Solutions to Four-Letter Words in Mathematics

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Abstract

By trial and error, and observing patterns in existing mathematics, we built our own unorthodox notation to solve mathematical challenges having no known solutions. The imaginary number i is real. We determined the significance and applicability of unbalanced equations such as one "equal" to zero, and one "equal" to negative one, solved for multiplications and divisions by zero. Computing coefficients from general solutions with initial conditions turns homogeneous equations into non-homogeneous ones. We computed coefficients from general solutions using periods from their own equations instead of initial conditions. Homogeneous linear second-order equations with real roots have companions with "imaginary" roots, and double-roots. We solved for logarithms having negative arguments, negative bases, or both without the absolute-value notation. The roots of homogeneous linear equations are the frequencies of their signals. Roots/frequencies and periods from linear homogeneous equations are related to electronics. The speed of light is not a limiting factor in special relativity.

Keywords: imaginary number i, special relativity, complementary solving methods, remnant, double-sign, double-root, reflection method, inverse of i, electronics.

1. Introduction

The core of this manuscript is the product of analyzing patterns in mathematics to build our own notation system. Once a pattern was discovered, by trial and error, we determined the applicability of its mechanics to solve mathematical challenges that historically had no known solutions. Our *unorthodox* notation is formed by a set of formulas found hidden within the backbone of existing traditional mathematics.

There are mathematical concepts which do not conform to standard notation: some of them are finding real values to the square root of a negative one; solving logarithms having negative arguments, negative bases, or both without using the absolute value notation; and giving proper interpretation to unbalanced expressions such as $\delta = -\delta$ and $\delta = 0$, etc., where δ is not equal to zero. Such challenges are considered to be *four-letter* words because they violate standard notation, thus having neither solutions nor interpretations–at least until now. We solve for those mathematical conundrums with our own notation system, conforming the results into standard notation. But first, we want to introduce the mechanics of our own notation dealing strictly with *double-signs*.

2. Double-Sign Notation

Simplifying

Let $a \neq 0$, and $b \neq 0$. Converting double-sign values into single-sign values is achieved reading the signs from top to bottom.

$$\pm a = -2a.$$

$$\mp a = +2a.$$

The $\pm a$ are two independent a's positioned linearly on the same plane on opposite sides of a common origin; $-a = +a$. We have two equally valid versions of conversions spanning in opposite directions. In the former equation the motion goes from positive to negative. If both a's are uniformly moved to the left until $+a$ rests on the origin, then the total value is $-2a$. In the latter equation the motion goes from negative to positive. If both a's are uniformly moved to the right until $-a$ rests on the origin, then the total value is $+2a$.

Addition

Double-sign expressions are added *horizontally*; top signs to top signs, and bottom signs to bottom signs.

Example 1

Terms with only one sign, $-a$ and $+a$, imply that they have identical double-signs, e. g., $-a = -a$, and $+a = +a$, respectively.

$$\begin{aligned} &+ a \pm a \\ &= +2a \end{aligned}$$

Example 2

$$\pm a \mp a = 0$$

Example 3

$$\pm a \pm a = \pm 2a$$

Multiplication

Double-sign expressions are multiplied diagonally.

Example 1

$$\begin{aligned} (\pm a)^2 &= (\pm a)(\pm a) \\ &= (+a)(-a) \\ &= -a^2 \end{aligned}$$

Example 2

$$(\pm a)(\mp b) = +ab$$

Example 4

$$(\pm a + b)^2$$

We can solve this expression using two different notation methods.

Method (1)

$$\begin{aligned} &(\pm a + b)(\pm a + b) \\ &= (\pm a)(\pm a) + 2b(\pm a) + b^2 \\ &= -a^2 \pm 2ba + b^2 \end{aligned}$$

We have two independent equations which can be simplified.

$$\begin{aligned} &= (-a^2 + 2ab + b^2) + (-a^2 - 2ab + b^2) \\ &= -a^2 + b^2. \end{aligned}$$

Method (2)

The second method is multiplying diagonally.

$$\begin{aligned} (\pm a + b)^2 &= (+a + b)(-a + b) \\ &= -a^2 + b^2. \end{aligned}$$

The two methods yield identical results. The *double-sign* notation will be applied to simplify expressions when the imaginary number i is replaced with its *real* number equivalent. Finding a real number for i is our next goal.

3. The Imaginary Number *i*

Theorem 3.1. *The imaginary number i is a real number:*

$$\begin{aligned} &i = \pm\sqrt{-1} = \mp 1 \\ &r_1 = -1, r_2 = +1 \end{aligned}$$

Proof. Equation (1) below, with real roots, will provide a new insight into the mechanics for solving β. Such mechanics will be applied to solve equation (2) converting its *imaginary* roots into *real* roots. Let $\beta > 0$, and $\delta > 0$.

$$f(\beta, \delta) = \beta^2 - \delta = 0 \tag{1}$$

$$g(\beta, \delta) = \beta^2 + \delta = 0 \tag{2}$$

Solve for β in (1).

$$\begin{aligned} \beta^2 - \delta &= 0 \\ \beta^2 &= +\delta \\ \beta^2 &= +\delta \\ \pm\beta &= \pm\sqrt{\delta} \end{aligned}$$

Moving the solution back, the negative sign reappears;

$$\pm\beta - (\pm\delta) = 0.$$

The roots are

$$\beta_1 = \pm\sqrt{\delta}, \beta_2 = \mp\sqrt{\delta}.$$

The signs of roots β_1 and β_2 are inverse of each other, therefore, there is a total of two single-root repeated; $\beta_{1a} = +\sqrt{\delta}$ and $\beta_{1b} = -\sqrt{\delta}$. Solving equation (1) with single-sign roots, we get

$$\begin{aligned} f(\beta_{1a}, \delta) &= (+\sqrt{\delta})^2 - \delta \\ &= +\delta - \delta = 0, \end{aligned}$$

and

$$\begin{aligned} f(\beta_{1b}, \delta) &= (-\sqrt{\delta})^2 - \delta \\ &= +\delta - \delta = 0. \end{aligned}$$

We now apply the mechanics of the *double-sign* notation to solve equation (2) without resorting to imaginary roots;

$$\begin{aligned} g(\beta_1, +\delta) &= (\pm\sqrt{+\delta})^2 + \delta \\ &= -\delta + \delta = 0. \end{aligned}$$

Root β_2 is also a solution to equation (2). Based on this evidence we obtain the following corollary.

Corollary 3.1. *1. We can obtain the square root of an equation on the same side of the equal sign. 2. The signs associated with the coefficients are not included under the radical, i.e.,* $x^2 + c = 0, \pm x + (\pm\sqrt{c}) = 0$*,* $c \neq 0$.

We apply Corollary 3.1 to solve for β in equation (2).

$$\begin{aligned} \beta^2 + \delta &= 0 \\ \pm\beta + (\pm\sqrt{\delta}) &= 0 \\ \pm\beta &= -(\pm\sqrt{\delta}) \\ \pm\beta &= \mp\sqrt{\delta} \end{aligned}$$

Therefore

$$\beta_1 = \mp\sqrt{\delta},$$

and

$$\beta_2 = \pm\sqrt{\delta}.$$

Except for the inversion of the signs, the real roots from equation (1) and the "imaginary" roots from equation (2) are identical. The only difference is that the real roots travel from *positive* to *negative*; $\pm\sqrt{+\delta}$, to a total distance of $-2\sqrt{+\delta}$, and the "imaginary" roots travel from *negative* to *positive*; $\mp\sqrt{+\delta}$ to a total distance of $+2\sqrt{+\delta}$, over the same interval implying the motion of a *pendulum*, with its axis at the center of the total distance between the roots. See Figure 1 below.

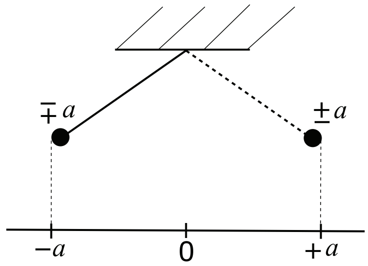

Figure 1. Pendulum

This is consistent with the standard counter-clockwise motion on the *unit-circle*.

The solution to equation (2) implies that the negative sign under the radical square is factored out without violating any mathematical rules, then

$$\begin{aligned} \pm\beta &= \pm\sqrt{-\delta} \\ &= \mp\sqrt{\delta}. \end{aligned}$$

Thus we conclude that

$$i = \pm\sqrt{-1} = \mp 1.$$

Hence

$$r_1 = -1,$$

and

$$r_2 = +1.$$

□

Imaginary numbers and complex equations are extraneous notation. Therefore, hereafter, the square roots of a negative number will be referred to as **alternative** roots, otherwise they will be called **standard** roots.

Solving equation (1) with the double-root method results in -2δ, and solving equation (2) with the single-root method results in $+2\delta$. Both procedures yield a **remnant**–one of the fundamental topics in this manuscript.

The concept of the *remnant*–$f(t)$– seems to be a side effect; it is not. The *remnant* explains some of the *four-letter* words in mathematics. Remnants will be incorporated into their own equations, and **general** solutions will be computed with *standard* and *alternative* roots.

3.1 Simplifying the i

In this section we replace the "imaginary" number i with ∓ 1. Solving the new expressions with our notation should yield *identical*, or simpler, results as those obtained with standard notation. The double-root method requires the plus or minus signs, of squared results, on both sides of the equation. In standard notation sometimes the plus or minus signs are not included with the radical square, in our notation we will include them.

Example 1

Standard notation.

$$(ai)^2 = -a^2$$

Double-sign notation.

$$\begin{aligned} [a(\mp 1)]^2 &= (\mp a)^2 \\ &= (\mp a)(\mp a) \end{aligned}$$

$$= -a^2$$

Example 2

Standard notation.

$$(ai)(-ai) = +a^2$$

Double-sign notation.

$$\begin{aligned} a(\mp 1)(-a(\mp 1)) &= (\mp a)(\pm a) \\ &= +a^2 \end{aligned}$$

Example 3
Standard notation.

$$(a+bi)^2 = a^2 + 2abi - b^2$$

Double-sign notation.

$$\begin{aligned} (a+b(\mp 1))^2 &= (a \mp b)(a \mp b) \\ &= a^2 \mp 2ab - b^2 \\ &= a^2 - b^2 \end{aligned}$$

Example 4
Standard notation.

$$(a+bi)(a-bi) = a^2 + b^2$$

Double-sign notation.

$$\begin{aligned} &(a+b(\mp 1))(a-b(\mp 1)) \\ &= (a \mp b)(a \pm b) \\ &= a^2 + (\pm ab) + (\mp ab) + (\mp b)(\pm b) \\ &= a^2 + b^2 \end{aligned}$$

Example 5
Standard notation.

$$\pm\sqrt{2i} = \pm(1+i)$$

Simplify both sides with the double-sign method. We solve the radical square first.

$$\begin{aligned} \pm\sqrt{2i} &= \pm\sqrt{2(\mp 1)} \\ &= \pm\sqrt{\mp 2} \end{aligned}$$

The ∓ 2 is going from *negative* to *positive*, shifting the total value to the positive side.

$$= \pm\sqrt{+4} = \pm 2.$$

Solving the right-hand side, we get

$$\pm(1 \mp 1) = \pm 2$$

The $i = \mp 1$ conforms complex expressions into standard notation, corroborating our notation as **valid**.

4. The Remnant–f(t)

Theorem 4.1. *Homogeneous linear second-order equations with standard roots and non-zero discriminants have linear independent counterparts with alternative roots.*

Proof. Consider equation (3) with standard roots.

$$ay'' + by' + cy = 0 \tag{3}$$

Let $a \neq 0$. Applying the quadratic formula to the auxiliary equation, $f(x) = ax^2 + bx + c = 0$, the roots are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

with $b^2 > 4ac$. Find the remnant $f(t)$.

Solution

Solving with its double-root, we obtain

$$f(\frac{-b\pm\sqrt{b^2-4ac}}{2a}) = a(\frac{-b\pm\sqrt{b^2-4ac}}{2a})^2 + b(\frac{-b\pm\sqrt{b^2-4ac}}{2a}) + c$$

$$= a(\frac{-b\pm\sqrt{b^2-4ac}}{2a})(\frac{-b\pm\sqrt{b^2-4ac}}{2a}) + b(\frac{-b\pm\sqrt{b^2-4ac}}{2a}) + c$$

$$= a(\frac{b^2+2(-b)(\pm\sqrt{b^2-4ac})+(\pm\sqrt{b^2-4ac})^2}{4a^2}) + b(\frac{-b\pm\sqrt{b^2-4ac}}{2a}) + c$$

$$= (\frac{b^2\mp 2b\sqrt{b^2-4ac}+(\pm\sqrt{b^2-4ac})(\pm\sqrt{b^2-4ac})}{4a}) + \frac{-2b^2\pm 2b\sqrt{b^2-4ac}}{4a} + c$$

$$= \frac{b^2\mp 2b\sqrt{b^2-4ac}-b^2+4ac-2b^2\pm 2b\sqrt{b^2-4ac}+4ac}{4a}$$

$$= \frac{8ac-2b^2\pm 2b\sqrt{b^2-4ac}\mp 2b\sqrt{b^2-4ac}}{4a}.$$

Adding up the top sign to the top sign and bottom sign to the bottom sign, the radical squares become zero. Then, the remnant is

$$f(t) = 2c - \frac{b^2}{2a},$$

or

$$f(t) = \frac{(-\sqrt{b^2-4ac})(\sqrt{b^2-4ac})}{2a}.$$

The remnant signifies that in order for equation (3) to be zero when applying the double-root, it should be expressed as non-homogeneous instead: thus,

$$ay'' + by' + cy = f(t). \tag{4}$$

In the auxiliary equation the y is not represented and the remnant does not depend on y'' or y'. So, we assume that the remnant is associated to the y term, therefore

$$ay'' + by' + (c - f(t))y = 0 \tag{5}$$

$$ay'' + by' + (\frac{b^2}{2a} - c)y = 0,$$

with $C = \frac{b^2}{2a} - c$. Its alternative roots are

$$r = \frac{-b\mp\sqrt{b^2-4ac}}{2a}.$$

Formula (5) states that it turns equations with imaginary roots into their equivalent counterparts with standard roots, and vice versa. Applying the remnant from equation (5) to itself, we should obtain the original equation back. So

$$\begin{aligned} &2C - \frac{b^2}{2a}, \\ &= 2(\frac{b^2}{2a} - c) - \frac{b^2}{2a}, \\ &= -2c + \frac{b^2}{2a} \\ &= -f(t). \end{aligned}$$

Setting up equation (5) equal to its remnant, equation (3) is recovered.

$$\begin{aligned} &ay'' + by' + Cy = -f(t), \\ &ay'' + by' + (\frac{b^2}{2a} - c)y = -2c + \frac{b^2}{2a}, \\ &ay'' + by' + cy = 0. \end{aligned}$$

□

The entire process can be stated as $ay'' + by' + cy = f(t) - f(t)$.
The linear independence from equations with standard roots and their counterparts with alternative roots are clear. The Wronskian further supports our findings.

Theorem 4.2. *There is an intrinsic ratio between individual terms from homogeneous linear second-order equations and their counterparts with alternative roots in the form of* $2a^2y'' + 2aby' + (b^2 - 2ac)y = 0$.

Proof. Let

$$a_iy'' + b_iy' + c_iy = 0 \tag{6}$$

be the equation with alternative roots counterpart of equation (3). Thus, we split up the roots from equation (3) into

$$r_{11} = -\frac{b}{2a}, \quad r_{12} = +\frac{\sqrt{b^2 - 4ac}}{2a}$$

and

$$r_{21} = -\frac{b}{2a}, \quad r_{22} = -\frac{\sqrt{b^2 - 4ac}}{2a}.$$

The Wronskian is

$$\begin{aligned} \mathbf{W} = \begin{vmatrix} r_{11} & r_{12} \\ r_{22} & r_{21} \end{vmatrix} &= r_{11}r_{21} - r_{22}r_{12} \\ &= \frac{b^2 - 2ac}{2a^2}. \end{aligned}$$

The non-zero result shows linear independence of a second equation having identical roots. But we need to find how our result relates to both equations. For that, we use the discriminant to find an identical outcome.

$$\begin{aligned} b^2 - 4ac &= 0 \\ b^2 - 2ac &= 2ac \\ b^2 - 2ac &= \frac{2a^2c}{a} \\ \frac{b^2 - 2ac}{2a^2} &= \frac{c_i}{a_i} \end{aligned}$$

Therefore $c_i = b^2 - 2ac$ and $a_i = 2a^2$. Our answer shows that the equation is in the form of $y'' + \frac{b_i}{a_i}y' + \frac{c_i}{a_i}y = 0$. Then, to find b_i, we set up the following ratio,

$$\begin{aligned} \frac{b_i}{a_i} &= \frac{b}{a}, \\ b_i &= \frac{a_ib}{a} \\ &= 2ab \end{aligned}$$

So, equation (6) is a linear independent from equation (3) with identical inverse roots;

$$2a^2y'' + 2aby' + (b^2 - 2ac)y = 0.$$

Equation (6) has alternative roots

$$r = \frac{-b \pm \sqrt{-b^2 + 4ac}}{2a}.$$

Applying the Wronskian to (6), without factoring the negative sign out of the radical, we should get equation (3) back.

$$\begin{aligned} r_{11} &= -\frac{b}{2a} \\ r_{12} &= +\frac{\sqrt{-b^2+4ac}}{2a} \\ r_{21} &= -\frac{b}{2a} \\ r_{22} &= -\frac{\sqrt{-b^2+4ac}}{2a} \end{aligned}$$

$$\mathbf{W} = \begin{vmatrix} r_{11} & r_{12} \\ r_{22} & r_{21} \end{vmatrix} = \frac{c}{a},$$

and

$$b = \frac{ab_i}{a_i} = b.$$

□

5. General Solutions

Theorem 5.1. *Homogeneous and non-homogeneous linear equations have an equal number of periods as frequencies.*

Proof. Consider equations (3) and

$$ay'' + by' + cy = f(t). \tag{7}$$

Each equation has two periods. Find the periods of the equation (7) independently of $f(t)$. Let $y'' = \frac{y}{t^2}$, $y' = \frac{y}{t}$ and $y = y$. Multiplying $\frac{t^2}{y}$ times the homogeneous side of (7), we get

$$\frac{t^2}{y}(ay'' + by' + cy) = 0,$$

$$ct^2 + bt + a = 0. \tag{8}$$

The descending powers are reversed. Applying the quadratic formula, the periods are

$$\begin{aligned} t &= \frac{-b \pm \sqrt{b^2-4ac}}{2c}, \\ t_1 &= \frac{-b + \sqrt{b^2-4ac}}{2c}, \end{aligned}$$

and

$$t_2 = \frac{-b - \sqrt{b^2-4ac}}{2c}.$$

□

What is the relationship between the roots and the periods? We found out the following;

$$\begin{aligned} r_1 &= \frac{1}{t_2} \\ \frac{-b + \sqrt{b^2-4ac}}{2a} &= \frac{2c}{-b - \sqrt{b^2-4ac}} \\ 4ac &= 4ac, \end{aligned}$$

and

$$\begin{aligned}
r_2 &= \frac{1}{t_1},\\
\frac{-b-\sqrt{b^2-4ac}}{2a} &= \frac{2c}{-b+\sqrt{b^2-4ac}}\\
4ac &= 4ac.
\end{aligned}$$

Our results indicate that the roots are the actual frequencies of the equation.

With the discovery of the periods and association to the remnant, we can expand our options to find more general solutions to linear equations.

The variable coefficients c_1, c_2, etc., in general solutions are computed by entering initial values to the position $y(t)$, and the speed $y'(t)$. We will expand the options to compute general solutions: converting homogeneous equations into their equivalent non-homogeneous equations; turning non-homogeneous equations into homogeneous by computing $f(t)$ with its periods and integrating them into the y term; by calculating the coefficients with the periods of the signal itself instead of initial conditions.

Theorem 5.2. *Homogeneous linear second-order equations, with standard roots, can be converted into two equivalent independent non-homogeneous equations when assigning initial conditions. Let equation (3) have standard roots, then* $ay'' + by' + cy = 1$, *and* $ay'' + by' + cy = t$ *can be computed.*

Proof. Suppose that equation (3) has general solution

$$y(t) = c_1e^{r_1t} + c_2e^{r_2t} = 0.$$

Differentiating it once, we get

$$y'(t) = c_1r_1e^{r_1t} + c_2r_2e^{r_2t} = 0.$$

To compute the coefficients, we have two cases frequently used in standard notation.

Case 1 Let initial conditions be $y(0) = 1$ and $y'(0) = 0$. Using the remnant method the general solution would be represented as $f(t) = 1$, and $f'(t) = 0$. Thus

$$\begin{aligned}
y(t) &= c_1e^{r_1t} + c_2e^{r_2t} = f(t),\\
y(t) &= c_1e^{r_1t} + c_2e^{r_2t} = 1,
\end{aligned}$$

and

$$\begin{aligned}
y'(t) &= c_1r_1e^{r_1t} + c_2r_2e^{r_2t} = f'(t),\\
y'(t) &= c_1r_1e^{r_1t} + c_2r_2e^{r_2t} = 0.
\end{aligned}$$

The $f(t)$ originates from non-homogeneous equations, then, with our method, equation (3) can be rewritten as

$$ay'' + by' + cy = 1. \tag{9}$$

Case 2 Let initial conditions be $y(0) = 0$, and $y'(0) = 1$. With our method the initial conditions would be represented as $f(t) = t$ and $f'(t) = 1$, then

$$\begin{aligned}
y(t) &= c_1e^{r_1t} + c_2e^{r_2t} = f(t),\\
y(t) &= c_1e^{r_1t} + c_2e^{r_2t} = t,
\end{aligned}$$

and

$$\begin{aligned}
y'(t) &= c_1r_1e^{r_1t} + c_2r_2e^{r_2t} = f'(t),\\
y'(t) &= c_1r_1e^{r_1t} + c_2r_2e^{r_2t} = 1.
\end{aligned}$$

Therefore equation (3) can optionally be rewritten as

$$ay'' + by' + cy = t. \tag{10}$$

□

We conclude that homogeneous second-order linear equations, with standard roots, become non-homogeneous by way of assigning initial conditions. Moreover, in standard notation, equations (9) and (10) call for particular solutions as well.

The concept of the remnant offers more options to find general solutions to equations (9) and (10). Applying the remnant to equation (9) simplifies to

$$\begin{aligned} &ay'' + by' + cy = f(t),\\ &ay'' + by' + (c - f(t))y = 0,\\ &ay'' + by' + (c - 1)y = 0. \end{aligned}$$

However, equation (10) requires the period-dependent $f(t)$ to be simplified first. Since it has two independent periods from equation (8), that translates into two equally valid equations, therefore

$$\begin{aligned} &ay'' + by' + (c - f(t_1))y = 0,\\ &ay'' + by' + (c - t_1)y = 0, \end{aligned}$$

and

$$\begin{aligned} &ay'' + by' + (c - f(t_2))y = 0,\\ &ay'' + by' + (c - t_2)y = 0. \end{aligned}$$

Our approach can be expanded into more complex non-homogeneous equations such as $ay'' + by' + cy = \beta te^t$, and $ay'' + by' + cy = cos(t)$. Let $f(t) = \beta te^t$, and $f(t) = cos(t)$, then follow the examples of the two cases to find general solutions.

5.1 Further Applications of the Remnant–f(t)

Theorem 5.3. *The variable coefficients of general solutions from homogeneous linear equations can be computed using their own period-based equations instead of initial conditions: $y(t) = f(t), y'(t) = f'(t)$ and $y''(t) = f''(t)$. There are three cases: case 1) equations with standard roots; case 2) equations with double-roots; case 3) equations with alternative roots.*

We use equation (8) as the $f(t)$ to compute the coefficients of general solutions from homogeneous equations. So

$$\begin{aligned} &f(t) = ct^2 + bt + a,\\ &f'(t) = 2ct + b, \end{aligned}$$

and

$$f''(t) = 2c.$$

We get three linear independent general equations.

Case 1 Let us suppose that equation (3) has standard frequencies $r = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ with $b^2 > 4ac$. So the general equation is

$$y(t) = c_1e^{r_1t} + c_2e^{r_2t} = 0. \tag{11}$$

Deriving it twice, we obtain

$$y'(t) = c_1r_1e^{r_1t} + c_2r_2e^{r_2t} = 0,$$

and

$$y''(t) = c_1r_1^2e^{r_1t} + c_2r_2^2e^{r_2t} = 0.$$

So, plugging the results back into (3) we find

$$\begin{aligned} &c_1e^{r_1t}(ar_1^2 + br_1 + c) + c_2e^{r_2t}(ar_2^2 + br_2 + c) = 0,\\ &0 = 0. \end{aligned}$$

The expression within the parenthesis are the auxiliary equations from the two roots, thus they are both zeros. Then,

$$\begin{aligned} &y(t) = f(t),\\ &y'(t) = f'(t),\\ &y''(t) = f''(t). \end{aligned}$$

Let $t = 0$ in all the results.

$$\begin{aligned} y(0) &= c_1 + c_2 = a, \\ y'(0) &= c_1 r_1 + c_2 r_2 = b, \\ y''(0) &= c_1 r_1^2 + c_2 r_2^2 = 2c. \end{aligned}$$

Combining two results at a time, we can compute three linearly independent general solutions:

1) $y(0)$ and $y'(0)$ yield $c_1 = \frac{ar_2 - b}{-r_1 + r_2}$ and $c_2 = \frac{b - ar_1}{-r_1 + r_2}$, then

$$y(t) = \frac{ar_2 - b}{-r_1 + r_2} e^{r_1 t} + \frac{b - ar_1}{-r_1 + r_2} e^{r_2 t} = 0;$$

2) $y(0)$ and $y''(0)$ yield $c_1 = \frac{ar_2^2 - 2c}{-r_1^2 + r_2^2}$ and $c_2 = \frac{2c - ar_1^2}{-r_1^2 + r_2^2}$, then

$$y(t) = \frac{ar_2^2 - 2c}{-r_1^2 + r_2^2} e^{r_1 t} + \frac{2c - ar_1^2}{-r_1^2 + r_2^2} e^{r_2 t} = 0;$$

3) $y'(0)$ and $y''(0)$ yield $c_1 = \frac{-2c + br_2}{-r_1^2 + r_1 r_2}$ and $c_2 = \frac{2c - br_1}{-r_1 r_2 + r_2^2}$, then

$$y(t) = \frac{-2c + br_2}{-r_1^2 + r_1 r_2} e^{r_1 t} + \frac{2c - br_1}{-r_1 r_2 + r_2^2} e^{r_2 t} = 0.$$

Since the coefficients are arbitrary, letting $c_1 = c_2 = 1$ in equation (11) generates a *non-period-based* general solution,

$$y(t) = e^{r_1 t} + e^{r_2 t} = 0. \tag{12}$$

This expression would be a *reference* general solution to help us analyze how the period-based general solutions relate to it.

Case 2 Setting the discriminant in equation (3) equal to zero and solve for c.

$$\begin{aligned} b^2 - 4ac &= 0 \\ c &= \frac{b^2}{4a}. \end{aligned}$$

Replacing c back in the equation, we find its counterpart with double-roots. Thus, equation (3) becomes

$$4a^2 y'' + 4aby' + b^2 y = 0 \tag{13}$$

with $b^2 = 4ac$ under the radical, thus having double-roots $r = -\frac{b}{2a}$,

$$\begin{aligned} f(t) &= b^2 t^2 + 4abt + 4a^2, \\ f'(t) &= 2b^2 t + 4ab, \end{aligned}$$

and

$$f''(t) = 2b^2.$$

The general solution is

$$y(t) = c_1 e^{rt} + c_2 t e^{rt} = 0. \tag{14}$$

Deriving it twice, we get

$$\begin{aligned} y'(t) &= c_1 r e^{rt} + c_2 e^{rt} + c_2 r t e^{rt}, \\ y''(t) &= c_1 r^2 e^{rt} + 2c_2 r e^{rt} + c_2 r^2 t e^{rt}. \end{aligned}$$

Plugging the results back into (13) we obtain

$$\begin{aligned}
&c_1e^{rt}(4a^2r^2+4abr+b^2)+c_2e^{rt}(8a^2r+4a^2r^2t+4ab+4abrt+b^2t)=0\\
&=c_1e^{rt}(4a^2(\frac{b^2}{4a^2})+4ab(-\frac{b}{2a})+b^2)\\
&+c_2e^{rt}(8a^2(-\frac{b}{2a})+4a^2(-\frac{b}{2a})+8ab+b^2(-\frac{2a}{b}))=0\\
&=c_1e^{rt}(b^2-2b^2+b^2)+c_2e^{rt}(-4ab-2ab+8ab-2ab)=0\\
&0=0.
\end{aligned}$$

Then,

$$y(t)=f(t),$$
$$y'(t)=f'(t),$$

and

$$y''(t)=f''(t).$$

Letting $t=0$ in all the results we obtain

$$\begin{aligned}
y(0)&=c_1=4a^2,\\
y'(0)&=c_1r+2c_2=4ab,\\
y''(0)&=c_1r^2+3c_2r=2b^2.
\end{aligned}$$

We find four linear independent general solutions.

(1) $y(0)$ yields

$$y(t)=4a^2e^{rt}.$$

(2) $y(0)$ and $y'(0)$ yield $c_1=4a^2$ and $c_2=6ab$. Thus

$$y(t)=(4a^2+6ab)e^{rt}.$$

(3) $y(0)$ and $y''(0)$ yield $c_1=4a^2$ and $c_2=-4ab$. Then

$$y(t)=(4a^2-4ab)e^{rt}.$$

(4) $y'(0)$ and $y''(0)$ yield $c_1=-24a^2$ and $c_2=-8ab$. So

$$y(t)=(-24a^2-8ab^2)e^{rt}.$$

Setting $c_1=c_2=1$, the *reference* general solution is

$$y(t)=(1+t)e^{rt}.$$

Alternatively, since $rt=1$, then $y'(t)=c_1re^{rt}+2c_2e^{rt}$, and $y''(t)=c_1r^2e^{rt}+3c_2re^{rt}$, at least four more linear-independent general solutions can be computed.

Case 3 Equation (3) has a counterpart

$$2a^2y''+2aby'+(b^2-2ac)y=0$$

with *alternative* frequencies $r=\frac{-b\pm i\sqrt{b^2-4ac}}{2a}$, *alternative* periods $t=\frac{-ab\pm ai\sqrt{b^2-4ac}}{b^2-2ac}$, and $b^2>4ac$. Here, the period-based $f(t)$ allows us to find three linear independent general solutions without using sines and cosines.

$$\begin{aligned}
f(t)&=(b^2-2ac)t^2+2abt+2a^2,\\
f'(t)&=(2b^2-4ac)t+2ab,\\
f''(t)&=2b^2-4ac.
\end{aligned}$$

Then the general solution is

$$y(t)=c_1e^{r_1t}+c_2e^{r_2t}. \tag{15}$$

Differentiating (15) twice, we get

$$y'(t) = c_1 r_1 e^{r_1 t} + c_2 r_2 e^{r_2 t},$$

and

$$y''(t) = c_1 r_1^2 e^{r_1 t} + c_2 r_2^2 e^{r_2 t}.$$

Then plugging in the results back into the original equation, we obtain

$$\begin{aligned}
&2a^2(c_1 r_1^2 e^{r_1 t} + c_2 r_2^2 e^{r_2 t}) + 2ab(c_1 r_1 e^{r_1 t} + c_2 r_2 e^{r_2 t}) + (b^2 - 2ac)(c_1 e^{r_1 t} + c_2 e^{r_2 t}) = 0\\
=&c_1 e^{r_1 t}(2a^2 r_1^2 + 2abr_1 + (b^2 - 2ac)) + c_2 e^{r_2 t}(2a^2 r_2^2 + 2abr_2 + (b^2 - 2ac)) = 0\\
=&c_1 e^{r_1 t}[2a^2(\frac{-b + i\sqrt{b^2 - 4ac}}{2a})^2 + 2ab(\frac{-b + i\sqrt{b^2 - 4ac}}{2a}) + (b^2 - 2ac)]\\
&+ c_2 e^{r_2 t}[2a^2(\frac{-b - i\sqrt{b^2 - 4ac}}{2a})^2 + 2ab(\frac{-b - i\sqrt{b^2 - 4ac}}{2a}) + (b^2 - 2ac)] = 0\\
=&c_1 e^{r_1 t}[\frac{b^2 - 2bi\sqrt{b^2 - 4ac} - b^2 + 4ac - 2b^2 + 2bi\sqrt{b^2 - 4ac} + 2b^2 - 4ac}{2}]\\
&+ c_2 e^{r_2 t}[\frac{b^2 + 2bi\sqrt{b^2 - 4ac} - b^2 + 4ac - 2b^2 - 2bi\sqrt{b^2 - 4ac} + 2b^2 - 4ac}{2}] = 0\\
&0 = 0.
\end{aligned}$$

We pair up

$$\begin{aligned}
y(t) &= f(t),\\
y'(t) &= f'(t),
\end{aligned}$$

and

$$y''(t) = f''(t).$$

Letting $t = 0$ in the equations above, we obtain

$$\begin{aligned}
y(0) &= c_1 + c_2 = 2a^2,\\
y'(0) &= c_1 r_1 + c_2 r_2 = 2ab,
\end{aligned}$$

and

$$y''(0) = c_1 r_1^2 + c_2 r_1^2 = 2b^2 - 4ac.$$

So,

(1) $y(0)$ and $y'(0)$ yield $c_1 = \frac{2a^2 r_2 - 2ab}{-r_1 + r_2}$ and $c_2 = \frac{2ab - 2a^2 r_1}{-r_1 + r_2}$. Then, our first general solution is

$$y(t) = \frac{2a^2 r_2 - 2ab}{-r_1 + r_2} e^{r_1 t} + \frac{2ab - 2a^2 r_1}{-r_1 + r_2} e^{r_2 t}.$$

(2) $y(0)$ and $y''(0)$ yield $c_1 = \frac{2a^2 r_2^2 - 2b^2 + 4ac}{-r_1^2 + r_2^2}$ and $c_2 = \frac{2b^2 - 4ac + 2a^2 r_1^2}{-r_1^2 + r_2^2}$. So

$$y(t) = \frac{2a^2 r_2^2 - 2b^2 + 4ac}{-r_1^2 + r_2^2} e^{r_1 t} + \frac{2b^2 - 4ac + 2a^2 r_1^2}{-r_1^2 + r_2^2} e^{r_2 t}.$$

(3) $y'(0)$ and $y''(0)$ yield $c_1 = \frac{-2b^2 + 4ac + 2abr_2}{-r_1^2 + r_1 r_2}$ and $c_2 = \frac{2b^2 - 4ac - 2abr_1}{-r_1 r_2 + r_2^2}$. Then

$$y(t) = \frac{-2b^2 + 4ac + 2abr_2}{-r_1^2 + r_1 r_2} e^{r_1 t} + \frac{2b^2 - 4ac - 2abr_1}{-r_1 r_2 + r_2^2} e^{r_2 t}.$$

Now let us solve the traditional complex general solution.

$$\begin{aligned}
y(t) &= c_1 e^{(\alpha + i\beta)t} + c_2 e^{(\alpha - i\beta)t}\\
&= c_1 e^{\alpha t}(cos\beta t + isin\beta t) + c_2 e^{\alpha t}(cos\beta t - isin\beta t).
\end{aligned}$$

Replace the i with its real counterpart ∓ 1.

$$y(t) = c_1 e^{\alpha t}(cos\beta t \mp sin\beta t) + c_2 e^{\alpha t}(cos\beta t \pm sin\beta t)$$

The double-sign notation allows us to find as many combinations as possible; they should all be linear independent general solutions.

Each term within the brackets are two independent equations. Adding up each term we obtain

$$y(t) = 2c_1 e^{\alpha t} cos\beta t + 2c_2 e^{\alpha t} cos\beta t.$$

Since the coefficients are arbitrary, we let $c_1 = c_2 = \frac{1}{2}$, then the general solution is $y(t) = e^{\alpha t} cos\beta t$, one of the *existing* linear independent solutions.

We now combine expressions from c_1 and c_2 in as many ways as possible.

$$\begin{aligned}
y(t) =& c_1 e^{\alpha t}(cos\beta t - sin\beta t) + c_2 e^{\alpha t}(cos\beta t + sin\beta t)\\
y(t) =& c_1 e^{\alpha t}(cos\beta t - sin\beta t) + c_2 e^{\alpha t}(cos\beta t - sin\beta t)\\
y(t) =& c_1 e^{\alpha t}(cos\beta t + sin\beta t) + c_2 e^{\alpha t}(cos\beta t + sin\beta t)\\
y(t) =& c_1 e^{\alpha t}(cos\beta t + sin\beta t) + c_2 e^{\alpha t}(cos\beta t - sin\beta t)
\end{aligned}$$

The first and last expressions are identical. The only difference is that their coefficients are reversed.

Following the examples above, the coefficients can be computed to find three linearly independent general solutions.

In all three cases particular solutions can be computed with $y(t) = f(t)$. Our notation offers more general solutions directly associated to their original equations.

We must note that the periods on the exponents are variables. The roots are not variables, as they are predetermined by the equation itself. Therefore, letting $t = 0$ to find the coefficients of an expression such as $y'(t) = c_1 r_1 e^{r_1 t} + c_2 r_2 e^{r_2 t} = f'(t)$ does not set $r = \frac{1}{0}$.

6. Roots, Periods, and Equations

In this section, we will add, multiply roots/frequencies, and periods. The results will be expressed as their equivalent *partial sections* of the equation from which they originated. Analyzing the roots, periods and the *homogeneous linear second-order* equations themselves give us a further insight into the behavior of the signal produced by the equation. The computing mechanics suggest that the results are directly related to electronics. Consider equation (3) and its frequencies

$$r_1 + r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} \tag{16}$$

$$= -\frac{b}{a}$$

The addition of the frequencies can also be expressed as a partial expression of the equation itself. Let $c = 0$ in equation (3), then we have the frequency

$$\begin{aligned}
&ay'' + by' = 0,\\
&y'' = -\frac{by'}{a},\\
&\frac{y''}{y'} = -\frac{b}{a},\\
&r_3 = -\frac{b}{a}.
\end{aligned}$$

A negative period or negative frequency only means that the signal is traveling in the negative direction from a reference point. See section 9, *Special Relativity*.

$$(r_1)(r_2) = (\frac{-b + \sqrt{b^2 - 4ac}}{2a})(\frac{-b - \sqrt{b^2 - 4ac}}{2a}) \tag{17}$$

$$= \frac{c}{a}$$

$$(\frac{1}{t_2})(\frac{1}{t_1}) = (\frac{2c}{-b - \sqrt{b^2 - 4ac}})(\frac{2c}{-b + \sqrt{b^2 - 4ac}}) \tag{18}$$

$$= \frac{c}{a}$$

The results from (17) and (18) are identical, corroborating the direct relationship between roots and periods. Let $b = 0$ in (3), then

$$y'' = -\frac{cy}{a}.$$

$$\frac{y}{y''} = t^2 = -\frac{a}{c},$$

with periods

$$\pm t = \mp\sqrt{\frac{a}{c}}.$$

Splitting up the periods we get

$$t_4 = \mp\sqrt{\frac{a}{c}},$$

and

$$t_5 = \pm\sqrt{\frac{a}{c}}.$$

What accounts for the shifting of the signs between the periods t_4 and t_5? They reveal a *period-pendulum*.

$$t_1 + t_2 = \frac{2c}{-b + \sqrt{b^2 - 4ac}} + \frac{2c}{-b - \sqrt{b^2 - 4ac}} \tag{19}$$

$$= -\frac{b}{c}$$

We can obtain the frequency in (17) by dividing $(\frac{16}{19})$. Or by letting $c = 0$ in equation (3), then, $y'' = -\frac{by'}{a}$, and letting $a = 0$ in equation (3), so that, $y' = -\frac{cy}{b}$. Finally, replacing y' into y'';

$$y'' = (-\frac{b}{a})(-\frac{cy}{b}) = \frac{cy}{a}.$$

The inverses of additions and multiplications of frequencies and periods reveal a direct association to electronics.

Example 1 Compute the inverse of (16). Solution,

$$\frac{1}{r_1 + r_2} = \frac{1}{\frac{1}{t_2} + \frac{1}{t_1}}$$

$$= \frac{(t_1)(t_2)}{t_2 + t_1} = -\frac{a}{b} \tag{20}$$

Example 2

Compute the inverse of (19). Solution,

$$\frac{1}{t_1 + t_2} = \frac{1}{\frac{1}{r_2} + \frac{1}{r_1}}$$

$$= \frac{(r_1)(r_2)}{r_2 + r_1} = -\frac{c}{b} \tag{21}$$

In electronics, structures (20) and (21), a period and a frequency, respectively, both represent the addition of two parallel resistors, and the addition of two capacitors in series.

7. The Reflection Method

In standard notation the equal sign means that the values on both sides of the equation are identical to each other. Sometimes mathematical results are expressions such as $\delta = 0$ and $\delta = -\delta$ where δ is not zero. They are interpreted as a number equal to zero, and a number equal to its negative counterpart, respectively. These results are discarded as nonsense because they violate the long held belief that two horizontal bars means equality and equality only, and no further attempts to interpret new meanings are made. We introduce an alternate interpretation of such results; *The Reflection Method*. This concept stems out of need to interpret these "four-letter" words in mathematics. However, this concept is not new. It is a fundamental approach which has always been used in mathematics. But no one has ever specifically addressed it.

As students, we are introduced to mathematical concepts that for a lack of better understanding extraneous notation, such as the *absolute-value*, is introduced. For example, a typical mathematical problem where the absolute value is introduced goes as follows: A person walks distance +D in a straight line. He then doubles back to his original starting point. What is the total distance that he walked? The person walked the same distance in opposite directions. In standard notation, adding up the two distances would cancel each other out. Meaning that the person did not walk at all, which would be incorrect. Therefore, the incorrect interpretation of the problem introduces the unnecessary notation of the *absolute-value* to find a solution;

$$D + |-D| = +2D.$$

There is no need for the introduction of extraneous notation. A closer analysis of the problem produces the correct results in both the positive and negative directions. When the person walks in a particular direction we define an origin relative to the motion. Once he reaches distance +D, then doubles back changing the direction of the motion at 180 degrees. This action means that we just defined a second origin independent of the first one, -D. We can redraw the two distances joined with a common origin, Figure 2 below. Adding up graphs a) and b) at their origin produces graph c). Now we have a graph to help us explain *The Reflection Method*, as distance -D travels to the positive side, without introducing extraneous notation. Here, we define an alternative mathematical meaning associated with the graph in Figure 2. As a value travels to the opposite side of the equal sign/origin, the sign or polarity of the value changes. The same happens when a length, graph a), is transfered or *reflected* onto the opposite side in graph b). With *The Reflection Method* we will express each graph in Figure 2 as a mathematical "equation", with the equal sign as the origin/zero on the graph.

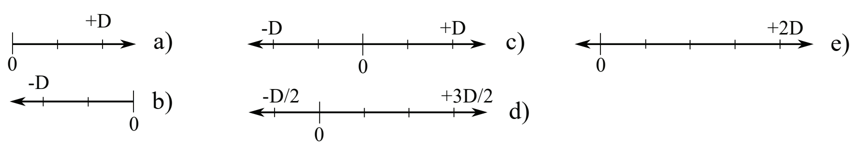

Figure 2. Reflection Scale

$$0 = +D \qquad a)$$

$$-D = 0 \qquad b)$$

$$-D = +D \qquad c)$$

$$-\frac{D}{2} = +\frac{3D}{2} \qquad d)$$

$$0 = +2D \qquad e)$$

The expressions in c) and d) are not implying that the length of +2D is occupying two different *dimensions* or two *parallel* universes simultaneously. This only means that as the distance -D moves to the right of its *reference* point, from c) to d), that portion acquires a positive sign maintaining its original overall length. Graph c) can be written more compactly in two different ways: motion from left to right is expressed as $\mp D = +2D$; motion from right to left is expressed as $\pm D = -2D$. In both cases the total distance is 2D.

Using these similarities we can redefine the equal sign in Figure 2, as a *reflection* sign.

Definition 7.1. *The Reflection Method has one value on each side never identical to each other, e.g.* $-D = +D, +D = 0$*, etc.. The sign a value, or a length, acquires reflects its position relative to a frame of reference.*

We have solved the problem without resorting to the use of the *absolute-value* notation. The man walked a total distance of 2D, regardless of the direction from his frame of reference.

What appears to contradict logic is the fact that the transitioning of a value over to the opposite side of a reference point is a *natural* process. Mathematics is not an exception. That process is not instantaneous no matter how small that value is, or how fast the processing system is. The transfer is always gradual, not instantaneous.

In mathematics, some computations reveal *uneven* equations proving the existence of *The Reflection Method*. This is the topic of our next section.

7.1 The Reflection Method in Mathematics

Using standard notation, we will compute some mathematical concepts and formulas to show the existence of *The Reflection Method* as the backbone in mathematics. Ironically, it has always been derided as a *four-letter* word.

An area where *The Reflection Method* is most prevalent is in sinusoidal waves, typically expressed as

$$x = Asin(t\omega - \theta).$$

The cycle starts at

$$\omega t - \theta = 0, \tag{22}$$

and ends at

$$\omega t = 2\pi. \tag{23}$$

Solving for t in (22) and replacing t in (23), we get

$$\omega(\frac{\theta}{\omega}) - \theta = 2\pi,$$

$$0 = 2\pi.$$

This **four-letter** word expression is actually pointing at the origin of the unit-circle $(2\pi, 0)$.

There are many instances in mathematics where the result of a computation is a non-zero number "equal" to zero. *The Reflection Method* is nowhere more evident than on the unit-circle itself.

The circle in Figure 3, a) displays every point reflected on the graph exactly at 180 degrees. Using its origin as the

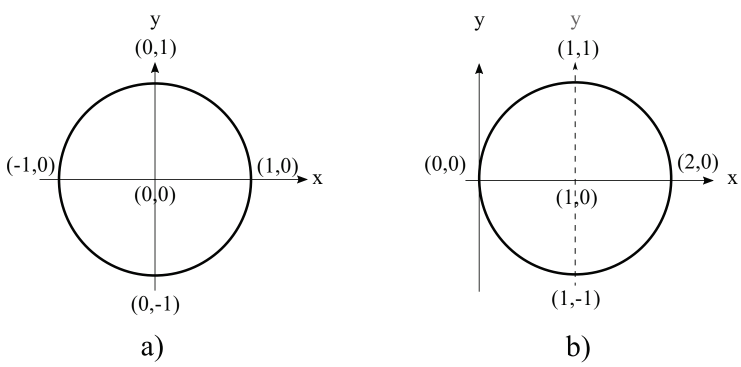

Figure 3. Double-Circle

"equal" sign, the *xy*-axes are expressed in their reflected form. Moving the entire circle one unit over to the right in Figure 3, b), the result along the *x*-axis is exactly as expected;

$$x) \quad -1 = 1,$$

$$0 = 2.$$

Transferring the unit-circle partially or entirely into any quadrant, the results would be similar, without compromising the integrity of the circle. All these results corroborating *The Reflection Method* appear to be sheer coincidence. They are not. We will next provide two examples on their existence in current mathematics.

Theorem 7.1. *Homogeneous second-order linear equations contain The Reflection Method; the radius of the unit-circle.*

Proof. We solve for the frequency y'' and the period y, individually, from equation (3) and replace them back into the equation itself. Let $c = 0$, so that $y'' = -\frac{by'}{a}$, then, let $a = 0$, so that $y = -\frac{by'}{c}$. Plugging the results back into the equation, we obtain

$$a(-\frac{by'}{a})+by' + c(-\frac{by'}{c}) = 0,$$
$$-by' = 0,$$
$$1 = 0.$$

□

Theorem 7.2. *The complex expression $e^{\frac{i\pi}{2}} = i$ contains the four major points on the unit-circle.*

Proof. Replace the i with ∓ 1 in the expression $e^{\frac{i\pi}{2}} = i$ and solve for each individual plus or minus signs.

$$e^{\mp\frac{\pi}{2}} = \mp 1,$$

$$e^{\mp\frac{\pi}{2}} = -1, \quad e^{\mp\frac{\pi}{2}} = +1.$$

Compute+1.

$$\mp\frac{\pi}{2} = ln(+1),$$

$$\mp\frac{\pi}{2} = 0,$$

$$-\frac{\pi}{2} = 0, \quad +\frac{\pi}{2} = 0.$$

The $-\frac{\pi}{2}$ is a reference to $+\frac{3\pi}{2}$ on the x-axis of the unit-circle in a clockwise direction. And the $\frac{\pi}{2}$ is on the *unit-circle* in a counter-clockwise direction, so $(+\frac{\pi}{2}, 0)$ and $(+\frac{3\pi}{2}, 0)$.

Compute -1.

$$\mp\frac{\pi}{2} = ln(-1),$$
$$\mp\pi = ln((-1)^2),$$
$$\mp\pi = 0.$$

This result points at the same spot on the unit-circle in a clockwise direction, $(-\pi, 0)$, and in a counter-clockwise direction $(+\pi, 0)$. Alternatively, dividing $\mp\pi = 0$ by π yields $\mp 1 = 0$, providing the end-points of the unit-circle along the x-axis at $(-1, 0)$, and $(+1, 0)$. Identical results may be obtained for the x-axis with $e^{\mp\frac{\pi}{2}} = +1$. □

What has always been thought of as a **four-letter** word is, in fact, a **complementary** solution method in mathematics.

8. The Inverse of *i*

The imaginary number i has a particular behavior that no other mathematical expression has. It is its own *negative inverse*. The usefulness of the mechanics is not immediately obvious; however, its peculiar behavior can help us find solutions to expressions in mathematical environments which do not conform to standard notation.

Theorem 8.1. *Multiplying a one times any quantity results in the quantity itself. The number 1 can also be expressed as* $(-1^{-1})(-1^{-1}) = +1^{+1}$. *Note that* $(-1^{-1})(-1^{-1}) \neq -1^{-2}$.

Proof. For $-\infty < a < +\infty$.

$$
\begin{aligned}
&a = a\\
&(-1^{-1})(-1^{-1})a = a\\
&(-1^{-1})(-a^{-1}) = a\\
&(-1^{-1})(\frac{1}{a}) = -a\\
&-a = -a\\
&a = a.
\end{aligned}
$$

This is true even when $a = 0$. □

Theorem 8.2. *Multiplying -1^{-1} times polynomial expressions, for example $ax^m + bx^n - x^q$, independently inverts each individual term on its own horizontal axis.*

Proof.

$$
\begin{aligned}
&(ax^m + bx^n - x^q)(-1^{-1})\\
&= (-ax^m - bx^n + x^q)^{-1}\\
&= (-ax^m)^{-1} + (-bx^n)^{-1} + (x^q)^{-1}\\
&= -\frac{1}{ax^m} - \frac{1}{bx^n} + \frac{1}{x^q}.
\end{aligned}
$$

Which implies

$$
\begin{aligned}
&(-ax^m - bx^n + x^q)^{-1}\\
&= \frac{1}{-ax^m - bx^n + x^q}\\
&= \frac{1}{-ax^m} - \frac{1}{bx^n} + \frac{1}{x^q}.
\end{aligned}
$$

□

Theorem 8.3. *$log(wv) = k$ is equivalent to $log(\frac{1}{wv}) = -k$.*

Proof. Using Theorems 8.1, and 8.2, we find the inverse equivalent of $log(wv) = k$. Let $wv > 0$.

$$
\begin{aligned}
&log(wv) = k\\
&log(w) + log(v) = k\\
&log[(-1^{-1})(-1^{-1})(w)] + log(v) = k\\
&log[(-1^{-1})(-w^{-1})] + log(v) = k\\
&-log[(-1^{-1})(-w)] + log(v) = k\\
&log[(-1^{-1})(-w)] - log(v) = -k\\
&log(w^{-1}) - log(v) = -k
\end{aligned}
$$

$$
log(\frac{1}{wv}) = -k \tag{24}
$$

$$
\begin{aligned}
&-log(\frac{1}{wv}) = k\\
&log(wv) = k.
\end{aligned}
$$

The equivalent inverse is equation (24), which is simplified to its original expression in the last step. Note that if $wv = 1$ then $k = 0$. So the equivalent inverse of (24) is negative zero. Negative zeros are used in the addition of binary numbers; however, in mathematics, a negative zero is still a zero. □

Theorem 8.4. *Simplifying* $i(i) = -1$ *also proves the existence of The Reflection Method.*

Proof.

$$
\begin{aligned}
&i(i) = -1\\
&i = -\frac{1}{i}\\
&i = -i^{-1}\\
&i = (i)(-1^{-1})\\
&\frac{i}{i} = -1^{-1}\\
&1 = -1^{-1}\\
&1 = -1\\
&2 = 0\\
&1 = 0.
\end{aligned}
$$

□

Step six is Theorem 8.1 "divided" by two; $(-1^{-1})(-1^{-1})(-1^{-1}) = -1^{-1}$. We shall call this process **The Inverse of i** notation.

Corollary 8.1. *We will generalize step three from the proof of Theorem 8.4, which states that* ***any*** *number has its own negative inverse.*

Proof.

$$
\begin{aligned}
&\beta = \beta\\
&(-1^{-1})\beta = \beta\\
&= -\frac{1}{\beta} = \beta\\
&= \frac{1}{\beta} = -\beta
\end{aligned}
$$

For $-\infty < \beta < +\infty$. When $\beta = 0$, then

$$
\begin{aligned}
&\frac{1}{0} = -0\\
&= \frac{1}{0} = -0 = 0.
\end{aligned}
$$

□

Corollary 8.2. $\beta^x = 0$. *For* $-\infty < \beta < +\infty$. *And for any value of x within the range* $-\infty < x < +\infty$ *solves the "equation", and can be expressed as its basic form* $1 = 0$*; When* $0^0 = 0$, *we apply theorem 8.4, then* $1^1 = 0$.

Corollary 8.3. *From step nine of proof from Theorem 8.4,* $(0)(0) = 1$.

Corollary 8.4. *From step nine of proof from Theorem 8.4,* $\frac{0}{0} = \frac{1}{1} = 1$.

Corollary 8.5. *From step nine of proof from Theorem 8.4,* $1^1 = 1$, *hence* $0^0 = 1$.

Definition 8.1. *Mathematical expressions having no solutions, undefined, or violating standard notation are presumed to be multiplied times The Inverse of i; and in order to find their equivalents in standard notation, they must be multiplied times The Inverse of i once more, then solve.*

Theorem 8.5. $\lim_{x\to 0} \frac{1}{x} = 0$.

Proof. Applying Corollary 8.1, and Definition 8.1.

$$\lim_{x\to+0} \frac{1}{x}$$
$$\lim_{x\to+0} (-1^{-1})\frac{1}{x}$$
$$\lim_{x\to+0} -x = -0 = 0,$$

and

$$\lim_{x\to-0} \frac{1}{x}$$
$$\lim_{x\to-0} (-1^{-1})\frac{1}{x}$$
$$\lim_{x\to-0} -x = 0.$$

Thus, $\lim_{x\to 0} \frac{1}{x} = 0$. □

Theorem 8.6. *1.* $\lim_{x\to-\infty} \frac{1}{x} = +\infty$, *and 2.* $\lim_{x\to+\infty} \frac{1}{x} = -\infty$.

Proof. We apply Definition 8.1, and Corollary 8.1.

$$1.\ \lim_{x\to-\infty} \frac{1}{x}$$
$$\lim_{x\to-\infty} (-1^{-1})\frac{1}{x}$$
$$\lim_{x\to-\infty} -x = +\infty,$$

and

$$2.\ \lim_{x\to+\infty} \frac{1}{x}$$
$$\lim_{x\to+\infty} (-1^{-1})\frac{1}{x}$$
$$\lim_{x\to+\infty} -x = -\infty.$$

□

Theorem 8.7. $\sqrt{-a} = \frac{1}{\sqrt{a}}$.

Proof. Let $a > 0$. We apply Definition 8.1, and Corollary 8.1.

$$\sqrt{-a} = \sqrt{-a}$$
$$= (-1^{-1})(-a)^{\frac{1}{2}} = \sqrt{-a}$$
$$= (a)^{-\frac{1}{2}} = \sqrt{-a}$$
$$= \frac{1}{\sqrt{a}} = \sqrt{-a}$$

Or alternate notation;

$$\begin{aligned}
&\sqrt{-a}\\
&= (-a)^{\frac{1}{2}}\\
&= (-a^{\frac{1}{2}})\\
&= -\sqrt{a}\\
&= (-1^{-1})(-a^{\frac{1}{2}})\\
&= (a^{\frac{1}{2}})^{-1}\\
&= \frac{1}{\sqrt{a}}.
\end{aligned}$$

□

Step four corroborates our previous results; factoring a negative sign out of the radical square.

Postulate 8.1. *Integrals require their integrands and their dx's to be in a non-fraction form relative to each other, i.e.* xdx, *or* $\frac{1}{xdx}$.

Theorem 8.8. *Standard notation.*

$$\int x^r \mathrm{d}x = \frac{x^{r+1}}{r+1} + c.$$

Solve the integral applying Theorems 8.1, 8.2, Definition 8.1, and Postulate 8.1 for $-\infty < r < +\infty$.

Proof.

$$\begin{aligned}
&\int x^r \mathrm{d}x\\
&= \int (-1^{-1})(-1^{-1})(x^r \mathrm{d}x)\\
&= (-1^{-1}) \int (-x^r \mathrm{d}x)^{-1}\\
&= (-1^{-1}) \int -\frac{1}{x^r \mathrm{d}x}\\
&= (-1^{-1})[-\frac{1}{\frac{x^{r+1}}{r+1} + c}]\\
&= (-1^{-1})[-\frac{r+1}{x^{r+1} + c(r+1)}]\\
&= [\frac{r+1}{x^{r+1} + c(r+1)}]^{-1}\\
&= (\frac{r+1}{x^{r+1}})^{-1} + (\frac{r+1}{c(r+1)})^{-1}\\
&= \frac{x^{r+1}}{r+1} + c.
\end{aligned}$$

If $r = -1$, then the solution is $\frac{1}{0} + c = 0 + c$. An alternate notation is

$$\begin{aligned} &\int x^r \mathrm{d}x \\ &= \int (x^r \mathrm{d}x)^{(-1)(-1)} \\ &= \int (\frac{1}{x^r \mathrm{d}x})^{-1} \\ &= (1^{-1}) \int (\frac{1}{x^r \mathrm{d}x}) \\ &= (\frac{1}{\frac{x^{r+1}}{r+1} + c})^{-1} \\ &= \frac{x^{r+1}}{r+1} + c. \end{aligned}$$

□

Theorem 8.9.

$$\int \frac{1}{x^r}\mathrm{d}x = \frac{r+1}{x^{r+1}} + c.$$

For $-\infty < r < +\infty$.

Proof. We apply Theorems 8.1, and 8.2, Definition 8.1, Corollary 8.1, and Postulate 8.1. In this case, the integrand, $\frac{1}{x^r}$, is not in standard form relative to dx. So, we apply Theorem 8.1 to the integrand only.

$$\begin{aligned} &\int \frac{1}{x^r}\mathrm{d}x \\ &= (-1^{-1}) \int (-1^{-1})(\frac{1}{x^r})\mathrm{d}x \\ &= (-1^{-1}) \int -x^r \mathrm{d}x \\ &= (-1^{-1})(-\frac{x^{r+1}}{r+1} + c) \\ &= (\frac{x^{r+1}}{r+1} - c)^{-1} \\ &= \frac{r+1}{x^{r+1}} - \frac{1}{c}. \end{aligned}$$

Applying Corollary 8.1, we obtain

$$\frac{r+1}{x^{r+1}} + c.$$

□

We create a tabular from $r = -3$ to $r = 3$ to show the patterns of integrals from Theorems 8.8 and 8.9. Let $c = 0$.

	$\int x^r \mathrm{d}x$	$\int \frac{1}{x^r}\mathrm{d}x$
r	$\frac{x^{r+1}}{r+1}$	$\frac{r+1}{x^{r+1}}$
-3	$-\frac{1}{2x^2}$	$-2x^2$
-2	$-\frac{1}{x}$	$-x$
-1	$\frac{1}{0} = -0$	$\frac{0}{1} = 0$
0	x	$\frac{1}{x}$
1	$\frac{x^2}{2}$	$\frac{2}{x^2}$
2	$\frac{x^3}{3}$	$\frac{3}{x^3}$
3	$\frac{x^4}{4}$	$\frac{4}{x^4}$

Multiplying $(\int x^r \mathrm{d}x)(\int \frac{1}{x^r}\mathrm{d}x)$, at any point, for $-\infty < r < +\infty$, should be equal to 1. When $r = -1$, by Corollary 8.4, we get

$$\begin{aligned}
&(\frac{1}{0})(\frac{0}{1}) = 1\\
&= \frac{0}{0} = 1\\
&= 1 = 1.
\end{aligned}$$

Theorem 8.10. $\int \mathrm{d}x = 1$. *We apply Theorems 8.1, 8.2, 8.9, Postulate 8.1, and Definition 8.1.*

Proof. For $-\infty < r < +\infty$.

$$\begin{aligned}
&\int \mathrm{d}x = 1\\
&= \int (\frac{x^r}{x^r})\mathrm{d}x = 1\\
&= (\int x^r \mathrm{d}x)(\int x^{-1}\mathrm{d}x) = 1\\
&= (\frac{x^{r+1} + c(r+1)}{r+1})(\frac{r+1}{x^{r+1} + c(r+1)}) = 1\\
&= 1 = 1.
\end{aligned}$$

Alternatively,

$$\int_0^{2\pi} \mathrm{d}x = 1$$

$$= \int_0^{2\pi} (\frac{cosx}{cosx})\mathrm{d}x = 1 \tag{25}$$

$$= (\int_0^{2\pi} cosx\mathrm{d}x)(\int_0^{2\pi} (cosx)^{-1}\mathrm{d}x) = 1$$

$$= \frac{sinx}{sinx}|_0^{2\pi} = 1. \tag{26}$$

□

At every point equation (25) is equal to one, even at points $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$, where integral results in $\frac{0}{0} = 1$, representing the radius of the unit-circle on the y-axis; $(0, 1)$. At every point equation (26) is equal to one, including at points $x = 0, 2\pi$, and $x = \pi$, where integral results in $\frac{0}{0} = 1$, representing the radius of the unit-circle on the x-axis $(1, 0)$. However, these results are correct: they represent The reflection method.

The inverse of i notation already exists in mathematics. If the product of slopes $m_1m_2 = -1$, then the slopes are perpendicular to each other. Such expression implies that one of the i's is the *negative inverse* of the other. For example, let

$$m_1 = m,$$

and

$$m_2 = (-1^{-1})(m) = -\frac{1}{m},$$

then

$$m_1m_2 = -1 = i^2.$$

This identity asserts that the i's are not equal to each other: they represent slopes perpendicular to each other, or 2-D vectors perpendicular to each other.

8.1 General Logarithms

Our findings allow us to solve some of the long-standing challenges in mathematics: logarithms having negative arguments, negative bases, or both. Instead of utilizing the theorems and corollaries developed above, we want to introduce an alternate way to solving these challenges.

Theorem 8.11. *1. $log_b(a) = k$ is equivalent to 2. $log_{-b}(-a) = k$. For $a > 0$ and $b > 0$.*

Proof. Solve for a in logarithm 2. Let $p = -b$.

$$
\begin{aligned}
&-a = p^k \\
&-a = -b^k \\
&\pm \sqrt[k]{-a} = \mp b \\
&\mp \sqrt[k]{a} = \mp b
\end{aligned}
$$

Multiply a negative sign across.

$$
\begin{aligned}
\pm \sqrt[k]{a} &= \pm b \\
a &= b^k \\
log_b(a) &= k.
\end{aligned}
$$

□

There is an intrinsic relationship between the signs of base and the signs of the argument of the logarithms, from which we develop the following corollary.

Corollary 8.6. *If the base of a logarithm is multiplied times a negative one, so shall its argument, and vice versa.*

Theorem 8.12. *$log_{-b}(a) = k_1$ is equivalent to $log_b(-a) = k_1$. For $a > 0$ and $b > 0$.*

Proof.

$$log_{-b}(a) = k_1$$

Applying Corollary 8.6, we obtain

$$
\begin{aligned}
&log_b(-a) = k_1 \\
&log_b(\frac{1}{a}) = k_1.
\end{aligned}
$$

□

9. Special Relativity

Regarding the Theory of Special Relativity, the speed of light is no longer a limiting factor. The speed v can exceed the speed of light many times over without violating the laws of physics. Our notation also solves the misinterpretation that once speed v is greater than the speed of light, the relative time Δt_2 becomes imaginary.

$$\Delta t_2 = \frac{\Delta t_1}{\pm\sqrt{1 - \frac{v^2}{c^2}}} \tag{27}$$

In equation (27), we have particle p_c with speed c, and particle p_v with speed v. For speed $v < c$, Δt_2 is associated with the plus sign from the radical. Particle p_c is traveling faster, increasing its distance relative to particle p_v. When $v = c$, time Δt_2 is undefined–both particles are traveling in the same direction and their relative distance to each other is not changing. When $v > c$, a negative sign can be factored out of the radical square in (27), thus

$$\Delta t_2 = \frac{\Delta t_1}{\mp\sqrt{\frac{v^2}{c^2} - 1}}. \tag{28}$$

Particle p_c maintains its original speed. But now relativity is *inverted*; Δt_2 is associated with the negative sign. We can get rid of the negative sign applying *The Inverse of i* to the periods.

$$-\Delta t_2 = \frac{\Delta t_1}{\pm\sqrt{\frac{v^2}{c^2}-1}}$$

$$\pm\sqrt{\frac{v^2}{c^2}-1} = (-1^{-1})(-\frac{\Delta t_1}{\Delta t_2})$$

$$\pm\sqrt{\frac{v^2}{c^2}-1} = \frac{\Delta t_2}{\Delta t_1}$$

$$\Delta t_1 = \frac{\Delta t_2}{\pm\sqrt{\frac{v^2}{c^2}-1}}.$$

Multiplying *The Inverse of i* times the discriminant of (27), we obtain identical results.

$$\Delta t_2 = \frac{\Delta t_1}{\pm\sqrt{(-1^{-1})(1-\frac{v^2}{c^2})}}$$

$$\Delta t_2 = \frac{\Delta t_1}{\pm\sqrt{(\frac{v^2}{c^2}-1)^{-1}}}$$

$$\Delta t_2 = \frac{\Delta t_1}{\pm\sqrt{\frac{1}{\frac{v^2}{c^2}-1}}}$$

$$\Delta t_2 = \frac{\Delta t_1}{\frac{1}{\pm\sqrt{\frac{v^2}{c^2}-1}}}$$

$$\Delta t_1 = \frac{\Delta t_2}{\pm\sqrt{\frac{v^2}{c^2}-1}}$$

The inversion of the signs means that the motion of particle p_c is *negative*–moving back relative to particle p_v. Applying *The Inverse of i* allows us to rewrite the formula in standard notation: now the relative time is Δt_1.

The absolute-value vertical bars are used to solve equations with logarithms with negative arguments, but they have never been used to solve the negative argument in a radical square. Placing an absolute-value around the argument in the radical square in equation (27) results in

$$\Delta t_2 = \frac{\Delta t_1}{\pm\sqrt{|1-\frac{v^2}{c^2}|}}. \tag{29}$$

Once speed v is greater than c, Δt_2 would still be positive, but relativity must be reinterpreted by associating opposite sign to the period Δt_2.

An alternative notation, to demystify the concept of the "imaginary" time, is by replacing the i by its equivalent plus or minus signs in equation (27).

$$\Delta t_2 = \frac{\Delta t_1}{-(\mp 1)\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Delta t_2 = \frac{\Delta t_1}{-i\sqrt{1-\frac{v^2}{c^2}}} \tag{30}$$

Therefore, when $v > c$, we get the square root $i\beta$, with $\beta > 0$. Then, $(-i)(i\beta) = +\beta$. So (30) becomes

$$\Delta t_2 = \frac{\Delta t_1}{+\beta}. \tag{31}$$

The period Δt_2 is not imaginary, it is **real**. Hence, equations (27), (28), (29), (30), and (31) are equivalents to each other. Proper interpretation of the equations should yield identical results.

The relative speeds of equation (3) can be found using the theory of special relativity. We replace the constant c for c_1 in equation (3) to avoid confusion with the speed of light c.

The inverse of (18) is

$$t_1 t_2 = \frac{a}{c_1}.$$

So (27) becomes

$$\pm\sqrt{1-\frac{v^2}{c^2}} = \frac{t_1}{t_2}.$$

Replace $t_1 = \frac{a}{t_2 c_1}$.

$$\pm\sqrt{1-\frac{v^2}{c^2}} = \frac{a}{t_2^2 c_1}$$

$$\pm v = \pm c\sqrt{1-\frac{a^2}{t_2^4 c_1^2}} \tag{32}$$

We simplify equation (28) and compare the results to those of equation (32).

$$-t_2 = \frac{t_1}{\pm\sqrt{\frac{v^2}{c^2}-1}}$$

$$\pm\sqrt{\frac{v^2}{c^2}-1} = -\frac{t_1}{t_2}$$

Applying Corollary 8.1,

$$\pm\sqrt{\frac{v^2}{c^2}-1} = \frac{t_2}{t_1}$$

and replace t_1,

$$\pm\sqrt{\frac{v^2}{c^2}-1} = \frac{t_2^2 c_1}{a},$$

$$\pm v = \pm c\sqrt{1+\frac{t_2^4 c_1^2}{a^2}}. \tag{33}$$

Applying Corollary 8.1 to the negative frequency under the square root in equation (32) becomes equation (33).

We have simplified the theory of special relativity using the periods of the homogeneous linear second-order equation. Perhaps the confusion about special relativity stems out the fact that the square root of a negative one was *defined* to be i for *imaginary* number. But it has never been proven mathematically that it is so. Defining the square root of negative one with any other symbol would have been less confusing.

We can assert that the speed of light is **not** a limiting factor. When a particle exceeds the speed of light, time does not become imaginary, instead special relativity is *reversed*–retaining its properties. The square root of a negative number **always** produces negative inverse **standard** results.

10. Conclusion

Based on this evidence, we can attest that the "imaginary" number i is real. *The Inverse i notation*, *The Reflection Method*, *The Remnant*, and *The Double-Root Method* are **The Complementary Solving Methods**, changing the status quo in mathematics, physics and science in general. Textbooks will have to be re-written from the ground up. Each component of *The Complementary Solving Methods* originated from a different mathematical source, however, they are all one and the same. Each one has a unique set of mechanical properties enabling us to find solutions to historical challenges in math and physics, offering new insights into realms which were previously deemed to be non-existent. These discoveries will also play a significant role in advances in our technological society. Every aspect in electronics, computing, software development, transportation, telecommunications, space exploration, medical devices, aerospace and engineering will be updated and/or re-calibrated, creating, perhaps, more efficient and cost-effective systems and products. *The Complementary Solving Methods* might be a herald to a *Second Industrial Revolution*.