1k+1}|^2}{a_{11}} \end{pmatrix} > 0.$$

By induction, we have

$$\begin{pmatrix} a_{22}x_{22} & a_{23}x_{23} & \cdot & a_{2k+1}x_{2k+1} \\ \bar{a}_{23}x_{23}^{*} & a_{33}x_{33} & \cdot & a_{3k+1}x_{3k+1} \\ \cdot & \cdot & \cdot & \cdot \\ \bar{a}_{2k+1}x_{2k+1}^{*} & \bar{a}_{3k+1}x_{3k+1}^{*} & \cdot & a_{k+1k+1}x_{k+1k+1} \end{pmatrix} > \begin{pmatrix} \frac{|a_{12}|^2}{a_{11}}x_{22} & \frac{\bar{a}_{12}a_{13}}{a_{11}}x_{23} & \cdot & \frac{\bar{a}_{12}a_{1k+1}}{a_{11}}x_{2k+1} \\ \frac{\bar{a}_{13}a_{12}}{a_{11}}x_{23}^{*} & \frac{|a_{13}|^2}{a_{11}}x_{33} & \cdot & \frac{\bar{a}_{13}a_{1k+1}}{a_{11}}x_{3k+1} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\bar{a}_{1k+1}a_{12}}{a_{11}}x_{2k+1}^{*} & \frac{\bar{a}_{1k+1}a_{13}}{a_{11}}x_{3k+1}^{*} & \cdot & \frac{|a_{1k+1}|^2}{a_{11}}x_{k+1k+1} \end{pmatrix}.$$

Since

$$\begin{pmatrix} \frac{|a_{12}|^2}{a_{11}} & \frac{\bar{a}_{12}a_{13}}{a_{11}} & \cdot & \frac{\bar{a}_{12}a_{1k+1}}{a_{11}} \\ \frac{\bar{a}_{13}a_{12}}{a_{11}} & \frac{|a_{13}|^2}{a_{11}} & \cdot & \frac{\bar{a}_{13}a_{1k+1}}{a_{11}} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\bar{a}_{1k+1}a_{12}}{a_{11}} & \frac{\bar{a}_{1k+1}a_{13}}{a_{11}} & \cdot & \frac{|a_{1k+!}|^2}{a_{11}} \end{pmatrix} = \frac{1}{a_{11}}(a_{12}a_{13}...a_{1k+1})^{*}(a_{12}a_{13}...a_{1k+1}) \geq 0,$$

we have

$$\begin{pmatrix} \frac{|a_{12}|^2}{a_{11}}x_{22} & \frac{\bar{a}_{12}a_{13}}{a_{11}}x_{23} & \cdot & \frac{\bar{a}_{12}a_{1k+1}}{a_{11}}x_{2k+1} \\ \frac{\bar{a}_{13}a_{12}}{a_{11}}x_{23}^{*} & \frac{|a_{13}|^2}{a_{11}}x_{33} & \cdot & \frac{\bar{a}_{13}a_{1k+1}}{a_{11}}x_{3k+1} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\bar{a}_{1k+1}a_{12}}{a_{11}}x_{2k+1}^{*} & \frac{\bar{a}_{1k+1}a_{13}}{a_{11}}x_{3k+1}^{*} & \cdot & \frac{|a_{1k+1}|^2}{a_{11}}x_{k+1k+1} \end{pmatrix}$$

$$\geq \begin{pmatrix} \frac{|a_{12}|^2}{a_{11}}x_{12}^{*}x_{11}^{-1}x_{12} & \frac{\bar{a}_{12}a_{13}}{a_{11}}x_{12}^{*}x_{11}^{-1}x_{13} & \cdot & \frac{\bar{a}_{12}a_{1k+1}}{a_{11}}x_{12}^{*}x_{11}^{-1}x_{1k+1} \\ \frac{\bar{a}_{13}a_{12}}{a_{11}}x_{13}^{*}x_{11}^{-1}x_{12} & \frac{|a_{13}|^2}{a_{11}}x_{13}^{*}x_{11}^{-1}x_{13} & \cdot & \frac{\bar{a}_{13}a_{1k+1}}{a_{11}}x_{13}^{*}x_{11}^{-1}x_{1k+1} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\bar{a}_{1k+1}a_{12}}{a_{11}}x_{1k+1}^{*}x_{11}^{-1}x_{12} & \frac{\bar{a}_{1k+1}a_{13}}{a_{11}}x_{1k+1}x_{11}^{-1}x_{13} & \cdot & \frac{|a_{1k+1}|^2}{a_{11}}x_{1k+1}^{*}x_{11}^{-1}x_{1k+1} \end{pmatrix}.$$

Hence

$$\begin{pmatrix} a_{11}x_{11} & a_{12}x_{12} & . & a_{1k+1}x_{1k+1} \\ \overline{a}_{12}x_{12}^{*} & a_{22}x_{22} & . & a_{2k+1}x_{2k+1} \\ . & . & . & . \\ \overline{a}_{1k+1}x_{1k+1}^{*} & \overline{a}_{2k+1}x_{2k+1}^{*} & . & a_{k+1k+1}x_{k+1k+1} \end{pmatrix} > 0.$$

This proves the Theorem.

Example 4.3. Applying Corollary 2.11, we know that

$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ is strictly positive. If $I \geq T^*T + TT^*$, from Corollary 3.2, we have

$\begin{bmatrix} I & T & 0 \\ T^* & I & T \\ 0 & T^* & I \end{bmatrix} \geq 0$. By Theorem 4.2, the matrix $\begin{bmatrix} 2 & T & 0 \\ T^* & 2 & T \\ 0 & T^8 & I \end{bmatrix}$ is strictly positive.

Corollary 4.4. If X_n is strictly positive in Theorem 4.2, then the inflated Schur product S_{X_n} is strictly positive.

Corollary 4.5. If $X_{n_1} > X_{n_2} > 0$ in $B(H_1 \oplus H_2 \oplus ... \oplus H_n)$ and the main diagonal elements of $X_{n_1} - X_{n_2}$ are strictly positive, then $X_{n_1} \circ (a_{ij}) > X_{n_2} \circ (a_{ij})$ for every $(a_{ij}) > 0$ in $M_n(C)$.

Corollary 4.6. Let X_n be strictly positive in $B(H_1 \oplus H_2 \oplus ... \oplus H_n)$ and $(a_{ij}) > 0 \in M_n(C)$. If X_n is invertible, then $X_n^{-1} \circ (a_{ij})^{-1} \geq (X_n \circ (a_{ij}))^{-1}$.

Proof. From[3, Theorem 7.7.17(a)] we know

$$\begin{pmatrix} X_n & I \\ I & X_n^{-1} \end{pmatrix} \circ \begin{pmatrix} (a_{ij}) & I \\ I & (a_{ij})^{-1} \end{pmatrix} = \begin{pmatrix} X_n \circ (a_{ij}) & I \\ I & X_n^{-1} \circ (a_{ij})^{-1} \end{pmatrix}.$$

Corollary 4.7. Let X_n be positive in $B(H_1 \oplus H_2 \oplus ... \oplus H_n)$ and the main diagonal elements of X_n are invertible, then the inflated Schur product S_{X_n} is strictly completely positive.

Proof. From Theorem 4.2 the inflated Schur product is strictly positive. Let

$\begin{pmatrix} A_{11} & A_{12} & . & A_{1n} \\ A_{12}^{*} & A_{22} & . & A_{2n} \\ . & . & . & . \\ A_{1n}^{*} & A_{2n}^{*} & . & A_{nn} \end{pmatrix}$ be strictly positive in $M_n(M_n(C))$, then

$$(S_{X_n} \otimes I_n)\left(\begin{pmatrix} A_{11} & A_{12} & . & A_{1n} \\ A_{12}^{*} & A_{22} & . & A_{2n} \\ . & . & . & . \\ A_{1n}^{*} & A_{2n}^{*} & . & A_{nn} \end{pmatrix}\right) = \begin{pmatrix} X_n \circ A_{11} & X_n \circ A_{12} & . & X_n \circ A_{1n} \\ X_n \circ A_{12}^{*} & X_n \circ A_{22} & . & X_n \circ A_{2n} \\ . & . & . & . \\ X_n \circ A_{1n}^{*} & X_n \circ A_{2n}^{*} & . & X_n \circ A_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} X_n & X_n & . & X_n \\ X_n & X_n & . & X_n \\ . & . & . & . \\ X_n & . & X_n & X_n \end{pmatrix} \circ \begin{pmatrix} A_{11} & A_{12} & . & A_{1n} \\ A_{12}^* & A_{22} & . & A_{2n} \\ . & . & . & . \\ A_{1n}^* & A_{2n}^* & . & A_{nn} \end{pmatrix} > 0.$$

Hence

S_{X_n} is strictly completely positive.

5. Conclusion

We provide a test of strictly positive matrices in $M_n(C)$ and equivalent conditions for $I > T^*T + TT^*$. Also, we prove that a Schur product of a positive matrix of operators with invertible elements in the main diagonal is strictly completely positive.

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