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A New Flexible Version of the Lomax Distribution with Applications

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Abstract

A new version of the Lomax model is introduced and studied. The major justification for the practicality of the new model is based on the wider use of the Lomax model. We are also motivated to introduce the new model since the density of the new distribution exhibits various important shapes such as the unimodal, the right skewed and the left skewed. The new model can be viewed as a mixture of the exponentiated Lomax distribution. It can also be considered as a suitable model for fitting the symmetric, left skewed, right skewed, and unimodal data sets. The maximum likelihood estimation method is used to estimate the model parameters. We prove empirically the importance and flexibility of the new model in modeling two types of aircraft windshield lifetime data sets. The proposed lifetime model is much better than gamma Lomax, exponentiated Lomax, Lomax and beta Lomax models so the new distribution is a good alternative to these models in modeling aircraft windshield data.

Keywords: Lomax model, Order Statistics, Maximum Likelihood Estimation, Quantile function, Generating Function, Moments

1. Introduction

A random variable (rv) W has the Lomax (Lx) distribution with two parameters λ and β if it has cumulative distribution function (CDF) (for $w > 0$) given by

$$G_{\lambda,\beta}(w) = 1 - \left(1 + w\beta^{-1}\right)^{-\lambda}, \tag{1}$$

where $\lambda > 0$ and $\beta > 0$ are the shape and scale parameters, respectively. Then the corresponding PDF of (1) is

$$g_{\lambda,\beta}(w) = \lambda\beta^{-1}\left(1 + w\beta^{-1}\right)^{-(\lambda+1)}. \tag{2}$$

In the literature, the Lomax (Lx) or Pareto type **II** (Pa**II**) model (see Lomax (1954)) was originally pioneered for modeling business failure data. The Lx distribution has found a wide application in many fields such as biological sciences, ctuarial science, engineering, size of cities, income and wealth inequality, amedical and reliability modeling. It has been applied to model data obtained from income and wealth (Harris (1968) and Atkinson and Harrison (1978)), firm size (Corbellini et al., (2007)), reliability and life testing (Hassan Al-Ghamdi (2009)), Hirschrelated statistics (Glanzel (2008)), for modeling gauge lengths data (Afify et al., (2015)), for modeling bladder cancer patients data and remission times data (Yousof et al., (2016) and Yousof et al., (2018)). According to Yousof et al. (2016) the CDF of the Burr X generator (BrX-G) is

$$F_{\theta,\xi}(x) = 2\theta\int_0^{\frac{G(x;\xi)}{\overline{G}(x;\xi)}} t\left[1-\exp(-t^2)\right]^{\theta-1}\exp\left(-t^2\right)dt = \left\{-\exp\left[-\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^2\right]+1\right\}^{\theta}. \tag{3}$$

The PDF of the BrX-G is given by

$$f_{\theta,\xi}(x) = 2\theta g(x;\xi)\overline{G}(x;\xi)^{-3}G(x;\xi)\exp\left[-\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^2\right]\left\{-\exp\left[-\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^2\right]+1\right\}^{\theta-1}, \tag{4}$$

where θ is the shape parameter, $g(x;\xi)$ and $G(x;\xi)$ denote the PDF and CDF of the baseline model with parameter vector ξ and $1 - G(x;\xi) = \overline{G}(x;\xi)$. Inserting (1) in to (3) we get the the CDF of the Burr X Lomax (BrXLx) as

$$F_{\theta,\lambda,\beta}(x) = \left(1 - \exp\left\{-\left[\frac{1-\left(1+x\beta^{-1}\right)^{-\lambda}}{\left(1+x\beta^{-1}\right)^{-\lambda}}\right]^2\right\}\right)^{\theta}. \tag{5}$$

The PDF of the BrXLx is given by

$$\begin{aligned} f_{\theta,\lambda,\beta}(x) &= 2\theta\lambda\beta^{-1}\left(1+x\beta^{-1}\right)^{2\lambda-1}\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right] \\ &\quad \times\exp\left\{-\left[\frac{1-\left(1+x\beta^{-1}\right)^{-\lambda}}{\left(1+x\beta^{-1}\right)^{-\lambda}}\right]^2\right\}\left(1-\exp\left\{-\left[\frac{1-\left(1+x\beta^{-1}\right)^{-\lambda}}{\left(1+x\beta^{-1}\right)^{-\lambda}}\right]^2\right\}\right)^{\theta-1}. \end{aligned} \tag{6}$$

We are motivated to introduce the new model since the PDF of the new distribution exhibits various important shapes such as the unimodal, the right skewed and the left skewed (see figurue 1). The new model can be viewed as a mixture of the exponentiated Lx distribution (see Subsection 2.1). It can also be considered as a suitable model for fitting the symmetric, left skewed, right skewed, and unimodal data sets (see aplications Section). The maximum likelihood estimation method is used to estimate the model parameters. We prove empirically the importance and flexibility of the new model in modeling two types of aircraft windshield lifetime data sets. The proposed lifetime model is much better than gamma Lx, beta Lx, exponentiated Lx and Lx models so the new model is a good alternative to these models in modeling aircraft windshield data.

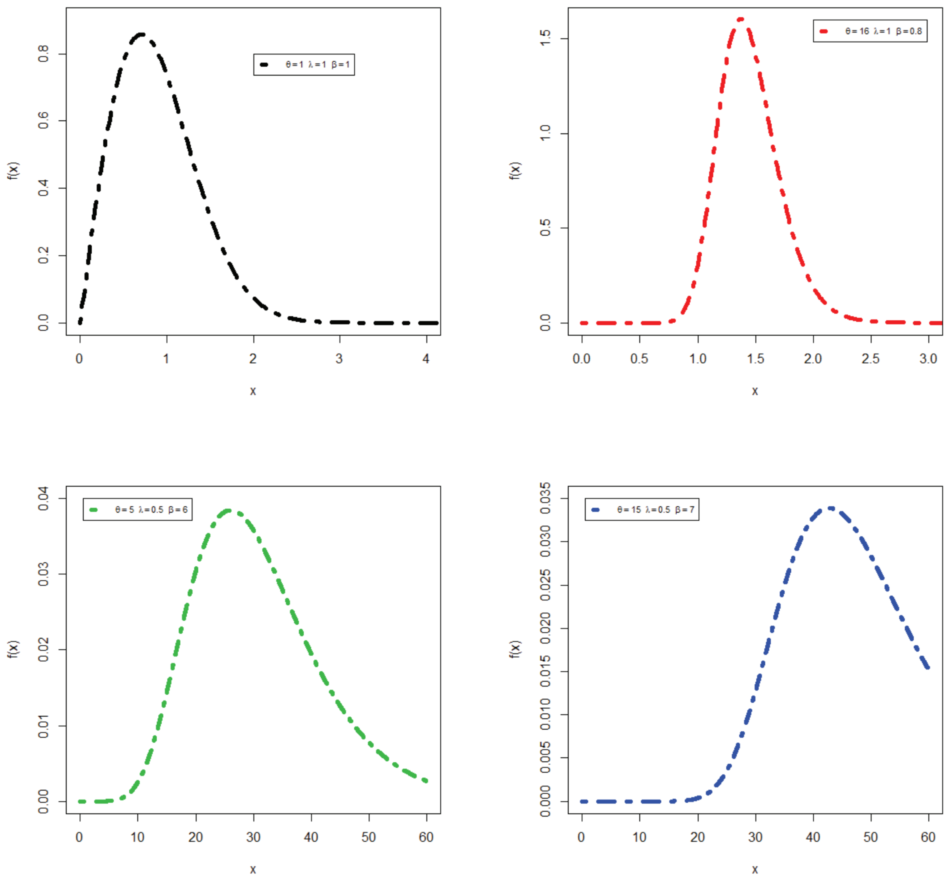

Figure 1. Plots of the BrXLx PDF

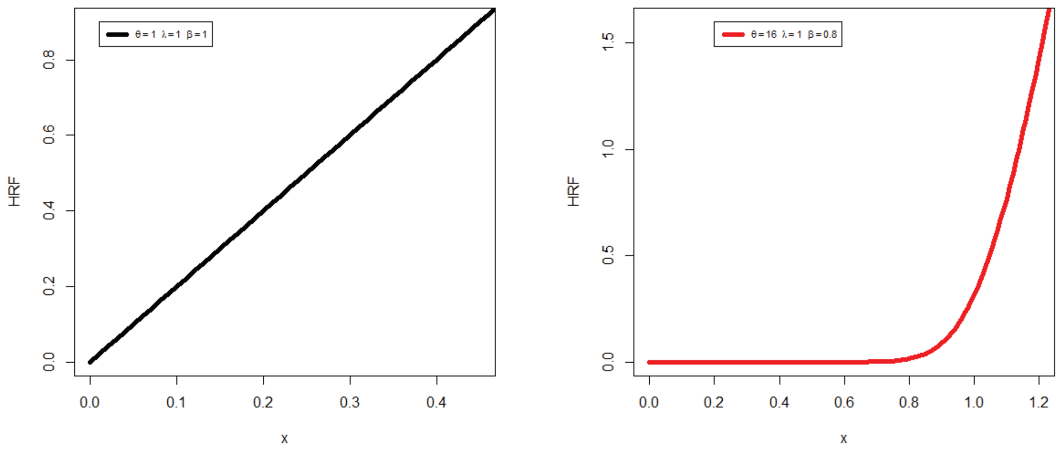

Figure 2. Plots of the BrXLx HRF

The reliability function (RF) ($R_{\theta,\lambda,\beta}(x)$), hazard rate function (HRF) ($h_{\theta,\lambda,\beta}(x)$), reversed hazard rate function (RHRF) ($r_{\theta,\lambda,\beta}(x)$) and cumulative hazard rate function (CHRF) ($H_{\theta,\lambda,\beta}(x)$) of X are given, respectively, by

$$R_{\theta,\lambda,\beta}(x) = 1 - \left(1 - \exp\left\{-\left[\frac{1-\left(1+x\beta^{-1}\right)^{-\lambda}}{(1+x\beta^{-1})^{-\lambda}}\right]^2\right\}\right)^{\theta},$$

$$\begin{aligned} h_{\theta,\lambda,\beta}(x) &= 2\theta\lambda\beta^{-1}\left(1+x\beta^{-1}\right)^{2\lambda-1}\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right] \\ &\times\left[1-\left(1-\exp\left\{-\left[\frac{1-\left(1+x\beta^{-1}\right)^{-\lambda}}{(1+x\beta^{-1})^{-\lambda}}\right]^2\right\}\right)^{\theta}\right]^{-1} \\ &\times\exp\left\{-\left[\frac{1-\left(1+x\beta^{-1}\right)^{-\lambda}}{(1+x\beta^{-1})^{-\lambda}}\right]^2\right\}\left(1-\exp\left\{-\left[\frac{1-\left(1+x\beta^{-1}\right)^{-\lambda}}{(1+x\beta^{-1})^{-\lambda}}\right]^2\right\}\right)^{\theta-1}, \end{aligned}$$

$$\begin{aligned} r_{\theta,\lambda,\beta}(x) &= 2\theta\lambda\beta^{-1}\left(1+x\beta^{-1}\right)^{2\lambda-1}\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right] \\ &\times\exp\left\{-\left[\frac{1-\left(1+x\beta^{-1}\right)^{-\lambda}}{(1+x\beta^{-1})^{-\lambda}}\right]^2\right\}\left(1-\exp\left\{-\left[\frac{1-\left(1+x\beta^{-1}\right)^{-\lambda}}{(1+x\beta^{-1})^{-\lambda}}\right]^2\right\}\right)^{-1} \end{aligned}$$

and

$$H_{\theta,\lambda,\beta}(x) = -\left\{\log\left[1-\left(1-\exp\left\{-\left[\frac{1-\left(1+x\beta^{-1}\right)^{-\lambda}}{(1+x\beta^{-1})^{-\lambda}}\right]^2\right\}\right)^{\theta}\right]\right\}.$$

2. Mathematical and Statistical Properties

2.1 Linear Representation

In this section, we provide a very useful linear representation for the BX-G density function. If $|s| < 1$ and $b > 0$ is a real non-integer, the power series holds

$$(1-s)^{a-1} = \sum_{h=0}^{\infty} \left\{(-1)^h\, \Gamma(a)\, s^h / [h!\, \Gamma(a-h)]\right\}. \tag{7}$$

Applying (7) to (6) we have

$$f(x) = 2\theta\lambda\beta^{-1} \frac{\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]}{(1+x\beta^{-1})^{1-2\lambda}} \sum_{i=0}^{\infty} \frac{(-1)^i\, \Gamma(\theta)}{i!\, \Gamma(\theta-i)} \exp\left\{-(1+i)\left[\frac{1-\left(1+x\beta^{-1}\right)^{-\lambda}}{(1+x\beta^{-1})^{-\lambda}}\right]^2\right\}. \tag{8}$$

Applying the power series to the term

$$\exp\left\{-(1+i)\left[\frac{1-\left(1+x\beta^{-1}\right)^{-\lambda}}{(1+x\beta^{-1})^{-\lambda}}\right]^2\right\}.$$

Equation (8) becomes

$$f(x) = 2\theta\lambda\beta^{-1}\left(1+x\beta^{-1}\right)^{-(\lambda+1)} \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}\, (i+1)^j\, \Gamma(\theta)}{i!\, j!\Gamma(\theta-i)} \frac{\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{2j+1}}{\left[(1+x\beta^{-1})^{-\lambda}\right]^{2j+3}}. \tag{9}$$

Consider the series expansion

$$(1-s)^{-a} = \sum_{w=0}^{\infty} \{s^w \Gamma(a+w) / [w! \Gamma(a)]\}\,|_{(|s|<1,\ a>0)}. \tag{10}$$

Applying (10) to (9) for the term $\left[\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{2j+3}$ we get

$$\begin{aligned} f(x) &= 2\theta \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j}\, (i+1)^j\, \Gamma(\theta)\, \Gamma(2j+k+3)\, [2j+k+2]}{i!\, j!k!\Gamma(\theta-i)\, \Gamma(2j+3)\, [2j+k+2]} \\ &\quad \times \underbrace{\lambda\beta^{-1}\left(1+x\beta^{-1}\right)^{-(\lambda+1)}}_{g_{\lambda,\beta}(x)} \underbrace{\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{2j+k+1}}_{G_{\lambda,\beta}(x)^{2j+k+1}}. \end{aligned}$$

This can be written as

$$f(x) = \sum_{j,k=0}^{\infty} \zeta_{j,k}\, \pi_{(2+2j+k),_{\lambda,\beta}}(x), \tag{11}$$

where

$$\zeta_{j,k} = \frac{2\theta\,(-1)^j\, \Gamma(\theta)\, \Gamma(2j+k+3)}{j!k!\Gamma(2j+3)\,(2+2j+k)} \sum_{i=0}^{\infty} \frac{(-1)^i\, (i+1)^j}{i!\, \Gamma(\theta-i)}$$

and

$$\pi_{(2+2j+k),_{\lambda,\beta}}(x) = (2+2j+k) \underbrace{\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{2j+k+1}}_{G_{\lambda,\beta}(x)^{2j+k+1}} \underbrace{\lambda\beta^{-1}\left(1+x\beta^{-1}\right)^{-(\lambda+1)}}_{g_{\lambda,\beta}(x)}.$$

The CDF of the BrXLx , similarly, can also be expressed as a mixture of exp-Lx CDFs as

$$F(x) = \sum_{j,k=0}^{\infty} \zeta_{j,k}\, \Pi_{(2+2j+k),_{\lambda,\beta}}(x), \tag{12}$$

where

$$\Pi_{(2+2j+k),_{\lambda,\beta}}(x) = \left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{2j+k+2}$$

is the CDF of the exp-Lx model with power parameter $2j+k+2$.

3. Moments and Generating Function

The $r-$th ordinary moment of X is given by

$$\mu'_r = \mathbf{E}(X^r) = \int_{-\infty}^{\infty} x^r f(x)\, dx.$$

Then we obtain

where $\mathbf{B}(\cdot,\cdot)$ is the complete beta function, setting $r = 1$ in (13), we have the mean of X

$$E(X) = \mu'_1 = \sum_{j,k=0}^{\infty} \sum_{w=0}^{r} \zeta_{j,k} (2 + 2j + k) \beta (-1)^w \binom{1}{w} \mathbf{B}\left((2 + 2j + k), 1 + \frac{w-1}{\lambda}\right)|_{(\lambda>1)}.$$

Setting $r = 2.3$ and 4 in (13), we have the 2-nd, 3-rd and the 4-th moments about the origin which can be used to obtain the central moments.

Table 1. Mean, variance, skewness and kurtosis of the BrXLx distribution with $\lambda = \beta = 0.5$ and different values of θ

θ	$Mean(X)$	$Variance(X)$	Skew(X)	Kur(X)
0.001	0.0031329	0.00401742	**35.22613**	**1685.78**6
0.01	0.03090983	0.03913622	11.13911	171.0975
0.1	0.27350260	0.3092640	3.534371	19.81156
1	1.3862270	0.9077153	1.310273	5.486999
5	2.6036300	0.9669048	1.004489	4.735616
10	3.1402080	0.9367001	0.9842919	4.730640
20	3.6686360	0.9007004	0.9865868	4.770655
50	4.3510240	0.8554498	1.002323	4.843780
100	4.8552040	0.8256333	1.016365	4.899162
200	5.3500750	0.7998305	1.029670	4.949684
400	5.8367520	0.7776043	1.041495	4.994194
800	6.3162710	0.7583990	1.051718	5.032806
1000	6.4692710	0.7527710	1.054685	5.044076

We proved, numerically, that the BrXLx model provides better fits than other four competitive extensions of the Lx models (see Section 6) so the BrXLx model is a exemplary alternative to these mosels. The skewness (Skew(X)) of the BrXLx distribution can range in the interval $(35.23, 0.17)$, whereas the kurtosis (Kur(X)) of the BrXLx distribution varies only in the interval $(1685.7, 2.61)$ also the mean of X ($Mean(X)$) increases as θ increases, the skewness is always positive (see Table 1 and 2).

Table 2. Mean, variance, skewness and kurtosis of the BrXLx distribution with $\lambda = \beta = 3.5$ and different values of θ

θ	$Mean(X)$	$Variance(X)$	Skew(X)	Kur(X)
0.001	0.0019254	0.000972489	23.04304	633.3245
0.01	0.0189181	0.009325541	7.190108	63.41473
0.1	0.1612188	0.06369933	1.989041	6.713972
1	0.6706015	0.08385418	0.2087107	**2.618290**
5	1.016599	0.03950313	**0.1743238**	3.001886
10	1.128713	0.02890308	0.2642751	3.103622
20	1.224394	0.02192567	0.3539608	3.207029
50	1.332142	0.01602527	0.4566714	3.347865
100	1.403011	0.01306238	0.520966	3.452703
200	1.466852	0.01089229	0.5750721	3.552538
400	1.525029	0.009253848	0.6207442	3.64536
800	1.578552	0.007983825	0.6595533	3.730876
1000	1.594923	0.007634884	0.6708122	3.756873

The moment generating function (MGF) $M_X(t) = \mathbf{E}\left(e^{tX}\right)$ of X. Clearly, the first one can be derived from equation (10) as

$$M_X(t) = \sum_{j,k=0}^{\infty} \sum_{w=0}^{r} \zeta_{j,k} \left[t^r/r!\right](2+2j+k)\beta^r(-1)^w \binom{r}{w} \mathbf{B}\left((2+2j+k), 1+\frac{w-r}{\lambda}\right)|_{(\lambda>r)},$$

4. Incomplete Moments and Mean Deviations

The $s-$th incomplete moment, say $I_s(t)$, of X can be expressed from (10) as $I_s(t) = \int_{-\infty}^{t} x^s f(x)\,dx$, we have

$$I_s(t) = \sum_{j,k=0}^{\infty} \sum_{w=0}^{s} \zeta_{j,k}\ (2+2j+k)\beta^s(-1)^w \binom{s}{w} \mathbf{B}_t\left((2+2j+k), 1+\frac{w-s}{\lambda}\right)|_{(\lambda>s)}. \quad (14)$$

The mean deviations about the mean

$$\mathbf{E}(|X-\mathbf{E}(X)|) = MD_{(mean)} = -2I_1(\mathbf{E}(X)) + 2\mathbf{E}(X)F(\mathbf{E}(X))$$

and about the median

$$\mathbf{E}(|X-Median(X)|) = MD_{(median)} = -2I_1(Median(X)) + \mathbf{E}(X)$$

of X , $F(\mathbf{E}(X))$ is easily calculated from (5) and $I_1(t)$ is the first incomplete moment given by (14) with $s=1$. Now, we provide two ways to determine $MD_{(mean)}$ and $MD_{(median)}$. The $I_1(t)$ can be derived from (14) as

$$I_1(t) = \sum_{j,k=0}^{\infty} \sum_{w=0}^{s} \zeta_{j,k}\ (2+2j+k)\beta(-1)^w \binom{1}{w} \mathbf{B}_t\left((2+2j+k), 1+\frac{w-1}{\lambda}\right)|_{(\lambda>1)}.$$

4.1 Probability Weighted Moments (PWM)

The $(s,r)-$th PWM of X following the BrXLx model, say $\rho_{s,r}$, is formally defined by

$$\rho_{s,r} = \mathbf{E}\left\{X^s F_{\theta,\lambda,\beta}(x)^r\right\} = \int_{-\infty}^{\infty} x^s F_{\theta,\lambda,\beta}(x)^r f_{\theta,\lambda,\beta}(x)\,dx.$$

Using (5), (6) we can write

$$F_{\theta,\lambda,\beta}(x)^r f_{\theta,\lambda,\beta}(x) = \sum_{j,k=0}^{\infty} v_{j,k}\pi_{2j+k+2}(x),$$

where

$$v_{j,k} = \frac{2\theta(-1)^j}{(2+2j+k)\,j!k!}\left[\Gamma(2j+k+3)/\Gamma(2j+3)\right]\sum_{i=0}^{\infty}(-1)^i(i+1)^j\binom{\theta(r+1)-1}{i}.$$

Then, the $(s,r)-$th PWM of X can be expressed as

$$\rho_{s,r} = \sum_{j,k=0}^{\infty} \sum_{w=0}^{s} v_{j,k}(2+2j+k)\beta^s(-1)^w \binom{s}{w} \mathbf{B}\left((2+2j+k), 1+\frac{w-s}{\lambda}\right)|_{(\lambda>s)}.$$

4.2 Moments of the Reversed Residual Life

The $n-$th moment of the reversed residual life, say

$$\Upsilon_n(t) = \mathbf{E}\left[(t-X)^n\ |_{(X\leq t,\, t>0,\, n=1,2,\ldots)}\right],$$

uniquely determines $F_{\theta,\lambda,\beta}(x)$. We obtain

$$\Upsilon_n(t) = \frac{\int_0^t (t-x)^n dF_{\theta,\lambda,\beta}(x)}{F_{\theta,\lambda,\beta}(x)}.$$

Then, the $n-$th moment of the reversed residual life of X becomes

$$\Upsilon_n(t) = F_{\theta,\lambda,\beta}(x)^{-1}\sum_{j,k=0}^{\infty}\sum_{r=0}^{n}\sum_{w=0}^{n}\zeta_{j,k,r,w}\beta^n\binom{n}{r}\binom{n}{w}\mathbf{B}_t\left((2+2j+k), 1+\frac{w-n}{\lambda}\right)|_{(\lambda>n)}.$$

where

$$\zeta_{j,k,r,w} = \zeta_{j,k}\,(-1)^r\,t^{n-r}\,(-1)^w\,(2+2j+k)\,,$$

and

$$\mathbf{B}_{a_3}(a_1;a_2) = \int_0^{a_3} z^{a_1-1}\,(1-z)^{a_2-1}\,dz =$$

is the incomplete beta function.

4.3 Reliability Estimation

The most widely approach used for reliability estimation is the stress-strength model (SSM), this model is used in many applications of physics and engineering such as strength failure and system collapse. In SSM, we have

$$\mathbf{R}\left(T_1, T_2|_{(T_2<T_1)}\right) = \Pr(T_2 < T_1),$$

is a measure of reliability of the system when it is subjected to random stress T_2 and has strength T_1. The system fails if and only if the applied stress is greater than its strength ($T_1 < T_2$). Other interpretation can be given as the reliability $\mathbf{R}\left(T_1, T_2|_{(T_2<T_1)}\right)$ of a system is the probability that the system is strong enough to overcome the stress imposed on it. Let T_1 and T_2 be two independent rvs with BrXLx(θ_1, λ, β) and BrXLx(θ_2, λ, β) distributions, respectively. The PDF of T_1 and the CDF of T_2 can be written from Equations (6) and (5), respectively as

$$f^{(1)}_{\theta_1,\lambda,\beta}(t) = 2\theta_1 \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j}\,(i+1)^j\,\Gamma(2j+k+3)\,\Gamma(\theta_1)}{i!\,j!k!\Gamma(2j+3)\,\Gamma(\theta_1-i)} \underbrace{\left[1-\left(1+t\beta^{-1}\right)^{-\lambda}\right]^{2j+k+1}}_{G_{\lambda,\beta}(t)^{2J+K+1}} \underbrace{\lambda\beta^{-1}\left(1+t\beta^{-1}\right)^{-(\lambda+1)}}_{g_{\lambda,\beta}(t)}$$

and

$$F^{(2)}_{\theta_2,\lambda,\beta}(t) = 2\theta_2 \sum_{h,w,m=0}^{\infty} \frac{(-1)^{h+w}\,(h+1)^w\,\Gamma(2w+m+3)\,\Gamma(\theta_2)}{h!\,w!m!\,(2w+m+2)\,\Gamma(2w+3)\,\Gamma(\theta_2-h)} \underbrace{\left[1-\left(1+t\beta^{-1}\right)^{-\lambda}\right]^{2j+k+2}}_{G_{\lambda,\beta}(t)^{2J+K+2}}.$$

Then, the reliability is defined by

$$\mathbf{R}\left(T_1, T_2|_{(T_2<T_1)}\right) = \int_0^{\infty} f^{(1)}_{\theta_2,\lambda,\beta}(x)\,F^{(2)}_{\theta_2,\lambda,\beta}(x)\,dx.$$

We can write

$$\mathbf{R}\left(T_1, T_2|_{(T_2<T_1)}\right) = \sum_{j,k,w,m=0}^{\infty} \Psi_{j,k,w,m} \int_0^{\infty} \pi_{4+2j+2w+k+m}(t)\,dt,$$

where

$$r_{j,k,w,m} = 4\theta_1\theta_2 \sum_{j,k,w,m=0}^{\infty} \frac{(-1)^{j+w}\,\Gamma(2w+m+3)\,\Gamma(2j+k+3)}{j!k!\,w!m!\Gamma(\theta_2-h)\,\Gamma(2w+3)\,\Gamma(2j+3)} \sum_{i,h=0}^{\infty} \frac{(-1)^{i+h}\,(h+1)^w\,(i+1)^j\,\binom{\theta_1-1}{i}\binom{\theta_2-1}{h}}{(2j+k+2w+m+4)\,(2w+m+2)}.$$

and

$$\pi_{4+2j+2w+k+m}(t) = (2j+k+2w+m+4)\underbrace{\lambda\beta^{-1}\left(1+t\beta^{-1}\right)^{-(\lambda+1)}}_{g_{\lambda,\beta}(x)} \underbrace{\left[1-\left(1+t\beta^{-1}\right)^{-\lambda}\right]^{2j+k+2w+m+3}}_{G_{\lambda,\beta}(x)^{2j+k+2w+m+3}}.$$

Thus, the reliability, $\mathbf{R}\left(T_1, T_2|_{(T_2<T_1)}\right)$, can be expressed as

$$\mathbf{R}\left(T_1, T_2|_{(T_2<T_1)}\right) = \sum_{j,k,w,m=0}^{\infty} r_{j,k,w,m}.$$

4.4 Order Statistics

Let $X_1, \ldots, X_n$ be a random sample (RS) from the BrXLx distribution and let $X_{1:n}, \ldots, X_{n:n}$ be the corresponding order statistics. The PDF of $i-$th order statistic, say $X_{i:n}$, can be expressed as

$$f_{\theta,\lambda,\beta}^{(i:n)}(x) = \mathbf{B}^{-1}(i, n-i+1)\, f_{\theta,\lambda,\beta}(x) \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F_{\theta,\lambda,\beta}(x)^{j+i-1}, \quad (15)$$

using (5), (6) and (15) we get

$$f_{\theta,\lambda,\beta}(x) F_{\theta,\lambda,\beta}(x)^{j+i-1} = \sum_{w,k=0}^{\infty} a_{w,k} \pi_{2w+k+2}(x),$$

where

$$a_{w,k} = \frac{2\theta(-1)^w \Gamma(2w+k+3)}{w!k!\Gamma(2w+3)(2w+k+2)} \sum_{m=0}^{\infty} (-1)^m (m+1)^w \binom{\theta(j+i)-1}{m}.$$

The PDF of $X_{i:n}$ can be written as

$$f_{\theta,\lambda,\beta}^{(i:n)}(x) = \sum_{w,k=0}^{\infty} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} a_{w,k} \pi_{2w+k+2}(x).$$

Then, the density function of the BrXLx order statistics is a mixture of eponentiated Lomax (ELx). The $q-$th moments of $X_{i:n}$ can be expressed as

$$\mathbf{E}\left(X_{i:n}^q\right) = \sum_{w,k=0}^{\infty} \sum_{j=0}^{n-i} \sum_{m=0}^{q} a_{w,k,j,m} \beta^q \binom{q}{m} \mathbf{B}\left((2w+k+2), 1 + \frac{m-q}{\lambda}\right)|_{(\lambda>q)}. \quad (12)$$

where

$$a_{w,k}(-1)^{j+m}\ (2+2w+k)\,\mathbf{B}^{-1}(i, n-i+1) \binom{n-i}{j} \binom{q}{m} = a_{w,k,j,m}$$

4.5 Quantile Spread (QS) Ordering

The QS of the rv U ~BrXLx(θ, λ, β) having CDF (5) is given by

$$QS_U(\xi)|_{(\xi\in(0.5,1))} = -F^{-1}(1-\xi) + F^{-1}(\xi),$$

which implies

$$QS_U(\xi) = -\left[S^{-1}(\xi)\right] + \left[S^{-1}(1-\xi)\right],$$

where

$$S(u) = 1 - F(u) \text{ and } F^{-1}(\xi) = S^{-1}(1-\xi)$$

is the survival function. The QS of a any distribution describes how the probability mass is placed symmetrically about its median and hence it can be used to formalize concepts such as peakedness and tail weight traditionally associated with the kurtosis. So, it allows use to separate concepts of the kurtosis and peakedness for asymmetric models. Let U_1 and U_2 be two rvs following the BrXLx model with QS_{U_1} and QS_{U_2}. Then U_1 is called smaller than U_2 in quantile spread order, denoted as $U_1 \leq_{\{QS\}} U_2$, if

$$QS_{U_1}(\xi)|_{(\xi\in(0.5,1))} \leq QS_{U_2}(\xi).$$

Following are some properties of the QS order which can be obtained.

The order $\leq_{\{QS\}}$ is a location-free

$$U_1 \leq_{\{QS\}} U_2 \text{ if } (U_1 + a) \leq_{\{QS\}} U_2\, |_{(a\in\mathbb{R})}.$$

The order $\leq_{\{QS\}}$is dilative

$$U_1 \leq_{\{QS\}} aU_1 \text{ whenever } a \geq 1 \text{ and } U_2 \leq_{\{QS\}} aU_2\, |_{(\mathbf{C}\geq 1)}.$$

Let F_{U_1} and F_{U_2} be symmetric, then

$$U_1 \leq_{\{QS\}} U_2 \text{ if, and only if } F_{U_1}^{-1}(\xi) \leq F_{U_2}^{-1}(\xi)|_{(\xi\in(0.5,1))}.$$

The order $\leq_{\{QS\}}$ implies ordering of the mean absolute deviation around the median, say $\pi(U_i)|_{(i=1,2)}$,

$$\mathbf{E}\left[|U_1 - Median(U_1)|\right] = \pi(U_1)$$

and

$$\mathbf{E}\left[|U_2 - Median(U_2)|\right] = \pi(U_2),$$

where

$$\pi(U_1) \leq_{\{QS\}} \pi(U_2) \Leftarrow U_1 \leq_{\{QS\}} U_2,$$

finally

$$U_1 \leq_{\{QS\}} U_2 \text{ if, and only if } -U_1 \leq_{\{QS\}} -U_2.$$

4.6 Entropies

The Rényi entropy is defined by

$$I_\delta(X)|_{(\delta>0 \text{ and } \delta\neq 1)} = \frac{\log \int_{-\infty}^{\infty} f_{\theta,\lambda,\beta}(x)^\delta dx}{1-\delta}.$$

Using PDF (4), we can write

$$f(x)^\delta|_{(\delta>0 \text{ and } \delta\neq 1)} = \sum_{j,k=0}^{\infty} \tau_{j,k}\, G_{\lambda,\beta}(x)^{\delta+2j+k} g_{\lambda,\beta}(x)^\delta,$$

where

$$\tau_{j,k} = \sum_{i=0}^{\infty} \frac{2^\delta \theta^\delta \left[(\theta\delta-\delta)_i\right](-1)^{i+j}\,\Gamma(3\delta+2j+k)}{i!\,j!\,k!\,\Gamma(3\delta+2j)\,(\delta+i)^{-j}}.$$

Then, the Rényi entropy of the BrXLx is given by

$$I_\delta(X) = \frac{1}{1-\delta}\log\left\{\sum_{j,k=0}^{\infty}\tau_{j,k}\int_0^{\infty} G_{\lambda,\beta}(x)^{\delta+2j+k} g_{\lambda,\beta}(x)^\delta dx\right\},$$

The δ-entropy, say $E_\delta(X)$, can be obtained as

$$E_\delta(X) = \frac{1}{\delta-1}\log\left\{1-\left[\sum_{j,k=0}^{\infty}\tau_{j,k}\int_{-\infty}^{\infty} G_{\lambda,\beta}(x)^{q+2j+k} g_{\lambda,\beta}(x)^q dx\right]\right\}.$$

The Shannon entropy of a rv X, say SE, is defined by

$$SE = E\left\{-\left[\log f(X)\right]\right\},$$

follows by taking the limit of $I_\delta(X)$ as δ tends to 1.

5. Parameter Estimation

Let $x_1,\dots,x_n$ be a RS from the BrXLx model with parameters θ and ξ. Let $\Theta = (\theta,\lambda,\beta)^\intercal$ be the 3×1 parameter vector. For determining the maximum likelihood estimation (MLE) of Θ, we have the log-likelihood (Log L) function

$$\begin{aligned}\ell &= \ell(\Theta) = n\log 2 + n\log\theta + n\log\lambda - n\log\beta + (2\lambda-1)\sum_{i=1}^{n}\log\left(1+x_i\beta^{-1}\right)\\ &-\sum_{i=1}^{n}\left[\frac{1-\left(1+x_i\beta^{-1}\right)^{-\lambda}}{(1+x_i\beta^{-1})^{-\lambda}}\right]^2 + (\theta-1)\sum_{i=1}^{n}\log\left(1-\exp\left\{-\left[\frac{1-\left(1+x_i\beta^{-1}\right)^{-\lambda}}{(1+x_i\beta^{-1})^{-\lambda}}\right]^2\right\}\right)\end{aligned}$$

The components of the score vector, $\mathbf{U}(\Theta) = \left(\frac{\partial}{\partial\theta}\ell(\Theta), \frac{\partial}{\partial\lambda}\ell(\Theta), \frac{\partial}{\partial\beta}\ell(\Theta)\right)^\intercal$, are availble if needed, Via setting the nonlinear system of equations $U_\theta = 0, U_\gamma = 0$ and $U_\beta = 0$ and solving them simultaneously yields the MLE $\widehat{\Theta} = (\widehat{\theta},\widehat{\lambda},\widehat{\beta},)^\intercal$. To solve those equations, it is usually more convenient to use the nonlinear optimization methods such as the quasi-Newton algorithm to numerically maximize $\ell(\Theta)$.

6. Applications

In this section, we provide two applications to two real data sets to prove the importance and flexibility of the BrXLx distribution. We compare the fit of the BrXLx with competitve models namely: ELx model (Gupta et al., 1998), gamma Lomax (GLx) model (Cordeiro et al., 2015), beta Lomax (BLX) model (Lemonte and Cordeiro, 2013) and Lx model. The CDFs of these distributions are, respectively, given by (for $x > 0$ and $\alpha, \beta, \lambda, a > 0$):

$$\begin{aligned} F_{\alpha,\beta,\lambda}(x) &= \left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{\alpha}, \\ F_{\alpha,\beta,\lambda}(x) &= \Gamma\left(\alpha;\lambda\log\left[1+x\beta^{-1}\right]\right)\Gamma^{-1}(\alpha)\,, \text{ and} \\ F_{\alpha,\beta,\theta,\lambda}(x) &= \frac{\mathbf{B}_{\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]}(\alpha,\theta)}{\mathbf{B}(\alpha,\theta)}, \end{aligned}$$

where $\Gamma(\cdot;\cdot)$ is the incomplete gamma function.

The first real data set represents the data on failure times of 84 aircraft windshield given in Murthy et al. (2004). The data are:

4.1671.281,3, 4.035, 2.3000, 3.344, 4.602, 1.7570, 2.324, 2.625, 3.5780, 0.943, 4.121, 1.3030, 2.089, 2.632, 2.135, 2.962, 2.688,2.902, 0.557, 1.9110, 1.568, 3.5950, 1.0700, 4.2550, 1.8990, 2.610, 3.4780, ,1.248, 2.0100, 1.914, 1.505, 2.154, 2.9640, 4.278, 1.506,0.309, 1.2810, 1.9120, 3.9240, 2.190, 3.000, 4.3050, 3.3760, 2.6460, 3.699, 1.4320, 2.097, 2.934, 4.2400, 1.480, 2.1940, 3.103, 4.376, 1.615, 2.2230, 0.0400, 1.866, 2.3850, 3.443, 0.3010, 1.876, 2.4810, 3.467, 4.663, 2.0850, 2.890, 2.038, 2.820, 1.1240, 1.981, 2.661, 3.7790 3.114, 4.449, 1.6190, 2.224, 3.1170, 4.485, 1.652, 2.2290, 3.166, 4.570, 1.652.

The second real data set (recently studied by Tahir et al. (2015)) represents the data on service times of 63 aircraft windshield given in Murthy et al. (2004). The data are:

0.046, 1.436, 1.0030, 2.137, 3.500, 1.0100, 2.141, 3.6220, 1.085, 2.163, 2.592, 0.140, 1.492, 2.600, 0.150, 1.580, 2.670, 0.248, 1.7190, 2.717, 2.820, 0.389, 1.9200, 0.3130, 1.915, 1.1520, 2.2400, 4.015, 1.183, 2.878, 0.487, 1.9630, 2.950, 0.622, 1.978, 3.0030, 0.2800, 1.794, 2.819, 2.053, 3.1020, 0.952, 2.065, 3.3040, 0.9960, 0.9000, 1.092, 2.183, 3.695, 2.117, 3.483, 3.6650, 2.3410, 4.628, 1.2440, 2.435, 4.806, 1.249, 2.4640, 4.881, 1.262, 2.5430, 5.140.

In order to make a real comparison among the distributions, the estimated Log L values $(\widehat{\ell})$, Akaike Information Criteria (AIC), Cramer von Mises (W^*) and Anderson-Darling (A^*) goodness of-fit statistics were calculated for all competitive models. The statistics W^* and A^* were defined in Chen and Balakrishnan (1995) with details. In general, it can be chosen as the best model which has the smaller values of the W^*, A^* and AIC statistics and the larger values of $(\widehat{\ell})$. The below computations are obtained via the "maxLik" and "goftest" sub-routines using the R-software. The analysis results are listed in Tables 3, 4, 5 and 6. These results show obviously that the new distribution has the lowest W^* , AIC and A^* values and has the biggest estimated $-\widehat{\ell}$ among all the fitted models. Hence, it could be chosen as the best model under these criteria. From tables 3 and 5, the new model is much better than all competitive (BLx, ELx, GLx and Lx) models so the new model is a adequate alternative to these models in modeling aircraft windshield data.

Table 3. MLEs, standard erros of the estimates (in parentheses) for the first data set

Model	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\theta}$	$\widehat{\lambda}$
BrXLx		$5.237231\times e^{5}$	$8.229086\times e^{-1}$	$1.159193\times e^{5}$
		(0.000)	(0.1051083)	(5446.8)
BLx	3.6036	118.8374	33.6387	4.8307
	(0.6187)	(63.7145)	(9.2382)	(429.0000)
ELx	3.6261	26257.6808		20074.5097
	(0.6236)	(99.7417)		(2041.8263)
GLx	3.5876	37029		52001
	(0.5133)	(81.1644)		(7955)
Lx		131789		51425
		(296.1200)		(5933.49)

Table 4. $-\hat{\ell}$ and goodness-of-fits statistics for the first data set

Model	$-\hat{\ell}$	AIC	W^*	A^*
BrXLx	**127.665**	**261.3294**	**0.0764164**	**0.584438**
BLx	138.7177	285.4354	1.4084	0.1680
ELx	141.3997	288.7994	1.7435	0.2194
GLx	138.4042	282.8093	1.3667	0.1619
Lx	164.9900	333.9767	1.3976	0.1665

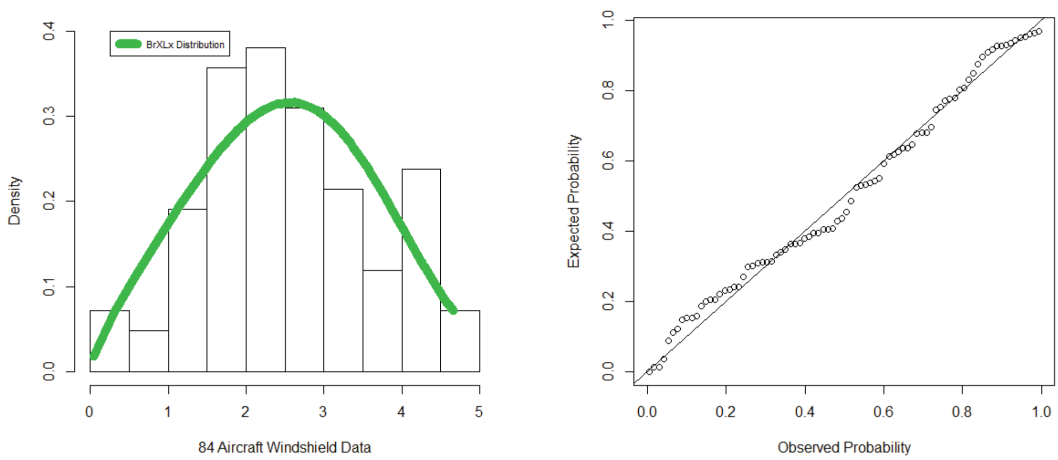

Figure 3. The fitted PDF and PP plot for the first data set

Table 5. MLEs, standard erros of the estimates (in parentheses) for the second data set

Model	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\theta}$	$\widehat{\lambda}$
BrXLx		0.6467194	0.5987192	1.6211236
		(0.0474756)	(0.3901375)	(0.9591999)
BLx	1.9218	169.5800	31.2595	4.9685
	(0.3185)	(339.2068)	(316.8413)	(50.5279)
ELx	1.9145	32881.9		22971.2
	(0.3483)	(162.2230)		(3209.5)
GLx	1.9073	39197.6		35842.4
	(0.3214)	(151.6530)		(6945)
Lx		207019		99269
		(301.2370)		(11863.5222)

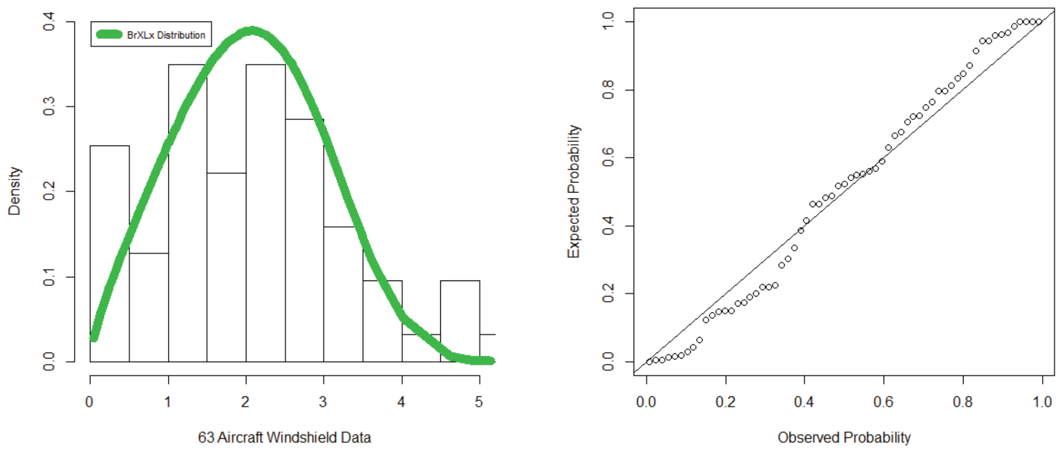

Figure 4. The fitted PDF and PP plot for the second data set

Table 6. $-\hat{\ell}$ and goodness-of-fits statistics for the second data set

Model	$-\hat{\ell}$	AIC	W^*	A^*
BrXLx	**98.10294**	**202.206**	**0.08763**	**0.527784**
BLx	102.9611	213.9223	1.1336	0.1872
ELx	103.5468	213.9223	1.2331	0.2037
GLx	102.8333	211.6664	1.1121	0.2038
Lx	109.2988	222.5976	1.1265	0.1861

7. Conclusions

In this work, a new lifetime model called the Burr X Lomax (BrXLx) is introduced and studied. The major justification for introducing and studying the BrXLx model is based on the wider use of the Lx model in applied fields. We are also motivated to introduce and study the BrXLx model since the density of the BrXLx distribution displays various important shapes such as the unimodal, the right skewed and the left skewed. The new model can be viewed as a mixture of the exponentiated Lx distribution. It can also be considered as a convenient model for fitting the symmetric, the left skewed, the right skewed, and the unimodal data sets. The maximum likelihood estimation method is used to estimate the BrXLx parameters. We prove empirically the importance and flexibility of the BrXLx in modeling two types of aircraft windshield lifetime data. The proposed BrXLx lifetime model is much better than gamma Lomax, beta Lomax, exponentiated Lomax and Lomax models so the exponentiated Lomax, model is a good alternative to these models in modeling aircraft windshield data.

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