Iteration of Holomorphic Function Systems on the Riemann Sphere

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Abstract

In this work, it is shown that a set of holomorphic functions, normal in any set, remains normal in the same set under the action of another holomorphic function. And therefore, we verify that Julia's sets of two-function composition (no matter the order of composition) coincide, up to a rescaling.

Keywords: holomorphic function, normal set, the Fatou and Julia set

1. Introduction

In 2005, Professor SHERETOV Vladimir Gueorguevich in (Grigorief, 2005), proposes to generalize the sets of Fatou and Julia on the case of a system of holomorphic functions on the sphere of Riemann $\overline{C}$.

By applying the classical results of the complex analysis (notably the convergence of a sequence of elements, notion of a normal family (cf. for example (Shabat, 1969) and (Goluzin, 1966)), the stability of the set of Julia or of Fatou by composition with a holomorphic function is verified.

This article is dedicated to the study of some properties of such a system. Indeed, we study the influence of a holomorphic function on a holomorphic family.

Let $f^{\circ n}$ be the n-th iteration of the holomorphic function f (which is different to any constant) of a surface of Riemann S defined in itself, so we call the set of Fatou for this function, the set on which the iterations family $\{f^{\circ n}\}$ will be normal (a pre-compact family of continuous functions)(cf.(Grigorief, 2005)).

We consider the system $\{f_1, f_2, \cdots, f_k\}$ of holomorphic functions, which induces the following series of compositions:

$$f_1, f_2 \circ f_1, f_3 \circ f_2 \circ f_1, \quad \cdots \quad , f_k \circ f_{k-1} \circ \cdots \circ f_1, \cdots \tag{1}$$

For a good understanding of this article, note the following definition:

Definition 1 *System* (1) *is normal at the point $z_0 \in \overline{C}$, if there exists a neighborhood U of this point such that, from every infinite sequence of elements of* (1)*, we can determine a subsequence, locally and uniformly convergent, either to a holomorphic function or to infinity.*

We conclude that , system (1) is normal on a set, if it is normal in every point of this set. This set will be denoted Fatou $\{f_1, \cdots, f_k\}$ (set of Fatou for system (1)) and its complementary $\overline{C} \setminus Fatou\{f_1, \cdots, f_k\}$, will be denoted Julia $\{f_1, \cdots, f_k\}$ (set of Julia for system (1)).

In this paper we will limit to the study of holomorphic systems.

The following theorem plays a vital role in this present work:

Theorem 1 *Consider the holomorphic applications f_ν, $\nu = 1, ..., k$, on $\overline{C}$. If the family $\{(f_k \circ f_{k-1} \circ ... \circ f_1)^{\circ n}\}, n \in N$, is normal to the neighborhood of z_0, so for any holomorphic function h, the family $\{h \circ (f_k \circ f_{k-1} \circ ... \circ f_1)^{\circ n}\}$ it is also normal in this same neighborhood.*

Proof. Let U be the neighborhood of this point z_0, such that the family $\{(f_k \circ f_{k-1} \circ ... \circ f_1)^{\circ n}\}$ be normal in U.

We prove that, $\{h \circ (f_k \circ f_{k-1} \circ ... \circ f_1)^{\circ n}\}$ is normal in U.

Let's put $g = f_k \circ f_{k-1} \circ ... \circ f_1$. It is clear that g is holomorphic.According to the hypotheses of the theorem, $g^{\circ n}$ is locally and uniformly convergent in U, either to a holomorphic function, or to infinity.

1st case. Supposose that, $g^{\circ n}(z)$ locally and uniformly convergent to a function f in U. So we have:

a) if h is bounded near the point $f(z_0)$, therefore in a neighborhood of z_0, the compound $h \circ f(z) \neq \infty$ and continue.

Let's analyze the expression $h \circ f^{\circ n} - h \circ f$:

h being holomorphic function and bounded near the point $f(z_0)$, then asymptotically, one can write

$$h(w + \Delta w) = h'(w)\Delta w + o(|\Delta w|).$$

h' is also holomorphic at the point $g(z_0)$, then $\exists M \in R$ such that, $M = \sup |h'|$ in the neighborhood of $f(z_0)$. We will have the following increase:

$$\begin{aligned} |h \circ (f)^{\circ n} - h \circ f| &= |h'\,(f(z))\,(g^{\circ n}(z) - f(z)) + o\,(g^{\circ n}(z) - g(z))\,| \\ &\leq M|g^{\circ n}(z) - f(z)| + o\,(|\,g^{\circ n}(z) - g(z)|)\,. \end{aligned} \tag{2}$$

The right hand side of system (2) tends to zero when $n \to \infty$, because $g^{\circ n}$ locally and uniformly converge towards f;

b) Assume that $h \circ f(z_0) = \infty$.

Let's take $p_n(z) = \dfrac{1}{h \circ g^{\circ n}(z)}$. p_n is well defined because, $h \circ f(z_0) = \infty$ is different to zero. Plus, $p_n \to 0$ when $n \to \infty$.

The family of functions p_n is holomorphic and bounded in a neighborhood of zero. So for the function $p(w) = \dfrac{1}{h(w)}$, there exist a real $k \in R$ such that, $k = \sup |p'| < \infty$ in the neighborhood of $f(z_0)$.

We have

$$\begin{aligned} |\,p_n \circ g^{\circ n}(z) - p \circ f(z)\,| &= |\,p'\,(f(z))\,(g^{\circ n}(z) - f(z))\,| + o\,(|\,g^{\circ n}(z) - f(z)\,|) \\ &\leq K\,|\,g^{\circ n}(z) - f(z)\,| + o\,(|\,g^{\circ n}(z) - f(z)\,|)\,. \end{aligned} \tag{3}$$

The right hand side of system (3) tends to zero locally and uniformly in $U(z_0)$, when n tend to ∞. It is for this reason that $h \circ g^{\circ n}$ locally and uniformly in $U(z_0)$ converge to ∞.

2nd case. Suppose the $g^{\circ n}$ locally and uniformly tends to ∞ in the neighborhood $U(z_0)$ of the point z_0.

Let's take $G^{\circ n} = \dfrac{1}{g^{\circ n}}$.

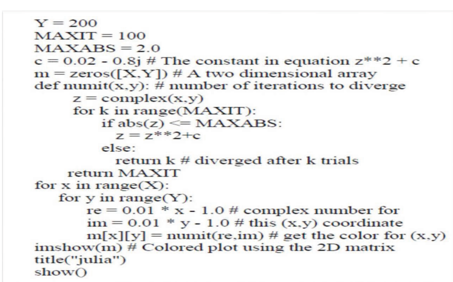
It is clear that for $n \to \infty$, the $G^{\circ n}$ locally and uniformly tends to zero. As before, we prove the local and uniform convergence of the sequence $h\left(\dfrac{1}{G^{\circ n}}\right)$, either to a holomorphic function or to the constant ∞.

The Theorem 1 is then proved.

Consequence.

Let f_1 and f_2 be two holomorphic functions, then Julia sets for the functions $f_1 \circ f_2$ and $f_2 \circ f_1$ are coincident.

The corresponding Pyzo program: Julia's set for the function $f_1(z) = z^2 + c$



```
Y = 200
MAXIT = 100
MAXABS = 2.0
c = 0.02 - 0.8j # The constant in equation z**2 + c
m = zeros([X,Y]) # A two dimensional array
def numit(x,y): # number of iterations to diverge
        z = complex(x,y)
        for k in range(MAXIT):
            if abs(z) <= MAXABS:
              z = z**2+c
            else:
              return k # diverged after k trials
       return MAXIT
for x in range(X):
     for y in range(Y):
          re = 0.01 * x - 1.0 # complex number for
          im = 0.01 * y - 1.0 # this (x,y) coordinate
          m[x][y] = numit(re,im) # get the color for (x,y)
imshow(m) # Colored plot using the 2D matrix
title("julia")
show()
```

Figure 1. Pyzo program

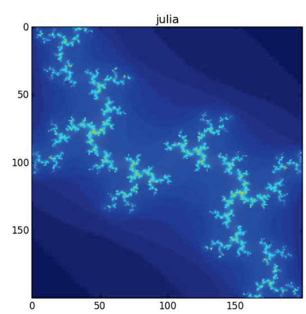


Figure 2. Julia set for the function $f_1(z) = z^2 + c$

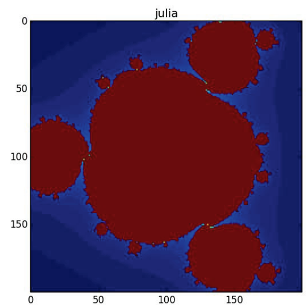


Figure 3. Julia's set for the function $f_2(z) = z^3 + c$

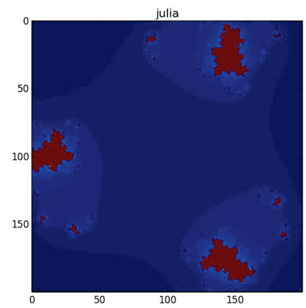


Figure 4. Julia's set for the function $f_1 \circ f_2$.

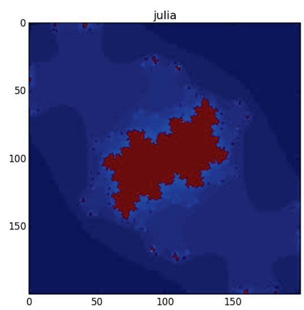


Figure 5. Julia's set for the function $f_2 \circ f_1$.

Remark 1 It may be noted that Figures 4 and 5 are similar to the difference of scales. So the consequence is true.

Taking inspiration from Theorem 1, we can state the following result:

Theorem 2 *The Fatou* $\{f_1, \cdots, f_k\}$ *set of the system* (1) *coincides with the Fatou set of the compound* $f = f_k \circ \cdots \circ f_1$.

Proof.

1. Let us first show that Fatou $(f_1, \cdots, f_k)$ is contained in Fatou $(f_k \circ \cdots \circ f_1)$ =Fatou(f).

   Let $z_0 \in Fatou(f_1, ..., f_k)$. Then the family $f_1, f_2 \circ f_1, ...$ is normal in a neighborhood $U(z_0)$ of point z_0. In particular, from the sequence , $f_k \circ ... \circ f_1, (f_k \circ ... \circ f_1)^{\circ n}$, functions of this family,we can extract a convergent subsequence, that is to say the family of iterations $(f_k \circ ... \circ f_1)^{\circ n}$ is normal in the same neighborhood of z_0. Therefore $Fatou(f_1, ..., f_k) \subset Fatou(f)$.

2. Suppose now that, $z_0 \in Fatou(f_k \circ ... \circ f_1) = Fatou(f)$. So any family sub-sequence (1) is constituted by the elements of the form $f^{\circ n}, f_1 \circ f^{\circ n}, f_2 \circ f_1 \circ f^{\circ n}, ..., f_k \circ ... \circ f_1 \circ f^{\circ n}$. It is also clear that we can choose a subsequence $f_l \circ ... \circ f_1 \circ f^{\circ n}$, $1 \leq l \leq k$. Taking $h = f_l \circ ... \circ f_1$, the family $\{f^{\circ n}\}$ is normal, so according to Theorem 1, $\{h \circ f^{\circ n}\}$ is also normal. Thus, from the elements of the family (1), it is possible to extract a subset locally and uniformly convergent, that is (1) is normal.

The theorem 2 is proved.

Remark 2 Theorems 1 and 2 could be stated relatively to Julia sets.

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References

Bisi, C. (2004) *On Commuting polynomial automorphisms of* $\mathbb{C}^2$ *: Publ. Mat.*, *48*(1), 227-239.

Bisi, C. (2008) *On Commuting polynomial automorphisms of* $\mathbb{C}^k$, $k \geqslant 3$ *: Math. Z.*, *258*(4), 875-891.

Blanchard, P. (1984) *Complex Analytic on the Riemann Sphere : Bull. A.M.S.*. 85-141.

Foster, O. (1980) *Les Surfaces de Riemann*.

Goluzin, G. M. (1966) *Théorie gométrique des fonctions à variable complexe*.

Grigorief, V. V., & Sheretov, V. G. (2005) *Certaines modifications des fractales de Mandelbrot et de Julia: Application Analyse Fonctionnelle dans la Théorie des Rapprochements*. 321-2. Tver.

Milnor, J. (2000). *Dynamique Holomorphe*. Izhevsk.

Paitgen, H. O. (1993) *La Beauté des Fractales. Images sur ordinateurs des systèmes dynamiques*.

Shabat, B. V. (1969). *Introduction à l'Analyse Complexe*.

Sheretov, V. G. (2005) *Cours sur les Surfaces de Riemann*. Tver.