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The Logarithmic Burr-Hatke Exponential Distribution for Modeling Reliability and Medical Data

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Abstract

In this work, we introduced a new one-parameter exponential distribution. Some of its structural properties are derived**.** The maximum likelihood method is used to estimate the model parameters by means of numerical Monte Carlo simulation study. The justification for the practicality of the new lifetime model is based on the wider use of the exponential model. The new model can be viewed as a mixture of the exponentiated exponential distribution. It can also be considered as a suitable model for fitting right skewed data. We prove empirically the importance and flexibility of the new model in modeling cancer patients data, the new model provides adequate fits as compared to other related models with small values for W^* and A^*. The new model is much better than the Modified beta-Weibull, Weibull, exponentiated transmuted generalized Rayleig, the transmuted modified-Weibull, and transmuted additive Weibull models in modeling cancer patients data. We are also motivated to introduce this new model because it has only one parameter and we can generate some new families based on it such as the the odd Burr-Hatke exponential-G family of distributions, the logarithmic Burr-Hatke exponential-G family of distributions and the generalized Burr-Hatke exponential-G family of distributions, among others.

Keywords: Burr-Hatke Distribution, Exponential Distribution; Modeling, Estimation, Moments, Simulation

1. Introduction

A random variable (RV) X is said to have the exponential (E) distribution if its probability density function (PDF) and cumulative distribution function (CDF) are given by

$$g_{\lambda}(x) = \lambda \exp\left(-\lambda\, x\right) \text{ and } G_{\lambda}(x) = 1 - \exp\left(-\lambda\, x\right), \tag{1}$$

respectively, where $\lambda > 0$ and $x > 0$. Maniu and Voda (2008) introduced and studied the Burr-Hatke (BH) distribution with CDF and PDF given by

$$F_{\theta|_{\theta=1}}(t) = 1 - (t+1)^{-1} \exp\left(-t\right)|_{(\,t>0)} \text{ and } f_{\theta|_{\theta=1}}(t) = (t+1)^{-2} \exp\left(-t\right)\left[(t+1)+1\right], \tag{2}$$

respectivily. In this work, we propose a new one parameter E distribution called the Logarithmic Burr-Hatke exponential (LBH-E) distribution using the BH model.

The justification for the practicality of the one-parameter LBHE lifetime model is based on the wider use of the E model. The new model can be viewed as a mixture of the exponentiated E (Exp-E) distribution. It can also be considered as a suitable model for fitting right skewed data (see Figure 1). The hazard rate function (HRF) of the new model exhibits the constant and the decreasing shapes (see Figure 1). We prove empirically the flexibility and the importance of the one-parameter LBHE model in modeling a real data set, the new model provides adequate fits as compared to other related models with small values for W^* and A^* and it is much better than the Weibull, Modified beta-Weibull, Transmuted modified-Weibull, exponentiated transmuted generalized Rayleig and transmuted additive Weibull models in modeling cancer patients data.

The paper is outlined as follows. In Section 2, we present the formulation, expansions, graphical presentation and motivation for the logarithmic Burr-Hatke exponential model . In Sec. 3, we derive some its mathematical properties. In Sec. 4, the model parameter is estimated by using maximum likelihood (ML) method. In Sec. 5, we assess the performance of the maximum likelihood estimators by means of a simulation study. An application to real data is given in Sec. 6 to illustrate the flexibility of the logarithmic Burr-Hatke exponential model. Finally, some concluding remarks are presented in Sec. 7.

2. The New Model

2.1 Formulation and Motivation

Starting from (2) and replacing t by the argument $\left\{-\log\left[\overline{G}_\lambda(x)\right]\right\}$, where $\overline{G}_\lambda(x) = 1 - G_\lambda(x) = \exp(-\lambda x)$, then the CDF of the new one parameter LBH-E distributions is defined by

$$F(x) = F_\lambda(x) = 1 - \exp(-\lambda x)(1 + \lambda x)^{-1} \quad (2)$$

The PDF corresponding to (2) is

$$f(x) = f_\lambda(x) = \lambda \exp(-\lambda x)(1 + \lambda x)^{-2}(2 + \lambda x). \quad (3)$$

The reliability function (rf) and HRF of new LBHE model are given by

$$R(x) = R_\lambda(x) = \exp(-\lambda x)(1 + \lambda x)^{-1},$$

and

$$h(x) = h_\lambda(x) = \lambda(1 + \lambda x)^{-1}(2 + \lambda x).$$

We are also motivated to introduce this new model because it has only one parameter and we can generate some new families based on it such as the the odd Burr-Hatke exponential-G (OBrHE-G) family, the logarithmic Burr-Hatke exponential-G (LogBrHE-G) family and the generalized Burr-Hatke exponential-G (GBrHE-G) with the following CDFs

$$F_\lambda^{(OBrHE-G)}(x) = 1 - \exp\left[-\lambda \frac{G(x;\mathbf{\Theta})}{1 - G(x;\mathbf{\Theta})}\right]\left[1 + \lambda \frac{G(x;\mathbf{\Theta})}{1 - G(x;\mathbf{\Theta})}\right]^{-1},$$

$$F_\lambda^{(LogBrHE-G)}(x) = 1 - \exp\left[-\lambda\left(-\log\left\{\left[1 - G(x;\mathbf{\Theta})\right]\right\}\right)\right]\left[1 + \lambda\left(-\log\left\{\left[1 - G(x;\mathbf{\Theta})\right]\right\}\right)\right]^{-1},$$

and

$$F_{\lambda,\alpha}^{(GBrHE-G)}(x) = 1 - \exp\left\{-\lambda \frac{1 - [1 - G(x;\mathbf{\Theta}))]^\alpha}{[1 - G(x;\mathbf{\Theta}))]^\alpha}\right\}\left\{1 + \lambda \frac{1 - [1 - G(x;\mathbf{\Theta}))]^\alpha}{[1 - G(x;\mathbf{\Theta}))]^\alpha}\right\}^{-1},$$

respectively. Where $G(x;\mathbf{\Theta})$ is the baseline CDF and $\mathbf{\Theta}$ is the vector of parameters. As a future work we will study these new families in a separate articles.

2.2 Expansions

Consider the following expansions,

$$(1 - \zeta)^\tau = \sum_{k=0}^{\infty} (-1)^k \binom{\tau}{k} \zeta^\tau |_{(|\zeta|<1)}, \quad (4)$$

and

$$\log(1 - \zeta) = -\sum_{k=0}^{\infty} \left[\zeta^{k+1} / (k+1)\right] |_{(|\zeta|<1)}. \quad (5)$$

The CDF (2) can be rewritten as

$$F(x;\lambda) = 1 - \left[\overbrace{\exp(-\lambda x)}^{\overline{G}_\lambda(x)} \div \underbrace{1 + \lambda x}_{\left[1 - \log \overline{G}_\lambda(x)\right]}\right].$$

Applying (4) for $\overline{G}_\lambda(x)$ in (2) we have

$$\exp(-\lambda x) = \sum_{k=0}^{\infty} a_k \left[1 - \exp(-\lambda x)\right]^k$$

where

$$a_k = (-1)^k \binom{1}{k},$$

applying (5) for $1 + \lambda x$, still in (2), we get

$$1 + \lambda x = 1 + \sum_{i=0}^{\infty} \left\{\left[1 - \exp(-\lambda x)\right]^{i+1} / (i+1)\right\} = \sum_{k=0}^{\infty} b_k \left[1 - \exp(-\lambda x)\right]^k,$$

where $b_0 = 1$ and for $k \geq 1$, $-k^{-1} = b_k$, then

$$F_\lambda(x) = 1 - \left[\sum_{k=0}^{\infty} a_k \left[1 - \exp(-\lambda\, x)\right]^k \div \sum_{k=0}^{\infty} b_k \left[1 - \exp(-\lambda\, x)\right]^k\right] = 1 - \sum_{k=0}^{\infty} c_k \left[1 - \exp(-\lambda\, x)\right]^k,$$

where $c_0 = a_o / b_0$ and, for $k \geq 1$, we have

$$c_k = \left(a_k - \frac{1}{b_0}\sum_{r=1}^{k} b_r c_{k-r}\right)\frac{1}{b_0}.$$

Finally the CDF (2) can be expressed as

$$F_\lambda(x) = \sum_{k=0}^{\infty} d_{k+1} \mathbf{\Pi}_{k+1}(x;\lambda), \tag{6}$$

where $1 - c_k = d_0$, for $k \geq 1$ we have $d_0 = -c_k$ and

$$\mathbf{\Pi}_\eta(x) = \underbrace{\left[1 - \exp(-\lambda\, x)\right]^\eta}_{[G_E(x;\lambda)]^\eta}$$

is the CDF of the Exp-E model with power parameter η. By differentiating (6), we obtain the same mixture representation

$$f_\lambda(x) = \sum_{k=0}^{\infty} d_{k+1} \pi_{k+1}(x;\lambda), \tag{7}$$

where

$$\pi_\eta(x;\lambda) = \eta \underbrace{\lambda \exp(-\lambda\, x)}_{g_\lambda(x)} \underbrace{\left[1 - \exp(-\lambda\, x)\right]^{\eta-1}}_{[G_\lambda(x)]^{\eta-1}}$$

is the Exp-E PDF with power parameter η. Equation (7) reveals that the PDF of the LBHE is a linear combination of Exp-E density. Thus, some of its structural properties can be immediately obtained from well-established properties of the Exp-E distribution (see Section 3). The properties of the Exp-E distribution have been studied by many authors. Many extensions for the E model can be cited by Korkmaz and Genç (2015), Yousof et al., (2015), Afify et al. (2016a,b,c), Nofal et al. (2016), Yousof et al. (2016), Aryal et al. (2017), Merovci et al. (2017), Korkmaz and Yousof (2017), Aryal and Yousof (2017), Afify et al. (2017), Cordeiro et al. (2017a,b), Nofal et al. (2017), Yousof et al. (2017a,b,c), Brito et al. (2017), Alizadeh et al., (2018a,b,c,d,e), Korkmaz et al. (2018), Hamedani et al. (2018a,b) and Yousof et al., (2018a,b,c), among others.

2.3 Graphical Presentation

In this subsection, we show the flexibility of the LBHE model using the graphical presentation for the PDF and HRF as follows

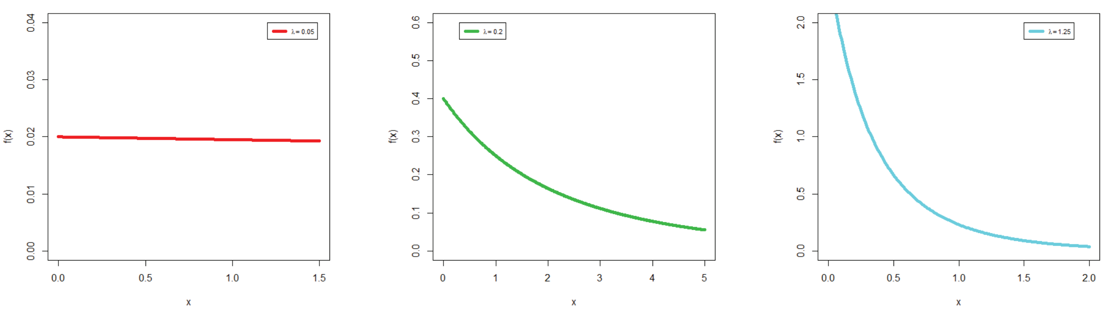

Figure 1. The PDF of the LBHE model

From Figure 1, we conclude that the PDF of the LBHE model exhibits the constant and the decreasing shapes so the new model can be used as a suitable model to fit the uniform and the right skewed data. From Figure 2, we conclude that the HRF of the new model also exhibits the constant and the decreasing shapes.

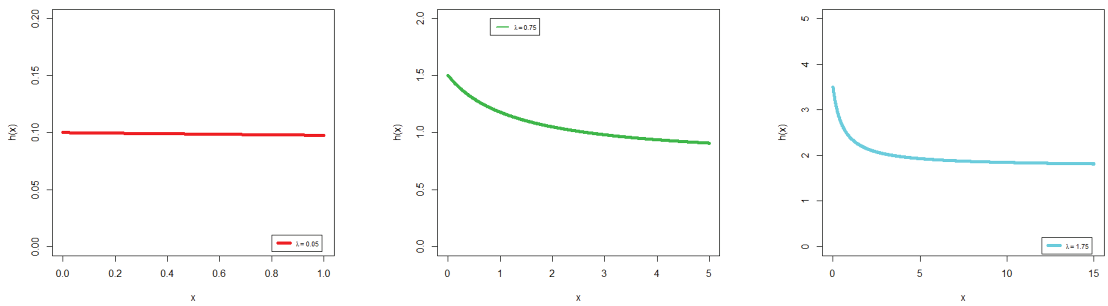

Figure 2. The HRF of the LBHE model

3. Properties

3.1 Asymptotics

Let $a = \inf\{x|F(x) > 0\}$ the asymptotics of CDF, PDF and HRF as $x \to a$ are given by

$$F_\lambda(x) \sim 1 - \exp(-\lambda x), \text{as} \quad x \to a,$$

$$f_\lambda(x) \sim \exp(-\lambda x)\lambda, \text{as} \quad x \to a,$$

and

$$h_\lambda(x) \sim \exp(-\lambda x)\lambda, \text{ as} \quad x \to a.$$

The asymptotics of CDF, PDF and HRF as $x \to \infty$ are given by

$$[1 - F_\lambda(x)] \sim -(-\lambda x)^{-1}\exp(-\lambda x), \text{ as} \quad x \to \infty,$$

$$f_\lambda(x) \sim \lambda\,(\lambda x)^{-2}\exp(-\lambda x)(1 - \lambda x), \text{ as} \quad x \to \infty,$$

and

$$h_\lambda(x) \sim \lambda(1 - \lambda x)(\lambda x)^{-1}, \text{ as} \quad x \to \infty.$$

The effect of the parameters on tails of distribution can be evaluated by means of above equations.

3.2 Moments

The $r^{(th)}$ moment (about the zero) of X is given by

$$\mu'_r = \mathbf{E}(X^r) = \int_{-\infty}^{\infty} f(x)\,x^r\,dx.$$

Then, we have

$$\mu'_r = \Gamma(1 + r)\sum_{k,h=0}^{\infty} \zeta_{k,h}^{(k+1,r)}\,|_{(r>-1)}, \tag{8}$$

where

$$\zeta_{k,h}^{(k+1,r)} = d_{k+1}\;\zeta_h^{(k+1,r)}$$

and

$$\zeta_h^{(k+1,r)} = (k + 1)\lambda^{-r}(-1)^i(h + 1)^{-(r+1)}\binom{k}{h}$$

Setting $r = 1, 2, 3, 4$ in (8), we have the mean of X , μ'_2, μ'_3 and μ'_4 respectively as

$$\mathbf{E}(X) = \mu'_1 = \Gamma(2)\sum_{k,h=0}^{\infty} \zeta_{k,h}^{(k+1,1)}\,|_{(r>-1)},$$

$$\mathbf{E}(X^2) = \mu'_2 = \Gamma(3)\sum_{k,h=0}^{\infty} \zeta_{k,h}^{(k+1,2)}\,|_{(r>-1)},$$

$$\mathbf{E}(X^3) = \mu'_3 = \Gamma(4)\sum_{k,h=0}^{\infty} \zeta_{k,h}^{(k+1,3)}\,|_{(r>-1)},$$

and

$$\mathbf{E}(X^4) = \mu_4' = \Gamma(5) \sum_{k,h=0}^{\infty} \zeta_{k,h}^{(k+1,4)} |_{(r>-1)},$$

which can be used to get the moment about the mean. The skewness $\left(\sqrt{\beta_1}\right)$ and kurtosis (β_2) measures also can be calculated from the ordinary moments using the well-known relationships. For the skewness and kurtosis coefficients, we have

$$\sqrt{\beta_1} = \sqrt{\mu_3^2/\mu_2^3} \quad \text{and} \quad \beta_2 = \mu_4/\mu_2^2,$$

respectively. Numerically, we prove that the LBHE distribution provides better fits than five models with (two, three, four and five parameters) (see Section 6) so the new model is a good alternative for modeling cancer patients data. Further, the LBHE density can be only right-skewed (see Figure1 and column 4 in Table 1). Whereas the LBHE HRF can be monotonically decreasing and constant (see Figure 2). The skewness of the LBHE distribution can range in the interval $(0.45, 2.53)$, whereas the kurtosis of the LBHE distribution varies only in the interval $(1.5, 13)$ also that the mean of X decreases as λ increases, the skewness is always positive (see Table 1).

Table 1. Mean, variance, skewness and kurtosis of the LBHE distribution with different values of λ

λ	*Mean*	*Variance*	*Skewness*	*Kurtosis*
0.001	596.3474	451675.1	2.532964	13.18679
0.01	59.63474	4516.751	2.532964	13.18679
0.1	5.963474	45.16751	2.532964	13.18679
1	0.5963474	0.4516751	2.532964	13.18679
5	0.1192695	0.01806701	2.532963	13.18679
10	0.0596347	0.004516751	2.532964	13.18679
20	0.0298174	0.001129188	2.532961	13.18681
50	0.0119269	0.00018067	2.532545	13.18857
100	0.00596347	$4.516812\times e^{-5}$	2.535457	13.16063
200	0.0029817	$1.069889\times e^{-5}$	3.028092	14.2233
400	0.0014908	$2.666303\times e^{-6}$	1.802343	17.63329
800	0.0007454	$1.372808\times e^{-6}$	0.4465059	1.551292
900	0.00066261	$1.148517\times e^{-6}$	0.6865397	1.610696
1000	0.00066261	$8.63845\times e^{-7}$	0.9798369	2.012945

3.3 Generating Function

The moment generating function (mgf) $M_X(t) = E\left(e^{tX}\right)$ of X. Clearly, the first one can be derived using Equation (7) as

$$M_X(t) = \Gamma(1+r) \sum_{k,r,h=0}^{\infty} \zeta_{k,r,h}^{(k+1,r)}|_{(r>-1)},$$

where

$$\zeta_{k,r,h}^{(k+1,r)} = d_{k+1}\,[t^r/r!]\,\zeta_h^{(k+1,r)}.$$

3.4 Order Statistics

Suppose $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$, is a random sample (RS) from the LBHE model. Let $X_{i:n}$ denote the $i^{(th)}$ order statistic. The PDF of $X_{i:n}$ can be expressed as

$$f_{i:n}(x) = \sum_{j=0}^{n-i} B^{-1}(i, n-i+1)(-1)^j \binom{n-i}{j} f(x)\,F(x)^{j+i-1}.$$

Following Gradshteyn and Ryzhik (2000) the result 0.314 for a power series raised to a positive integer n (for $n \geq 1$)

$$\left(\sum_{i=0}^{\infty} a_i u^i\right)^n = \sum_{i=0}^{\infty} c_{n,i}\, u^i,$$

where the coefficients $c_{n,i}\ \forall\ i = 1, 2, \ldots$ are determined from the recurrence equation (with $c_{n,0} = a_0^n$)

$$c_{n,i} = (i\,a_0)^{-1}\sum_{m=1}^{i}[m(n+1)-i]a_m\,c_{n,i-m}\ .$$

The $i^{(th)}$ order statistic of the LBHE model can be expressed as

$$f_{i:n}(x) = \sum_{\tau,k=0}^{\infty} b_{\tau,k}\,\pi_{\tau+k+1}(x), \quad (10)$$

where $\pi_{\tau+k+1}(x)$ denotes the Exp-E density function with parameter $(\tau + k + 1)$ and

$$b_{\tau,k} = [n!\,(\tau+1)\,(i-1)!\,d_{\tau+1}/(\tau+k+1)]\sum_{j=0}^{n-i}\left\{\varsigma_{j+i-1,k}(-1)^j\,/\,[(n-i-j)!\,j!]\right\},$$

and $d_{\tau+1}$ is given in subsection 3.2 and the quantities $\varsigma_{j+i-1,k}$ can be determined with $f_{j+i-1,0} = d_0^{j+i-1}$ and recursively for $k \geq 1$

$$f_{j+i-1,k} = (k\,d_0)^{-1}\sum_{m=1}^{k}[m\,(j+i)-k]\ d_m\,\varsigma_{j+i-1,k-m}.$$

the moments of $X_{i:n}$ can be expressed as

$$E\left(X_{i:n}^q\right) = \Gamma(1+q)\sum_{\tau,k,h=0}^{\infty} b_{\tau,k}\ \ \zeta_h^{(\tau+k+1,q)}|_{(q>-1)}, \quad (11)$$

where

$$\zeta_{\tau,k,h}^{(\tau+k+1,q)} = b_{\tau,k}\ \ \zeta_h^{(\tau+k+1,q)}.$$

3.5 Quantile Spread Ordering

The quantile spread (QS) of the RV W ~LBHE(λ) having CDF (2) is given by

$$QS_W(\xi)\,|_{(\xi\in(0.5,1))} = F^{-1}(\xi) - F^{-1}(1-\xi),$$

which implies

$$QS_W(\xi) = \left[S^{-1}(1-\xi)\right] - \left[S^{-1}(\xi)\right],$$

where

$$F^{-1}(\xi) = S^{-1}(1-\xi) \text{ and } S(w) = 1 - F(w)$$

is the survival function. The QS of a any distribution describes how the probability mass is placed symmetrically about its median and hence it can be used to formalize concepts such as peakedness and tail weight traditionally associated with the kurtosis. So, it allows us to separate concepts of the kurtosis and peakedness for asymmetric models.

Let W_1 and W_2 be two RVs following the LBHE model with QSs QS_{W_1} and QS_{W_2}, respectively. Then W_1 is called smaller than W_2 in quantile spread order, denoted as $W_1 \leq_{(QS)} W_2$, if

$$QS_{W_1}(\xi)\,|_{(\xi\in(0.5,1))} \leq QS_{W_2}(\xi)\,.$$

Following are properties of the QS order can be obtained:

- The order $\leq_{(QS)}$ is *location-free*

$$W_1 \leq_{(QS)} W_2 \text{ if } (W_1 + \mathbf{C}) \leq_{(QS)} W_2\,|_{(\mathbf{C}\in\mathbb{R})}.$$

- The order $\leq_{(QS)}$ is *dilative*

$$W_1 \leq_{(QS)} \mathbf{C}W_1 \text{ whenever } \mathbf{C} \geq 1 \text{ and } W_2 \leq_{(QS)} \mathbf{C}W_2\,|_{(\mathbf{C}\geq 1)}.$$

• Let F_{W_1} and F_{W_2} be symmetric, then

$$W_1 \leq_{(QS)} W_2 \text{ if, and only if } F^{-1}_{W_1}(\xi) \leq F^{-1}_{W_2}(\xi)\,|_{(\xi\in(0.5,1))}.$$

• The order $\leq_{(QS)}$ implies ordering of *the mean absolute deviation around the median*, say $\psi(W_i)$, $i = 1, 2$.

$$\psi(W_1) = \mathbf{E}\left[|W_1 - Median(W_1)|\right]$$

and

$$\psi(W_2) = \mathbf{E}\left[|W_2 - Median(W_2)|\right],$$

where

$$W_1 \leq_{(QS)} W_2 \text{ implies } \psi(W_1) \leq_{(QS)} \psi(W_2).$$

Finally

$$W_1 \leq_{(QS)} W_2 \text{ if, and only if } -W_1 \leq_{(QS)} -W_2.$$

3.6 Moments of Residual Life

The $n^{(th)}$ moment of the residual life, say

$$\tau_n(t) = \mathbf{E}[(X-t)^n\,|^{(n=1,2,...)}_{(X>t)}],$$

the $n^{(th)}$ moment of the residual life of X is given by

$$\tau_n(t) = R^{-1}(t)\int_t^\infty (x-t)^n dF(x),$$

for the LBHE model

$$\tau_n(t) = \Gamma\left(1+n, \frac{\lambda}{t}\right)R^{-1}(t)\sum_{k,h=0}^{\infty}\zeta_h^{*(k+1,n)},$$

where

$$\zeta_h^{*(k+1,n)} = d^*_{k+1}\ \zeta_h^{(k+1,n)},$$

$$d^*_{k+1} = d_{k+1}\sum_{r=0}^{n}\binom{n}{r}(-t)^{n-r} \text{ and } R(t) = 1 - F(t).$$

3.7 Moment of the Reversed Residual Life

The $n^{(th)}$ moment of the reversed residual life, say

$$T_n(t) = \mathbf{E}\left[(t-X)^n\,|^{(n=1,2,...)}_{(X\leq t\ ,\ t>0)}\right]$$

uniquely determines the CDF, the we have

$$T_n(t) = F^{-1}(t)\int_0^t (t-x)^n dF(x).$$

The $n^{(th)}$ moment of the reversed residual life of X becomes

$$T_n(t) = \gamma\left(1+n, \frac{\lambda}{t}\right)F^{-1}(t)\sum_{k,h=0}^{\infty}\zeta_h^{**(k+1,n)},$$

where

$$\gamma(\eta, x) = \Gamma_x(\eta) = \int_0^x t^{\eta-1}\exp(-t)\,dt = \sum_{\tau=0}^{\infty}\frac{(-1)^\tau x^{\eta+\tau}}{\tau!\,(\eta+\tau)},$$

$$\zeta_h^{**(k+1,n)} = d^{**}_{k+1}\ \zeta_h^{(k+1,n)},$$

and

$$d^{**}_{k+1} = d_{k+1}\sum_{r=0}^{n}(-1)^r\binom{n}{r}t^{n-r}.$$

4. Estimation

Let $x_1, \ldots, x_n$ be a random sample from the LBHE distribution the log-likelihood function is given by

$$\ell(\mathbf{\Phi}) = n \log \lambda - \lambda \sum_{i=1}^{n} x_i - 2 \sum_{i=1}^{n} \log(1 + \lambda x_i) + \sum_{i=1}^{n} \log[2 + \lambda x_i] \tag{12}$$

The maximum likelihood estimation (MLE) estimate $\widehat{\lambda}$ of λ is the solution of the non-linear equation

$$0 = \lambda^{-1} - \overline{x} - \frac{2}{n} \sum_{i=1}^{n} [x_i / (1 + \lambda x_i)] + \frac{1}{n} \sum_{i=1}^{n} [x_i / (2 + \lambda x_i)], \tag{13}$$

where $\bar{x}$ is the sample mean. To obtain the MLE of λ, we can maximize (12) directly with respect to λ or we can solve the non-linear equation (13). Note that maximum likelihood estimation (MLE) of the λ cannot be solved analytically; numerical iteration techniques, such as the Newton-Raphson algorithm, are adopted to solve the log-likelihood equation for which (12) is maximized.

5. Simulation Study

Here, in this Section, we will perform the simulation study using the LBHE distribution. To see the performance of MLEs of this distribution, we generate 1,000 samples of sizes 20, 50, 100 and 300 from LBHE using inverse of the its CDF. We also compute the biases and mean squared errors (MSE) of the MLEs with

$$Bias_{(\hat{\lambda})} = \sum_{i=1}^{1000} \frac{1}{1000} \left(\hat{\lambda}_i - \lambda\right)$$

and

$$MSE_{(\hat{\lambda})} = \sum_{i=1}^{1000} \frac{1}{10000} \left(\hat{\lambda}_i - \lambda\right)^2$$

respectively. To obtain the inverse of the CDF, We used the uniroot routine in the R programme for the random generation and used the optim routine for MLEs. Results of the simulation are reported in Table 2. From Table 2, we observe that when the sample size increases, biases$(\hat{\lambda})$ and MSEs$(\hat{\lambda})$ decrease in all the cases.

Table 2. Emprical mean, Bias and MSE of the estimator $\widehat{\lambda}$

Parameter		$n = 20$			$n = 50$	
λ	$\widehat{\lambda}$	$Bias_{(\hat{\lambda})}$	$MSE_{(\hat{\lambda})}$	$\widehat{\lambda}$	$Bias_{(\hat{\lambda})}$	$MSE_{(\hat{\lambda})}$
0.20	0.21113	2.24973	2.99110	0.20323	1.40344	2.11367
0.50	0.50242	2.25244	2.77051	0.55121	1.40563	2.13818
1.00	1.03436	2.30364	2.58224	1.00893	1.40970	1.14816
1.50	1.50177	2.31055	2.75542	1.53972	1.41456	2.15540
3.00	3.03018	2.41151	2.92255	3.19818	1.51653	2.25579
5.00	5.00915	2.69922	2.45917	5.01252	1.57917	2.39152
50.0	50.23429	2.99170	2.93710	50.01789	1.99981	2.72511
		$n = 100$			$n = 300$	
0.20	0.20093	0.08413	1.01342	0.20019	0.00172	0.00201
0.50	0.51012	0.08528	1.02097	0.50065	0.00173	0.00213
1.00	1.01210	0.11109	1.04815	1.02133	0.00186	0.00215
1.50	1.50139	0.12323	1.10547	1.50102	0.00198	0.00310
3.00	3.01898	0.17294	1.25575	3.00198	0.00226	0.00394
5.00	5.02313	0.22172	1.25430	5.01231	0.00229	0.00653
50.0	50.03223	0.28711	1.04168	50.00011	0.00331	0.00931

6. The LBHE Distribution for Modeling the Cancer Patients Data

We provide an application to show empirically the potentiality of the new model. In order to compare the fits of the LBH-E distribution with other competing distributions, we consider the Cramér-von Mises (W^*) and the Anderson-Darling (A^*) statistics. These two statistics are widely used to determine how closely a specific CDF fits the empirical distribution of a given data set. These statistics are given by

$$W^* = (1 + 1/2n)\left[(1/12n) + \sum_{j=1}^{n} [z_i - (2j-1)/2n]^2\right],$$

and

$$A^* = \left\{n + \frac{1}{n}\sum_{j=1}^{n}(2j-1)\log\left[z_i\left(1 - z_{n-j+1}\right)\right]\right\}\left(1 + \frac{9}{4n^2} + \frac{3}{4n}\right)$$

respectively, where $z_i = F\left(y_j\right)$ and the y_j's values are the ordered observations. The smaller these statistics are, the better the fit. The required computations are carried out using the R software. The MLEs and the corresponding standard errors (in parentheses) of the model parameters are given in Table 2. The numerical values of the statistics W* and A* are listed in Table3. The Estimated PDF, CDF, HRF and P-P plot of the new model are displayed in Figure 3. The cancer patients data set represents the remission times (in months) of a random sample of 128 bladder cancer patients as reported in Lee and Wang (2003). This data is given in Appendix A. We will compare the fits of the LBHE distribution with other competitive models, namely: the Weibull W (Weibull, 1951), the etransmuted additive Weibull distribution (TA-W) (Elbatal and Aryal, 2013), the Modified beta-Weibull (MB-W) (Khan, 2015), the transmuted modified-Weibull (TM-W) (Khan and King, 2013) and the exponentiated transmuted generalized Rayleigh (ETGR) (Afify et al., 2015) distributions.

Table 3. MLEs (standard errors in parentheses) for cancer patients data

Distribution	Estimates
LBHE$_{(\lambda)}$	0.0624
	(0.0061)
W$_{(\alpha,\beta)}$	9.5593, 1.0477
	(0.853), (0.068)
TM-W$_{(\alpha,\beta,\eta,\lambda)}$	0.1208, 0.8955, 0.0002, 0.2513
	(0.024), (0.626), (0.011), (0.407)
MB-W$_{(\alpha,\beta,a,b,c)}$	0.1502, 0.1632, 57.4167, 19.3859, 2.0043
	(22.4371), (0.0441), (37.3170), (13.492), (0.7891)
TA-W$_{(\alpha,\beta,\eta,\theta,\lambda)}$	0.1139, 0.9722, 3.0935×10^{-5}, 1.0065, −0.1634
	(0.032), (0.125), (6.106×10^{-3}), (0.035), (0.28)
ETG-R$_{(\alpha,\beta,\delta,\lambda)}$	7.37623, 0.0473, 0.04941, 0.1182
	(5.389), (3.965×10^{-3}), (0.036), (0.26)

Table 4. W^* and A^* for cancer patients data

Distribution	W^*	A^*
LBH-E$_{(\lambda)}$	**0.07000**	**0.43069**
W$_{(\alpha,\beta)}$	0.10553	0.66279
TM-W$_{(\alpha,\beta,\eta,\lambda)}$	0.12511	0.76028
MB-W$_{(\alpha,\beta,a,b,c)}$	0.10679	0.72074
TA-W$_{(\alpha,\beta,\eta,\theta,\lambda)}$	0.11288	0.70326
ETG-R$_{(\alpha,\beta,\delta,\lambda)}$	0.39794	2.36077

Based on Table 4 we note that the new model is much better than the TM-W, TA-W, MB-W, ETG-R and W, models with small values for W^* and A^* in modeling cancer patients data.

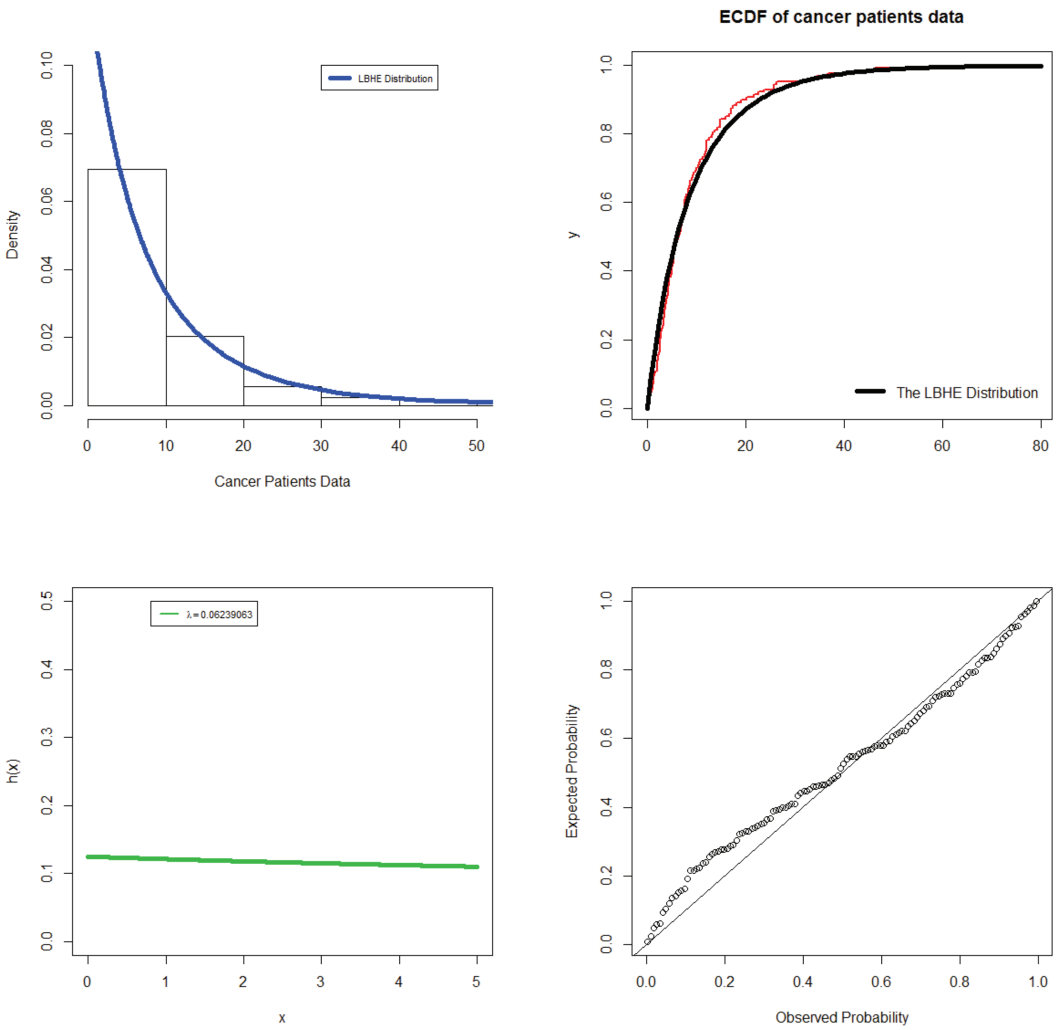

Figure 3. Estimated PDF, CDF, HRF and P-P plot for the cancer patients data

7. Concluding Remarks

In this work, we introduced a new one-parameter exponential distribution. Some of its structural properties are derived. The maximum likelihood method is used to estimate the model parameters by means of numerical Monte Carlo simulation study. The justification for the practicality of the LBHE lifetime model is based on the wider use of the exponential model. The new model can be viewed as a mixture of the exponentiated exponential distribution. It can also be considered as a suitable model for fitting right skewed data. We prove empirically the importance and flexibility of the LBHE model in modeling cancer patients data, the new model provides adequate fits as compared to other related models with small values for W^* and A^* is much better than the Modified beta-Weibull, Weibull, exponentiated transmuted generalized Rayleig, the transmuted modified-Weibull, and transmuted additive Weibull models in modeling cancer patients data. We are also motivated to introduce this new model because it has only one parameter and we can generate some new families based on it such as the the odd Burr-Hatke exponential-G family of distributions, the logarithmic Burr-Hatke exponential-G family of distributions and the generalized Burr-Hatke exponential-G family of distributions, among others. As a future work we will study these new families introduced in Subsection 2.1 in a separate articles.

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Appendix A:

{0.080, 2.09, 3.480, 4.870, 6.940, 8.660, 13.110, 23.63, 0.20, 2.230, 3.52, 4.980, 6.97, 9.020, 13.290, 0.40, 2.260, 3.57, 5.06, 7.090, 9.22, 13.80, 25.740, 0.50, 2.46, 3.640, 5.09, 7.260, 9.47, 14.240, 25.820, 0.510, 2.54, 3.70, 5.170, 5.41, 7.62, 10.750, 16.62, 43.010, 1.19, 2.750, 4.26, 5.410, 7.63, 17.120, 46.12, 1.260, 2.83, 4.330, 5.49, 7.660, 11.25, 17.140, 79.05, 1.350, 2.87, 5.620, 7.87, 11.640, 17.36, 1.40, 3.020, 4.34, 5.710, 7.28, 9.74, 14.760, 26.31, 0.810, 2.62, 3.82, 5.320, 7.32, 10.060, 14.77, 32.150, 2.64, 3.88, 5.320, 7.39,10.34, 14.830, 34.26, 0.90 , 2.690, 4.180, 5.340, 7.590, 10.66, 15.960, 36.660, 1.05, 2.690, 4.23, 7.930, 11.79, 18.10, 1.460, 4.40, 5.850, 8.26, 11.980, 19.13, 1.760, 3.25, 4.50, 6.250, 8.37, 12.02, 2.020, 3.310, 4.51, 6.54, 8.53, 12.030, 20.28, 2.02, 3.360, 6.76, 12.070, 21.730, 2.07, 3.360, 6.930, 8.650, 12.630, 22.690}

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