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Unit Roots in Time Series with Changepoints

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Abstract

Many financial time series are nonstationary and are modeled as ARIMA processes; they are integrated processes (I(n)) which can be made stationary (I(0)) via differencing n times. I(1) processes have a unit root in the autoregressive polynomial. Using OLS with unit root processes often leads to spurious results; a cointegration analysis should be used instead. Unit root tests (URT) decrease spurious cointegration. The Augmented Dickey Fuller (ADF) URT fails to reject a false null hypothesis of a unit root under the presence of structural changes in intercept and/or linear trend. The Zivot and Andrews (ZA) (1992) URT was designed for unknown breaks, but not under the null hypothesis. Lee and Strazicich (2003) argued the ZA URT was biased towards stationarity with breaks and proposed a new URT with breaks in the null. When an ARMA(p,q) process with trend and/or drift that is to be tested for unit roots and has changepoints in trend and/or intercept two approaches that can be taken: One approach is to use a unit root test that is robust to changepoints. In this paper we consider two of these URT's, the Lee-Strazicich URT and the Hybrid Bai-Perron ZA URT(Herranz, 2016.) The other approach we consider is to remove the deterministic components with changepoints using the Bai-Perron breakpoint detection method (1998, 2003), and then use a standard unit root test such as ADF in each segment. This approach does not assume that the entire time series being tested is all I(1) or I(0), as is the case with standard unit root tests. Performances of the tests were compared under various scenarios involving changepoints via simulation studies. Another type of model for breaks, the Self-Exciting-Threshold-Autoregressive (SETAR) model is also discussed.

Keywords: Unit root, structural breaks, changepoints, time series

1. Introduction

1.1 Time Series and ARIMA Models

A time series (TS) is an ordered sequence of values of a variable at equally spaced time intervals,

$$\{x_t\} = \{x_1, x_2, ..., x_n\}. \tag{1}$$

An auto-regressive integrated moving average (ARIMA(p,d,q)) model is a model of a time series x_t where we first difference the series d-times, resulting in $\Delta^d(x_t)$, and then we build an ARMA(p,q) model from the differenced series. Here the "I" stands for integrated. An ARMA(p,q) model is defined in terms of its lagged values x_t and its current and past innovations ϵ_t as :

$$x_t = \sum_{i=1}^{p} \phi_i x_{t-i} + \sum_{i=1}^{q} \theta_i \epsilon_{t-i} + \epsilon_t. \tag{2}$$

1.2 Stationary and Nonstationary Time Series

A weakly stationary time series is defined as having a constant mean, and an autocovariance function $\gamma(s, t)$ that depends on s and t only through their difference $|s - t|$.

A strictly stationary time series is one where the joint probability distribution does not change with time. A stationary time series is labeled I(0) if it is integrated order 0. A time series integrated order p, I(p), needs to be differenced p times to become stationary.

1.3 Unit Roots

The characteristic polynomial of the AR(p) part of Model (2) is defined as:

$$\phi(z) = 1 - \phi_1 z - ... - \phi_p z^p. \tag{3}$$

A unit root process is an ARMA(p,q) process (or AR(p) process) with 1 as a root of the characteristic polynomial equation. Time series with unit roots are non-stationary processes. In the case of an AR(1) process if $|\phi_1| = 1$ there will be a unit root.

A unit root process is an I(1) process; an example is a Gaussian random walk:

$$x_t = x_{t-1} + \epsilon_t; \quad x_t = \sum_{i=1}^{t} \epsilon_i; \quad x_0 = 0; \quad \Delta(x_t) = \epsilon_t\ ; \quad \epsilon_t \sim N(0,1). \tag{4}$$

As detailed in (Chan, Ngai Hang, 2010) if an AR(p) process has all of its characteristic polynomial roots with an absolute value greater than one, then such a process is defined to be causal, and will also be stationary.

Many financial time series, such as asset prices are modeled as unit root processes. As was originally proposed in the seminal work by (Fama, 1965), the logarithm of stock prices is often modeled as a random walk: $\log(S_t) = \log(S_{t-1}) + \epsilon_t$ which is equivalent to modeling log returns as a stationary process: $\log\left(\frac{S_t}{S_{t-1}}\right) = \epsilon_t$.

We define a cointegrating relationship between two or more time series each having unit roots (I(1)) if a linear combination exists that is stationary, i.e. I(0).

Prior to testing for cointegration of two or more series each one must be pretested to ensure they are all I(1) or this can lead to spurious cointegration.

2. Statistical Tests for Unit Roots Without Changepoints

See (Herranz, E., 2017) for an up to date literature review of unit root tests.

2.1 Unit Root Test Null and Alternative Hypotheses

Unit root tests of a time series address the null hypothesis that the series is *unit root* nonstationary(I(1)). The alternative hypothesis is that the time series is weakly stationary (I(0)). Consider the following AR(1) model with deterministic components of an intercept and a linear trend:

$$x_t = \phi_1 x_{t-1} + \epsilon_t; \quad y_t = \beta_0 + \beta_1 t + x_t. \tag{5}$$

In the AR(1) case the null hypothesis consists of $\phi_1 = 1$, and the alternative hypothesis that we will consider in this paper is $\phi_1 \neq 1$. Most unit root tests in the literature consider the alternative hypothesis to be $|\phi_1| < 1$.

2.2 Augmented Dickey Fuller Unit Root Test

(Said, S. E. and Dickey, D. A., 1984) extended the Dickey Fuller unit root test for ARMA models and not just AR(p) models; this is known as the Augmented Dickey Fuller(ADF) unit root test and is one of the most commonly used in the literature (Choi, In, 2010, p. 33). The ADF test regression is fitted using OLS:

$$\triangle y_t = \alpha + \delta t + \beta y_{t-1} + \sum_{i=1}^{n} \gamma_i \triangle y_{t-i} + \epsilon_t, \tag{6}$$

where $\triangle$ is the difference operator and ϵ_t represent 0-mean white-noise innovations. Under the null hypothesis y_t is considered to be I(1) which is equivalent to $\triangle y_t$ being I(0) in which case β would be zero. The test statistic is the standard regression t-statistic $t_\beta = \frac{\hat{\beta}}{s.e.(\hat{\beta})}$, where $\hat{\beta}$ is the standard coefficient estimate as derived using ordinary least squares and T is the time series length:

$$\hat{\beta} = \frac{\sum_{t=1}^{T} y_t y_{t-1}}{\sum_{t=1}^{T} y_{t-1}^2}. \tag{7}$$

The asymptotic quantiles of this test statistic are a functional of Brownian motions as detailed in Equation (8):

$$t_{\beta=1} \underset{D}{\Rightarrow} \frac{\int_0^1 W_t dW_t}{\left(\int_0^1 (W_t)^2\, dr\right)^{0.5}}. \tag{8}$$

This expression does not have a closed form solution, but it can be used to derive critical values via Monte Carlo simulation.

A normalized bias test statistic ($\hat{\delta}$) can be used as well:

$$\hat{\delta} = T(\hat{\beta} - 1). \tag{9}$$

The asymptotic quantiles of the normalized bias test statistic are also functionals of Brownian motions:

$$\hat{\delta}_{\beta=1} \underset{D}{\Rightarrow} \frac{\int_0^1 W_t dW_t}{\int_0^1 (W_t)^2 dr}. \tag{10}$$

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3. Removing Structural Changes in Intercept and Trend

3.1 Estimating Changepoints

The (Bai, J. and Perron, P., 1998) procedure for estimating structural breaks/change points in a linear model is implemented in the *breakpoints()* function of the *strucchange* R package (Zeileis,Achim and Leisch,Friedrich and Hornik,Kurt and Kleiber,Christian , 2002, p. 12).

This methodology assumes an underlying linear model with a dependent one dimensional variable y_t, a $p \times 1$ covariates vector $\boldsymbol{x_t}$ with corresponding coefficient vector $\boldsymbol{\beta}$, and the innovations ϵ_t:

$$y_t = \boldsymbol{x_t \beta} + \boldsymbol{\epsilon_t}. \tag{11}$$

If there are m change points in the coefficient this implies $m + 1$ regimes. Equation (11) can be rewritten as:

$$y_t = \boldsymbol{x_t \beta_j} + \boldsymbol{\epsilon_t} \; (j = T_{j-1} + 1, ..., T_j, j = 1, ..., m + 1). \tag{12}$$

The underlying idea is solving the problem by dividing it into independent optimally solvable sub-problems, whose solutions can be combined to solve the larger problem. In this case a triangular residual sum of squares (RSS) matrix is computed and stored in memory which can be reused over and over again to derive the residual sum of squares for a segment starting at observation t and ending at t' with $t < t'$

This approach is considerably faster than the brute force approach of computing the RSS for all possible sub-segments. The (Bai, J. and Perron, P., 1998) algorithm uses only $O(T^2)$ least squares operations for a number m of change points. The brute force approach would require $O(T^m)$ least squares operations.

3.2 Hybrid Bai-Perron ADF I(0)/I(1) Segment Procedure

(Herranz, Edward, 2016) proposed using the (Bai, J. and Perron, P., 1998) methodology of estimating structural break date/s based on finding the model specification that minimizes the RSS via a dynamic programming approach, and then to use the ADF URT to test each section to determine if it is likely I(0) or I(1). The following Regression Model (13) with the (Bai, J. and Perron, P., 1998) procedure which will be used to estimate structural breaks in the coefficients μ, μ_t, ϕ_1:

$$x_t = \mu + \mu_t t + \phi_1 x_{t-1} + \epsilon_t. \tag{13}$$

Figure 1 shows in red vertical bars the estimated break-points using this procedure with Model (13) on historical GE prices between 1-September-2016 and 1-September-2017. When the ADF URT was run on each of the five segments, the null hypothesis of a unit root was never rejected.

Table 1 summarizes the results of various simulations run with this methodology, as well as running the ADF test on the entire time series. Each time series has a length $l = 1000$, with $m = 100$ Monte Carlo simulations and a significance level of $\alpha = 0.05$ of the ADF tests used in each segment, and for the entire series. The following data generating process (DGP) was used with a single break time in all coefficients:

$$y_t = \phi_1^A y_{t-1} I\{t \leq T_B\} + \phi_1^B y_{t-1} I\{t > T_B\}. \tag{14}$$

We can see that this new Hybrid Bai-Perron-ADF testing procedure is sensitive to the location of the structural break. When compared to using a single unit root test on the entire time series, this approach can be significantly more accurate, as can be seen in the case with $\phi_1^A = 1$ and $\phi_1^B = 0.9$ where we can seem make a greater error if we assume that the entire series is homogeneous I(1) or I(0).

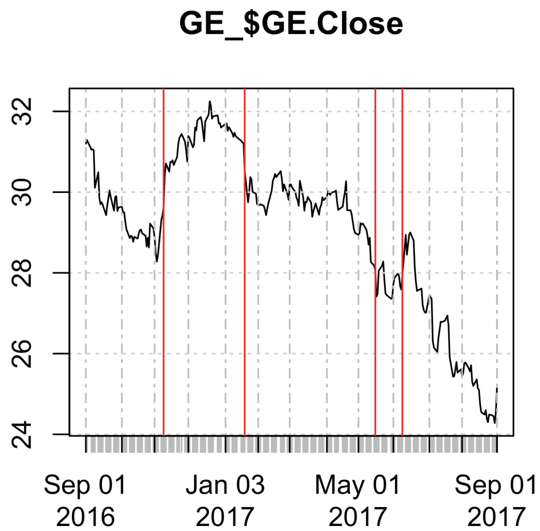

Figure 1. Estimated Changepoints in GE Prices.

Table 1. Proportions of Failures to Reject Null I(1) on a AR(1) Series with a Break in ϕ_1, intercept and linear trend with $l = 1000, m = 100$ and $\alpha = 0.05$

Break Propn	ϕ_1^A	ϕ_1^B	μ^A	μ^B	μ_t^A	μ_t^B	$\overline{\Gamma^{A}_{ADF} > CV_\alpha}$	$\overline{\Gamma^{B}_{ADF} > CV_\alpha}$	$\overline{\Gamma^{All}_{ADF} > CV_\alpha}$
0.50	0.000	0.000	10	-30	5	-2	0.00	0.00	1.00
0.50	1.000	1.000	10	-30	5	-2	0.95	0.97	1.00
0.50	1.000	0.500	10	-30	5	-2	0.96	0.00	1.00
0.50	**0.900**	**1.000**	10	-30	5	-2	**0.04**	0.97	1.00
0.50	**1.000**	**0.900**	10	-30	5	-2	0.95	**0.00**	1.00
0.50	1.000	0.950	10	-30	5	-2	0.95	0.12	1.00
0.50	1.000	0.980	10	-30	5	-2	0.95	0.65	1.00
0.25	0.000	0.000	10	-30	5	-2	0.00	0.00	0.03
0.25	1.000	1.000	10	-30	5	-2	0.98	0.96	0.02
0.25	1.000	0.500	10	-30	5	-2	0.98	0.00	0.10
0.25	0.900	1.000	10	-30	5	-2	0.53	0.96	0.00
0.25	1.000	0.900	10	-30	5	-2	0.98	0.00	0.00
0.25	1.000	0.950	10	-30	5	-2	0.98	0.01	0.01
0.25	1.000	0.980	10	-30	5	-2	0.98	0.60	0.01

4. Statistical Tests for Unit Roots in the Presence of Structural Changes

4.1 Unit Root Tests with Changepoints

A typical changepoints linear model for unit root testing, allowing breaks in β_0 and β_1 under both alternative and null is as follows:

$$\begin{aligned} x_t &= \phi_1 x_{t-1} + \epsilon_t \\ y_t &= \mu_1 + (\mu_2 - \mu_1)\mathbf{1}\{t > T_B\} + (t - T_B)(\beta_2 - \beta_1)\mathbf{1}\{t > T_B\} + \beta_1 t + x_t. \end{aligned} \tag{15}$$

A common assumption is that the auto-regressive multiplier ϕ_1 is constant. However, what if it exhibits one or more structural breaks? Then the series tested can have one or more changes from I(0) to I(1) or vice versa. We do not consider the case where ϕ_1 is random.

The ADF URT fails to reject a false null hypothesis of a unit root under the presence of structural changes in intercept and/or linear trend.

The Zivot and Andrews (ZA) (1992) URT was designed for unknown breaks, but not under the null hypothesis. (Lee, J. and Strazicich, M. C., 2001) argued the Zivot Andrews URT was biased towards stationarity with breaks and proposed a new URT with breaks in the null.

The new Hybrid Bai Perron Zivot Andrews unit root test proposed by (Herranz, Edward, 2016) also allows breaks under the null.

The one break ((Lee, J. and Strazicich, M.C., 2001) procedure and the two break (Lee, J. and Strazicich, M.C., 2003) procedure allows for the breaks to be determined endogenously from the data and breaks are allowed under both the null and the alternative hypothesis.

4.2 Zivot Andrews Unit Root Test

(Andrews, Donald and Zivot, Eric, 1992) developed a unit root test (ZA) that could deal with breakpoints in the drift and/or linear trend components. The test statistic of the ZA test is the Student t ratio. As detailed in (Pfaff, B., 2008, p. 110):

$$t_{\hat{\alpha}^i}[\hat{\lambda}^i_{\inf}] = \inf_{\lambda \in \Gamma} t_{\hat{\alpha}^i}(\lambda) \text{ for } i = A, B, C. \tag{16}$$

where Γ is a closed subset of (0, 1) and the A model has breaks in intercept,the B model has breaks in linear trend, and the C model has breaks in both intercept and trend. Depending on the model, the test statistic is inferred from one of these three regression models:

$$y_t = \hat{\mu}^A + \hat{\theta}^A DU_t(\hat{\lambda}) + \hat{\beta}^A t + \hat{\alpha}^A t y_{t-1} + \sum_{i=1}^{k} \hat{c}^A_i \Delta y_{t-i} + \hat{\epsilon}_t, \tag{17}$$

$$y_t = \hat{\mu}^B + \hat{\gamma}^B DT^*_t(\hat{\lambda}) + \hat{\beta}^B t + \hat{\alpha}^B t y_{t-1} + \sum_{i=1}^{k} \hat{c}^B_i \Delta y_{t-i} + \hat{\epsilon}_t, \tag{18}$$

$$y_t = \hat{\mu}^C + \hat{\theta}^C DU_t(\hat{\lambda}) + \hat{\beta}^C t + \hat{\alpha}^C t y_{t-1} + \hat{\gamma}^C DT^*_t(\hat{\lambda}) + \sum_{i=1}^{k} \hat{c}^C_i \Delta y_{t-i} + \hat{\epsilon}_t. \tag{19}$$

where $DU_t(\lambda) = 1$ if $t > T\lambda$ and 0 otherwise, and $DT^*_t(\lambda) = t - \lambda T$ for $t > T\lambda$ and 0 otherwise. The null hypothesis of the ZA unit root test does not allow structural breaks. Changepoints in the deterministic components are allowed only under the alternative hypothesis. (Glynn, J. and Perera, N. and Verma, R., 2007) criticize this since if there are breaks under the null ($\phi_1 = 1$) we can mistakenly conclude that the series is stationary (with breaks.)

$$\inf_{\lambda \in \Lambda} t_{\alpha=1} \underset{D}{\Rightarrow} \frac{\int_0^1 W^i(r,\lambda) dW(r)}{\left(\int_0^1 (W^i(\lambda, r))^2 dr\right)^{0.5}}. \tag{20}$$

The asymptotic critical values as $T \Rightarrow \infty$ for models $i = A, B, C$ are given by Equation (20) which is a functional of Brownian motions. The actual critical values used in the ZA test are derived via simulation of this formula. The critical values for the ZA unit root test for the intercept, trend and intercept and trend (both) models are detailed in Table 2. Notice the similarity of Equation (20) with the asymptotic critical value formula for the ADF model in Equation (8). The key difference is that in the ZA unit root test there is a minimization search for the parameter λ, the location in the series

Table 2. ZA Critical Values

Model	0.01	0.05	0.10
Intercept	-5.34	-4.80	-4.58
Trend	-4.93	-4.42	-4.11
Both	-5.57	-5.08	-4.82

Table 3. Proportions of Failures to Reject Null I(1) of URTs on AR(1) Processes with one Structural Break in Intercept(10,-10) and Linear Trend (-10,10) with $l = 1000$, $m = 500$, and $\alpha = 0.05$ and $s = 12345$

ϕ_1	ADF	ERS-Ptest	ERS-DFGLS	ZA
1.010	1.00	1.00	0.98	0.92
1.005	0.96	0.97	0.97	0.05
1.000	0.92	0.90	0.90	0.00
0.990	0.92	0.90	0.90	0.00
0.980	0.91	0.90	0.90	0.00
0.970	0.91	0.90	0.90	0.00
0.960	0.91	0.90	0.90	0.00
0.950	0.91	0.90	0.90	0.00
0.900	0.91	0.90	0.90	0.00
0.000	0.91	0.90	0.90	0.00

where a break can occur such that it results in the minimum possible t-statistic; the ZA regressions allow breaks in the deterministic components and the ADF regression does not.

4.3 Lee-Strazicich Unit Root Test

The one-break (Lee and Strazicich, 2004) procedure and the two-break (Lee, J. and Strazicich, M.C., 2003) unit root tests allow for structural breaks to be determined endogenously from the data and breaks are allowed under both the null and the alternative hypothesis. Consider the single break model:

$$y_t = \delta Z_t + X_t, \quad X_t = \beta X_{t-1} + \epsilon_t. \tag{21}$$

where Z_t contains exogenous variables. The null hypothesis is specified by $\beta = 1$.

“Model A” is the crash model that allows for a single change intercept where $Z_t = [1, t, D_t]'$ where $D_t = 1$ for $t \geq T_B + 1$, and zero otherwise. T_B is the time of the structural break and $\delta = (\delta_1, \delta_2, \delta_3)$

“Model C” allows for a shift in intercept and change in trend slope under the alternative hypothesis where $Z_t = [1, t, D_t, DT_t]'$ where $DT_t = t - T_B$ for $t \geq T_B + 1$, and zero otherwise. T_B is the time of the structural break and $\delta = (\delta_1, \delta_2, \delta_3)$.

The unit root test statistics are derived from the regression:

$$\Delta y_t = \delta' \Delta Z_t + \phi \tilde{S}_{t-1} + u_t, \tag{22}$$

$$\tilde{S}_t = y_t - \Psi_x - Z_t \tilde{\delta} \text{ , t=2,...,T.} \tag{23}$$

where $\tilde{\delta}$ are the coefficients estimated in the regression of Δy_t on ΔZ_t and $\tilde{\Psi}_x$ is the restricted MLE of $\Psi_x = \Psi + X_0$ given by $y_1 - Z_1 \tilde{\delta}$. The unit root null hypothesis consists of $\phi = 0$ and the LM(Lagrange multiplier) t-test statistic $\tilde{\tau}$ = t-statistic testing the null hypothesis $\phi = 0$. As in the case with the ADF test, a correction for auto-correlated innovations is made by adding lagged terms $\Delta \tilde{S}_{t-j}$ where $j = 1, ..., k$. The location of the break T_B is determined by searching across all breakpoints and picking the one with the most negative $\tilde{\tau}$.

$$\inf \tilde{\tau}(\tilde{\lambda}) = \inf_{\lambda} \tilde{\tau}(\lambda) \text{ where } \lambda = T_B/T. \tag{24}$$

If the DGP is that in (21), the ϵ_t satisfy certain regularity conditions, and $T_B/T \rightarrow \lambda$ as $T \rightarrow \infty$, then under the null hypothesis of $\beta = 1$

$$\inf \tilde{\tau}(\tilde{\lambda}) = \inf_{\lambda} \left(-\frac{1}{2} \int_0^1 \underline{V}(r)^2 dr \right)^{-1/2}, \tag{25}$$

where $\underline{V}(r)$ represents a demeaned Brownian bridge.

4.4 Hybrid Bai-Perron Zivot-Andrews Unit Root Test

(Herranz, Edward, 2016) proposed a new URT that allows structural breaks under the null hypothesis, which we refer to here as the Hybrid Bai-Perron-Zivot-Andrews (HBPZA) unit root test. The test is conducted as follows:

1. Given TS, we first use the Bai-Perron break-point estimation procedure in (Bai, J. and Perron, P., 2003) using Regression Model $x_t = \beta_0 + \beta_1 t + \beta_2 x_{t-1} + \epsilon_t$, to detect changes in the coefficients. This divides the TS into $k + 1$ segments given a total of k breakpoints.
2. For each segment within the TS we compute the ZA URT statistic (z_i).
3. A final test statistic is computed by weighing each sub-test statistic by the segment length ($\sum_i^{k+1} w_i z_i$).
4. If there are more than 3 breakpoints estimated, only the first three breakpoints are used to determine 4 segments.

Table 4. Simulated Quantiles of Zivot Andrews(ZA) and Hybrid Bai-Perron-Zivot-Andrews(HBPZA) Unit Root Test Statistics with $l = 1000$ and $m = 5000$

URT	Model	Breaks	ϕ_1	0.01	0.05	0.01
ZA	Trend	0	1.00	-5.00	-4.44	-4.17
ZA	Trend	0	0.95	-6.99	-6.52	-6.30
ZA	Both	0	1.00	-5.61	-5.10	-4.86
ZA	Both	0	0.95	-7.43	-7.03	-6.81
HBPZA	Trend	0	1.00	-4.95	-4.45	-4.21
HBPZA	Trend	0	0.95	-6.98	-6.52	-6.30
HBPZA	Trend	1	1.00	-5.80	-4.49	-4.21
HBPZA	Trend	1	0.95	-6.85	-5.95	-5.60
HBPZA	Trend	2	1.00	-5.87	-4.55	-4.24
HBPZA	Trend	2	0.95	-6.48	-5.63	-5.30

Table 4 shows the estimated 0.01,0.05 and 0.10 quantiles of the ZA and HBPZA test statistics for various combinations with simulations using $m = 5000$ replications with time series of length $l = 1000$ with the DGP in Equation (26) for 1 break, and another similar equation expanded to support two structural breaks. The quantiles in the unit root cases where $\phi_1 = 1$ can be used to derive the critical values.

Table 5 displays the percentage differences of the simulated critical values for ZA and HBPZA tests relative to the published ZA critical values. We can see that the ZA critical values derived from simulation are never more than 2% different from the published ZA critical values. The same is true when the HBPZA technique is used when there are no breaks; this shows empirically that the Bai-Perron breakpoint estimation procedure does not introduce any significant distortions when there are no breaks. We can also see that under 1 break and two breaks, the 0.05 and 0.10 critical values of the HBPZA test do not differ by more than 3% with respect to the corresponding ZA critical values. We see that there is more significant differences in the 0.01 critical values for HBPZA for 1 and 2 breakpoints relative to ZA where they are 18% and 19% respectively.

Table 5. Simulated Critical Values Percentage Difference Relative to ZA Critical Values With Trend Model

URT	Breaks	For 0.01	For 0.05	For 0.10
ZA	0	1	1	2
HBPZA	0	0	1	2
HBPZA	1	18	2	3
HBPZA	2	19	3	3

The Lee-Strazicich allowing both breaks in trend and intercept and the Hybrid Bai-Perron-Zivot-Andrews unit root tests allowing breaks in trend were compared in simulations using the following DGP:

$$\begin{aligned} y_t &= x_t + \alpha_1 + (\alpha_2 - \alpha_1) I\{t > t_u\} + \beta_1 t + (\beta_2 - \beta_1)(t - t_u) I\{t > t_u\} \\ x_t &= \phi_1 x_{t-1} + \epsilon_t \; ; \; \epsilon_t \sim N(0, 1) \; ; \; \mathrm{cor}(\epsilon_t, \epsilon_{t-1}) = 0 \\ t_u &\sim \mathrm{U}(3, l - 2) \; ; \; t = 1, ..., l \end{aligned} \tag{26}$$

The Lee-Strazicich test does not allow only breaks in trend; a more consistent comparison would have been to derive the Hybrid Bai-Perron-Zivot-Andrews unit root test using the "both" test statistic (allowing breaks in intercept and trend.) We note this as work to be done.

Table 6. Proportions of Failures to Reject the Unit Root H_0 on AR(1) with 1 break in intercept and trend with $l = 100$, $m = 1000$, and $\alpha = 5\%, \beta_0^A = 50, \beta_0^B = 1000, \beta_1^A = 1, \beta_1^B = 3$ and break=0.5; Lee-Strazicich implementation by (Gouvea and Teixeira, 2012)

ϕ_1	Hybrid Bai-Perron-Zivot-Andrews	Lee-Strazicich	Breakpoint Proportion
1.0100	0.97	0.99	0.25
1.0000	0.97	0.99	0.25
0.9700	0.96	0.99	0.25
0.9500	0.96	0.98	0.25
0.9000	0.96	0.97	0.25
0.7000	0.87	0.65	0.25
0.5000	0.53	0.20	0.25
0.4000	0.28	0.18	0.25
0.2500	0.05	0.21	0.25
0.1000	0.00	0.25	0.25
0.0000	0.00	0.27	0.25
1.0100	0.96	0.99	0.50
1.0000	0.96	0.99	0.50
0.9700	0.96	0.99	0.50
0.9500	0.96	0.99	0.50
0.9000	0.96	0.97	0.50
0.7000	0.83	0.62	0.50
0.5000	0.37	0.22	0.50
0.4000	0.18	0.18	0.50
0.2500	0.04	0.22	0.50
0.1000	0.01	0.26	0.50
0.0000	0.00	0.29	0.50

Simulations were preformed with AR(1) time series with one structural break using Model (26) with $\alpha_1 = 50, \alpha_2 = 1000, \beta_1 = 1, \beta_2 = 3$ under various levels of ϕ_1. The break times were 25% and 50% of the time series length. Two sets of tests were performed, one for each break ratio, with a time series of length 100 and are summarized in Table 6. We can see that the Lee-Strazicich URT has significantly less statistical power than the HBPZA URT when $\phi_1 < 0.4$.

5. Self Exciting Threshold Autoregressive Models

(Balke, Nathan S. and Fomby, Thomas B., 1997) proposed a general equilibrium model of z_t based on a self exciting threshold auto regressive framework (SETAR) such as the following with a low, middle and high regimes:

$$z_t = \begin{cases} \mu_h + \phi_h z_{t-1} + \epsilon_t, & \text{if } z_{t-1} > \theta_H \\ \mu_m + \phi_m z_{t-1} + \epsilon_t, & \text{if } \theta_L \leq z_{t-1} \leq \theta_H \\ \mu_l + \phi_l z_{t-1} + \epsilon_t, & \text{if } z_{t-1} < \theta_L \end{cases} \tag{27}$$

As (Stigler, 2010) points out the commonly assumed case where Equation (27) is stable is when $\phi_h < 1$ and $\phi_l < 1$. The middle regime can be nonstationary $\phi_m > 1$, but with sufficient time it is likely the low and high regimes will force the process to become stationary again.

The SETAR model exhibits structural breaks as transitions occur between the regimes. (Seo, B., 2006) developed a test for the linear no cointegration null hypothesis against threshold cointegration in a threshold vector error correction model with a sup-Wald type test and derived its null asymptotic distribution.

6. Conclusion

Unit roots are nonstationary ARMA(p,q) processes which have one of more roots of 1 of the auto-regressive polynomial. If deterministic parameters such as the intercept or linear breaks have structural breaks standard unit root tests such as

ADF will rarely reject the null of a unit root even when $\phi_1 < 1$. Unit root tests that allow breaks under the null hypothesis such as the Lee-Strazicich unit root test should be used in that case. The Hybrid Bai Perron Zivot Andrews unit root test is another such test which can have higher statistical power than the Lee-Strazicich test under certain conditions. These unit root tests such as the Lee-Strazicich test are robust to structural breaks in the deterministic terms.

Another approach when testing for unit roots under structural breaks is to estimate the breakpoints and use traditional unit root tests in each segment. One such approach is the Hybrid Bai-Perron ADF methodology. One strength of this approach is it can also potentially detect changepoints in the auto-regressive coefficient (such as ϕ_1 in the AR(1) case) which most unit root tests cannot support.

One more form of auto-regressive models with changepoints that should be considered are SETAR models where the previous level of the response variable determines the regime, and each regime can have different coefficients.

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