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Accelerated Life Test Sampling Plans under Progressive Type II Interval Censoring with Random Removals

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Abstract

This paper investigates the design of accelerated life test (ALT) sampling plans under progressive Type II interval censoring with random removals. For ALT sampling plans with two over-stress levels, the optimal stress levels and the allocation proportions to them are obtained by minimizing the asymptotic generalized variance of the maximum likelihood estimation of model parameters. The required sample size and the acceptability constant which satisfy given levels of producer's risk and consumer's risk are found. ALT sampling plans with three over-stress levels are also considered under some specific settings. The properties of the derived ALT sampling plans under different parameter values are investigated by a numerical study. Some interesting patterns, which can provide useful insight to practitioners in related areas, are found. The true acceptance probabilities are computed using a Monte Carlo simulation and the results show that the accuracy of the derived ALT sampling plans is satisfactory. A numerical example is also provided for illustrative purpose.

Keywords: accelerated life test, progressive Type II interval censoring, random removal, sampling plan

1. Introduction

The design of reliability sampling plans under Type II censoring schemes has been studied by many researchers (Fertig & Mann, 1980; Hosono, Ohta, & Kase, 1981; Kocherlakota & Balakrishnan, 1986; Schneider, 1989; Balasooriya, 1995; Wu, Hung, & Tsai, 2003). In practice, it is not uncommon that some units are removed during the test, which leads to progressive censoring schemes. Balasooriya and Saw (1998), Balasooriya and Balakrishnan (2000), and Balasooriya, Saw, and Gadag (2000) discussed reliability sampling plans for the two-parameter exponential, lognormal and Weibull distributions under progressive Type II censoring schemes, respectively.

The number of removals at each failure was assumed to be pre-fixed in those works. However, in practice it might be infeasible to pre-determine the removal pattern and the decision of removing any units is based on the status of the experiment at that specific time, such as excessive heat or pressure, reduction of budget and facility, etc. Therefore, the number of removals should be a random outcome (Yuen & Tse, 1996). Tse and Yang (2003) discussed the design of reliability sampling plans for the Weibull distribution under progressive Type II censoring with random removals, where the number of units removed at each failure was assumed to follow a binomial distribution. In recent years the feature of random removal has been adopted by many researchers in designing various kinds of progressive censoring schemes, such as Ashour and Afify (2007), Wu, Chen, and Chang (2007), and S. Dey and T. Dey (2014).

Units are supposed to run at use condition in traditional reliability sampling plans. When it is desired to test the acceptance of highly reliable products, it is impractical to use such reliability sampling plans due to time constraint. Wallace (1985) stressed the need for introducing ALT into reliability sampling plans. Bai, Kim, and Chun (1993) studied the design of failure censored ALT sampling plans for lognormal and Weibull distributions. Hsieh (1994) investigated reliability sampling plans with ALT under Type II censoring for exponential distribution. The optimal design of ALT sampling plans with a non-constant shape parameter under both Type I and Type II censoring schemes was given by Seo, Jung, and Kim (2009).

Note that continuous inspections were assumed in the above works. Nevertheless, sometimes it is inconvenient to conduct a test with continuous inspections due to the high cost and/or possible danger in monitoring the test continuously. Under these circumstances, the interval inspection schemes, in which only the number of failures between two successive inspections is recorded, would be more favorable. Studies on life test and/or accelerated life test which employ interval censoring schemes are numerous. To number some of them, Tse, Ding, and Yang (2008) investigated the optimal design of accelerated life test under interval censoring with random removals for Weibull distribution; Chen and Lio (2010) compared the maximum likelihood estimation, moment estimation and probability plot estimation of parameters in the generalized exponential distribution under progressive Type I interval censoring; Ding, Yang, and Tse (2010) discussed the design of optimal ALT sampling plans under progressive Type I interval censoring with random removals. Most recently, Ding and Tse (2013) investigated the design of optimal ALT plans under progressive Type II interval censoring with binomial removals for the Weibull distribution. However, as far as our knowledge goes, there is no relevant study that investigates the design of ALT sampling plans under similar experimental settings with a Type II censoring scheme.

The optimal reliability sampling plans which combine ALT, interval inspection and progressive Type II censoring with random removals are developed in this paper. This study can be noted as an extension to the work of Ding and Tse (2013) along three directions: (i) the research topic is extended from the design of optimal ALT plans to the design of optimal ALT reliability sampling plans, in which both the consumer's risk and the producer's risk are satisfied. In this sense this paper resolves a more practical problem; (ii) instead of minimizing the asymptotic variance of an estimated quantile of units' lifetime distribution, this paper minimizes the asymptotic generalized variance of the maximum likelihood estimation of model parameters. It enables us to compare the outcomes derived using two different criteria in optimization; (iii) the true acceptance probabilities of the derived optimal ALT sampling plans are simulated, which provides us a way to evaluate the accuracy of the proposed method.

The rest of this paper is organized as follows: Section 2 describes the basic model of the proposed scheme. The design of optimal ALT sampling plans under progressive Type II interval censoring with random removals is discussed in Section 3. A numerical study is conducted in Section 4 to examine the properties of the derived sampling plans. In Section 5 the accuracy of the proposed ALT sampling plans is evaluated by a Monte Carlo simulation. Section 6 provides a numerical example. Conclusions are drawn in Section 7.

2. Model Description

Consider an ALT with the following settings:

1. A total of n identical and independent units are available at the beginning of the test.

2. There are m over-stress levels, i.e., s_1, s_2, ..., s_m. Denote s_0 as the stress level at use condition.

3. Suppose that n_i units are randomly allocated to the i^{th} stress level ($i = 1, 2, ..., m$). Then the allocation proportion to the i^{th} stress level is given by $\alpha_i = n_i / n$.

4. A progressive Type II censoring scheme is employed, and the test on the i^{th} stress level will be terminated after c_i ($i = 1, 2, ..., m$) or more units fail.

5. Interval inspections are conducted at time points t_{i1}, t_{i2}, …, $t_{i,k(i)}$ and the number of failures x_{ij} between inspection interval $(t_{i,j-1}, t_{ij})$ is recorded. It should be pointed out that both the experiment time $t_{i,k(i)}$ and the number of inspections $k(i)$ are random variables.

6. Suppose that $r_{ij}\,(i = 1, 2, ..., m; j = 1, 2, ..., k(i) - 1)$ non-failed units are randomly removed at inspection time t_{ij}. To ensure that there are at least c_i failed units at the end of the test on stress level s_i, r_{ij} is restricted to be any integer value between 0 and $n_i - c_i - \sum_{l=1}^{j-1} r_{il}$. Further assume that r_{ij} follows a binomial distribution with probability p, then we have $r_{ij} \sim B\left(n_i - c_i - \sum_{l=1}^{j-1} r_{il},\, p\right)$. For notational convenience, denote $r_{i,k(i)} = n_i - \sum_{j=1}^{k(i)} x_{ij} - \sum_{j=1}^{k(i)-1} r_{ij}$ as the number of units left.

The process of this testing scheme is depicted in Figure 1.

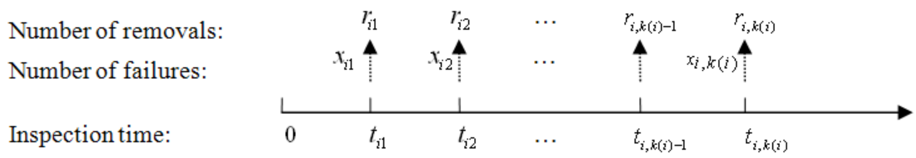

Figure 1. The testing scheme of an ALT under progressive Type II interval censoring with random removals (on the i^{th} stress level)

Suppose that the lifetime of a unit T follows a Weibull distribution with probability density function (pdf)

$$f(t) = (\delta/\theta)(t/\theta)^{\delta-1}\exp\left[-(t/\theta)^{\delta}\right],\quad t>0. \tag{1}$$

Further assume that the scale parameter θ and stress level s are related as

$$\theta = \exp(\beta_0+\beta_1 s), \tag{2}$$

where β_0 and β_1 are unknown constants and the shape parameter δ does not depend on s. Define $Y=\ln(T)$, then Y has an extreme value distribution with cumulative distribution function (cdf)

$$G(y) = 1-\exp\left[-\exp((y-\mu)/\sigma)\right],\quad -\infty<y<+\infty, \tag{3}$$

where $\mu=\ln\theta=\beta_0+\beta_1 s$, $\sigma=1/\delta$.

Given observations $(x_{ij}, r_{ij},\ i=1,2,\ldots,m;\ j=1,2,\ldots,k(i))$, the logarithm of the likelihood function can be derived as:

$$\begin{aligned}
&\ln L(\beta_0,\beta_1,\sigma,p;x_{ij},r_{ij},k(i))\\
&=\sum_{i=1}^{m}\left\{\ln\binom{n_i}{x_{i1}}+\ln\binom{n_i-x_{i1}-r_{i1}}{x_{i2}}+\ldots+\ln\binom{n_i-\sum_{j=1}^{k(i)-1}x_{ij}-\sum_{j=1}^{k(i)-1}r_{ij}}{x_{i,k(i)}}\right.\\
&\quad+\sum_{j=1}^{k(i)}\left[x_{ij}\ln\left[G(y_{ij})-G(y_{i,j-1})\right]+r_{ij}\ln\left[1-G(y_{ij})\right]\right]+\ln\left((n_i-c_i)!/\prod_{j=1}^{k(i)-1}r_{ij}!\left(n_i-c_i-\sum_{j=1}^{k(i)-1}r_{ij}\right)!\right)\\
&\quad\left.+\sum_{j=1}^{k(i)-1}r_{ij}\ln p+\left[(k(i)-1)(n_i-c_i)-\sum_{j=1}^{k(i)-1}(k(i)-j)r_{ij}\right]\ln(1-p)\right\}
\end{aligned}, \tag{4}$$

where $y_{ij}=\ln(t_{ij})$.

The maximum likelihood estimates of β_0, β_1, σ and p (denoted by $\hat\beta_0$, $\hat\beta_1$, $\hat\sigma$ and $\hat p$, respectively) can be solved from equations $\partial\ln L/\partial\beta_0=\partial\ln L/\partial\beta_1=\partial\ln L/\partial\sigma=\partial\ln L/\partial p=0$. Besides, the Fisher information matrix of $(\beta_0,\beta_1,\sigma,p)$ is given by

$$I(\beta_0,\beta_1,\sigma,p)=-E\begin{pmatrix}
\frac{\partial^2\ln L}{\partial\beta_0^2} & \frac{\partial^2\ln L}{\partial\beta_0\partial\beta_1} & \frac{\partial^2\ln L}{\partial\beta_0\partial\sigma} & 0\\
\frac{\partial^2\ln L}{\partial\beta_0\partial\beta_1} & \frac{\partial^2\ln L}{\partial\beta_1^2} & \frac{\partial^2\ln L}{\partial\beta_1\partial\sigma} & 0\\
\frac{\partial^2\ln L}{\partial\beta_0\partial\sigma} & \frac{\partial^2\ln L}{\partial\beta_1\partial\sigma} & \frac{\partial^2\ln L}{\partial\sigma^2} & 0\\
0 & 0 & 0 & \frac{\partial^2\ln L}{\partial p^2}
\end{pmatrix}=\begin{pmatrix} I_1(\beta_0,\beta_1,\sigma) & 0\\ 0 & I_2(p)\end{pmatrix}, \tag{5}$$

where $I_1(\beta_0,\beta_1,\sigma)$ is the upper left 3×3 sub-matrix of $I(\beta_0,\beta_1,\sigma,p)$ and $I_2(p)=-E\left(\partial^2\ln L/\partial p^2\right)$. The asymptotic covariance matrix of $(\hat\beta_0,\hat\beta_1,\hat\sigma)$ is then given by $I_1^{-1}(\beta_0,\beta_1,\sigma)$. The detailed formulation for the entries of Eq. (5) can be found in the Appendix of Ding and Tse (2013).

Given sample size n, use condition s_0 and high stress level s_m, removal probability p, predetermined number of failures c_i and inspections times $(t_{ij}, i=1,2,...,m; j=1,2,...,k(i))$ on each stress, the stress levels $(s_i, i=1,2,...,m-1)$ and the allocation proportions $(\alpha_i, i=1,2,...,m-1)$ to these stress levels are selected in such a way that the generalized asymptotic variance of $\hat{\beta}_0$, $\hat{\beta}_1$, and $\hat{\sigma}$, which is given by $\left|I_1^{-1}(\beta_0, \beta_1, \sigma)\right|$, is minimized.

3. Design of ALT Sampling Plans

Suppose that a sample of size n is randomly drawn from the lot and the test is conducted at the accelerated settings described in Section 2. Assume that the lifetime of a unit T follows a Weibull distribution $F(\theta,\delta)$, where the relationship between the scale parameter θ and the stress s is given by Eq. (2) and the shape parameter δ does not depend on s. Suppose that a unit with lifetime less than η is considered to be nonconforming. Define $Y=\ln(T)$, then Y follows an extreme value distribution $G(\mu,\sigma)$ and the lower specification limit for the log lifetime is given by $\varsigma=\ln(\eta)$.

Define $\kappa=\mu_0-d\sigma$, where μ_0 is the location parameter of $G(.)$ at use condition and d is the acceptability constant. Since the stresses can be standardized such that $s_0=0$, $s_m=1$ and $0<s_i<1\ (i=1,2,...,m-1)$, it follows from Eq. (3) that $\mu_0=\beta_0+\beta_1 s_0=\beta_0$. By the invariance principle of the maximum likelihood method, the MLE of κ is then given by $\hat{\kappa}=\hat{\mu}_0-d\hat{\sigma}=\hat{\beta}_0-d\hat{\sigma}$. To judge whether a lot should be accepted or not, $\hat{\kappa}$ is compared with the lower specification limit ς. If $\hat{\kappa}>\varsigma$, the lot is accepted; otherwise, it is rejected.

Define the nonconforming fraction of the lot by p_f, which is calculated as

$$p_f=P(Y<\varsigma)=1-\exp\left\{-\exp\left((\varsigma-\mu_0)/\sigma\right)\right\}. \tag{6}$$

The sample size n and the acceptability constant d are determined such that lots with nonconforming fraction $p_f<p_\partial$ are accepted with a probability of at least $1-\partial$ and lots with $p_f>p_\partial$ are rejected with a probability of at least $1-\beta$, where ∂ and β are the given levels of producer's and consumer's risks, respectively.

It follows from $\hat{\kappa}=\hat{\beta}_0-d\hat{\sigma}$ that $Var\left(\hat{\kappa}\right)=Var(\hat{\beta}_0)-2d\times Cov(\hat{\beta}_0,\hat{\sigma})+d^2Var(\hat{\sigma})$.

Since $U=\left[\hat{\kappa}-(\beta_0-d\sigma)\right]/\left[Var(\hat{\kappa})\right]^{1/2}$ is parameter-free and asymptotically standard normal, the operating characteristic (OC) curve is given by

$$O(p_f)=P(\hat{\kappa}>\varsigma)=1-\Phi\left(\left[\sigma\ln\left(-\ln(1-p_f)\right)+d\sigma\right]/\sqrt{Var(\hat{\kappa})}\right), \tag{7}$$

where $\Phi(.)$ is the cdf of the standard normal distribution.

The sample size n and the acceptability constant d are determined such that the OC curve goes through two points $(p_\partial, 1-\partial)$ and (p_β, β). This implies

$$\begin{aligned}1-\partial&=1-\Phi\left(\left[\sigma\ln\left(-\ln\left(1-p_\partial\right)\right)+d\sigma\right]/\sqrt{Var\left(\hat{\kappa}\right)}\right),\\ \beta&=1-\Phi\left(\left[\sigma\ln(-\ln(1-p_\beta))+d\sigma\right]/\sqrt{Var(\hat{\kappa})}\right).\end{aligned} \tag{8}$$

It follows that

$$\begin{aligned}d&=\left[u_{1-\beta}\ln\left(-\ln\left(1-p_\partial\right)\right)-u_\partial\ln\left(-\ln\left(1-p_\beta\right)\right)\right]/\left(u_\partial-u_{1-\beta}\right),\\ Var\left(\hat{\kappa}\right)&=\left[\sigma^2\left(\ln\left(-\ln\left(1-p_\partial\right)\right)+d\right)^2\right]/u_\partial^2,\end{aligned} \tag{9}$$

where $u_z=\Phi^{-1}(z)$. The acceptability constant d is calculated directly from the first part of Eq. (9), while the required sample size n can be obtained by a search method from the second part (the detailed algorithm is provided in Section 4.1).

4. Numerical Study

4.1. ALT Sampling Plans with Two Over-stress Levels

The properties of the derived ALT sampling plans under different parameter values are evaluated by a numerical study in this section. The following settings are made:

1. Two over-stress levels s_1, s_2 are employed, i.e., $m=2$.

2. The inspections on each stress level are equally spaced, i.e., $t_{i0}=0$, $t_{ij}=t_{i,j-1}+l_i$, $(i=1,2,...,m; j=1,2,...,k(i))$, where l_i is the inspection length on the i^{th} stress level. Define $MT_i=\exp(\beta_0+\beta_1 s_i)\Gamma(1+1/\delta)$ as the mean of units' lifetime distribution on the same stress and $\tau_i=l_i/MT_i$ as the proportion of the inspection length to the corresponding mean. τ_i, which is proportional to the inspection length l_i, is used in this numerical study since it is more convenient to use a relative value than an absolute one. The case of $\tau_1=\tau_2=\tau$ is considered.

3. Define the censoring fraction on the i^{th} stress level as $f_{ci}=c_i/n_i(i=1,2)$. The cases of both $f_{c1}=f_{c2}$ and $f_{c1}<f_{c2}$ are considered since units are much easier to fail on the high stress level than on the low one.

Without loss of generality, set $s_0=0$, $s_m=1$. In practice, it is often difficult for an experimenter to give prior estimates of parameters β_0 and β_1. On the contrary, based on the experimenters' experiences and/or the information collected from preliminary or similar studies, the estimation of the probability that a unit falls into a certain interval is much easier. Define $P_u=P$ (a unit's lifetime T falls into (0, 1) at use condition) and $P_h=P$ (a unit's lifetime T falls into (0, 1) on high stress level), then we have

$$\beta_0=-\sigma\ln\left(-\ln\left(1-P_u\right)\right),$$
$$\beta_1=\sigma\left(\ln\left(-\ln\left(1-P_u\right)\right)-\ln\left(-\ln\left(1-P_h\right)\right)\right). \tag{10}$$

In order to obtain an optimal ALT sampling plan under progressive Type II interval censoring with random removals, the values of n, d, s_1 and α_1 have to be determined. The acceptability constant d depends on $(p_\partial, 1-\partial)$ and (p_β, β) only, and it can be calculated from Eq. (9) directly. The determination of the other three parameters requires the combination of a grid search method and the Monte Carlo simulation. For the sake of simplicity, let Δ denote $\sigma^2\left(\ln\left(-\ln\left(1-p_\partial\right)\right)+d\right)^2/u_\partial^2$. Then, n, s_1 and α_1 are calculated using the following algorithm:

1. Set an initial value $n^{(0)}$ for n. Consider the smallest sample size and set $n^{(0)}=2$. Find the optimal (s_1^*, α_1^*) which minimizes $\left|I_1^{-1}(\beta_0,\beta_1,\sigma)\right|$ using a grid search method over unit square $(0,1)\times(0,1)$. Calculate the corresponding value of $Var(\hat{\kappa})$ at (s_1^*, α_1^*).
2. Set $n^{(1)}=2n^{(0)}$, find (s_1^*, α_1^*) based on sample size $n^{(1)}$ and compute $Var(\hat{\kappa})$ accordingly.
3. Repeat step 2 until for $n^{(i)}$, $Var(\hat{\kappa})>\Delta$ and for $n^{(i+1)}$, $Var(\hat{\kappa})<\Delta$. Define $n^{(l)}=n^{(i)}$ and $n^{(u)}=n^{(i+1)}$.
4. Set $n^{(i+2)}=(n^{(l)}+n^{(u)})/2$. Find (s_1^*, α_1^*) and calculate $Var(\hat{\kappa})$. If $Var(\hat{\kappa})>\Delta$, set $n^{(l)}=n^{(i+2)}$; otherwise, set $n^{(u)}=n^{(i+2)}$.
5. Repeat step 4 until $Var(\hat{\kappa})=\Delta$ approximately holds or $n^{(u)}-n^{(l)}\le 2$.

For given parameter values, specifically $(p_\partial, 1-\partial)$ =(0.00041, 0.95); (p_β, β) =(0.01840, 0.10); P_u =0.01; $P_h=0.1$; $\delta=$ 0.5, 1, 2; (f_{c1}, f_{c2}) =(0.5, 0.5), (0.8, 0.8), (0.5, 0.7), (0.5, 0.9); p =0, 0.05, 0.1, 0.3 and τ =0.02, 0.05, 0.1, 0.3, the optimal ALT sampling plans $(n, d, s_1^*, \alpha_1^*)$ are determined using the algorithm described above. A consistent pattern emerged based on the results of these combinations. The required sample size n decreases as the censoring fractions f_{c1}, f_{c2} increase. In order to provide a better insight on the effect of p (the probability of random removal) and τ (which is proportional to the inspection length), some cases are selected for illustration and the corresponding results are depicted in Figure 2.

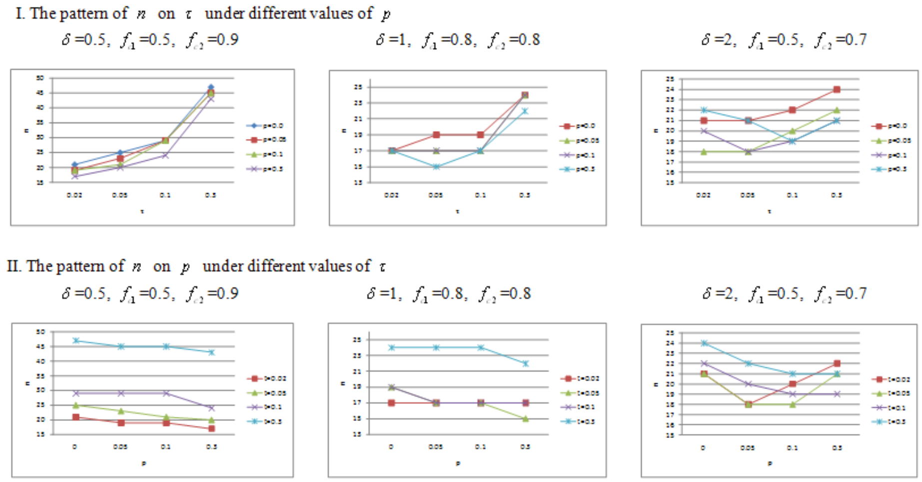

Figure 2. Two over-stress levels ALT sampling plans under progressive Type II interval censoring with random removals

The following patterns are observed:

a. For the cases of $\delta = 0.5$, n increases as τ increases for all values of p. For the cases of $\delta = 1$, when p=0, 0.05 and 0.1, n increases as τ increases; when p=0.3, n first decreases and then increases as τ increases. For the cases of $\delta = 2$, when p=0 and 0.05, n increases as τ increases; when p=0.1 and 0.3, n first decreases and then increases as τ increases. This pattern can be interpreted in this way: Larger τ means wider inspection intervals, from which the collected information on units' lifetime is less accurate and thus more units are required to judge whether to accept the lot or not. However, when $p > 0$, a larger τ also implies that units are less likely to be removed at the early stage of the test. Consequently, more information on the lifetime distribution is collected and the required sample size n is decreased. Taking these two kinds of effect into consideration, shorter inspection interval doesn't always yield smaller required sample size for ALT sampling plans under progressive Type II interval censoring with random removals.

b. For the cases of $\delta = 0.5$ and 1, n decreases as p increases for all values of τ. For the cases of $\delta = 2$, when τ =0.1 and 0.3, n decreases as p increases; when τ =0.02 and 0.05, n first decreases and then increases as p increases. This pattern is caused by the two-sided effects of the removal probability p. Generally speaking, a test is likely to be prolonged as p increases. Thus more information on the lifetime distribution can be observed and the required sample size n is decreased. Nevertheless, when the inspection intervals are too small, a non-zero removal probability p also causes more units being removed at the early stage of the test. In this case, less data can be collected and thus n is increased. In conclusion, except for several cases ($\delta = 2$ and τ =0.02/0.05), the removal probability p is helpful in reducing the required sample size n.

4.2. ALT Sampling Plans with Three Over-stress Levels

ALT plans with three over-stress levels are useful in practice since they can provide a way to check the assumed straight-line relationship between distribution parameter μ and stress level s by adding a middle stress. The design of three over-stress levels ALT sampling plans under progressive Type II interval censoring with random removals is discussed in this section. They are developed under the following settings:

1. Three over-stress levels, s_1, s_2 and s_3 are employed. In particular, set $s_0 = 0$, $s_3 = 1$ and $s_2 = (s_1 + s_3)/2$.
2. The allocation proportions to three over-stress levels $(\alpha_1, \alpha_2, \alpha_3)$ are set to be $(1/3, 1/3, 1/3)$ and $(0.5, 0.3, 0.2)$.
3. Three settings of censoring fractions are considered, namely, (f_{c1}, f_{c2}, f_{c3}) equals $(0.5, 0.5, 0.5)$, $(0.8, 0.8, 0.8)$ and $(0.5, 0.7, 0.9)$.
4. The proportion of the inspection length to the corresponding mean, that is, τ_i is set to be equal on three over-stress levels, i.e., $\tau_1 = \tau_2 = \tau_3 = \tau$.

A numerical study is conducted to determine ALT sampling plans with three over-stress levels under equally spaced inspection times. For given parameters, the optimal low stress level s_1^* is found by a grid search method over interval (0, 1). The step size is 0.01. For different parameter values, specifically, $(p_{\partial},1-\partial)$ =(0.00041, 0.95); (p_{β},β) =(0.01840, 0.10); P_u =0.01; $P_h=0.1$; δ =0.5, 1, 2; p =0, 0.05, 0.1, 0.3 and τ =0.02, 0.05, 0.1, 0.3, the optimal low stress level s_1^* and the required sample size n are calculated. The effects of p and τ on the required sample size n are depicted in Figure 3. We note that:

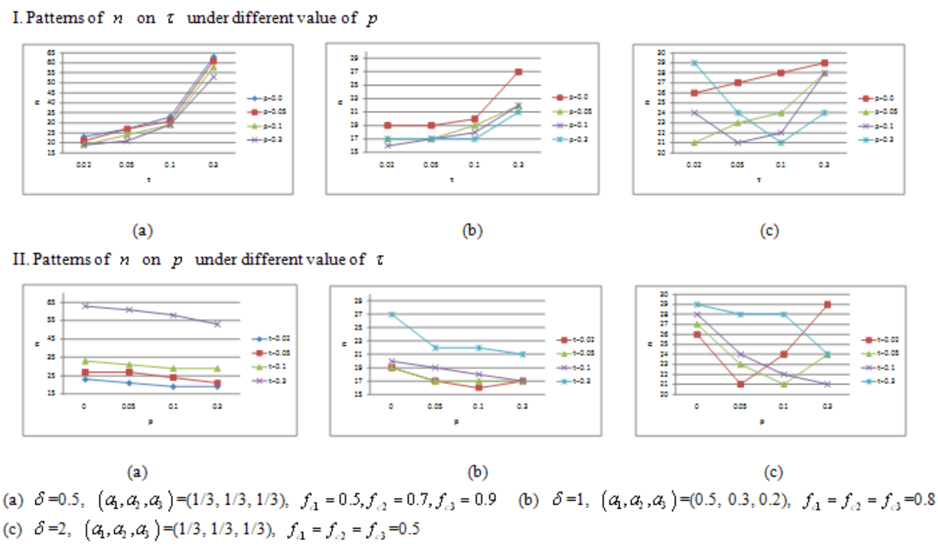

Figure 3. Three over-stress levels ALT sampling plans under progressive Type II interval censoring with random removals

a. For the cases of $\delta=0.5$ and 1, n increases as τ increases. For the cases of $\delta=2$, when p =0 and 0.05, n increases as τ increases; when p =0.1 and 0.3, n first decreases and then increases as τ increases.

b. For the cases of $\delta=0.5$, n decreases as p increases. For the cases of $\delta=1$, when τ =0.05, 0.1 and 0.3, n decreases as p increases; when $\tau=0.02$, n first decreases and then increases as p increases. For the cases of $\delta=2$, when τ =0.1 and 0.3, n decreases as p increases; when τ =0.02 and 0.05, n first decreases and then increases as p increases.

Note that these patterns are similar to those observed in the two over-stress levels case.

5. Accuracy of Large Sample Approximation

Since the proposed ALT sampling plans are derived based on asymptotic theory, there is a need to evaluate the finite sample behavior of them. The accuracy of the derived ALT sampling plans is assessed by a simulation study. The OC curve is set to go through two points, that is, $(p_{\partial},1-\partial)$ =(0.00041, 0.95) and (p_{β},β) =(0.01840, 0.10). For each combination of parameters, the nonconforming fraction of a lot under given acceptance probability (99% and 95%) on the pre-defined OC curve is computed, and then the true acceptance probability of a lot with that corresponding nonconforming fraction is calculated by a Monte Carlo simulation with 1000 runs. The results for ALT sampling plans with two over-stress levels and three over-stress levels are presented in Table 1 and Table 2, respectively. Actually, several different values of δ ($\delta=0.5$, 1, 2) are considered in this numerical study. Since they show similar patterns, only parts of the results of $\delta=1$ are provided for simplicity.

We note from Table 1 and Table 2 that the simulated acceptance probabilities are close to their nominal values in most cases. This indicates that the optimal ALT sampling plans derived based on asymptotic approximation have satisfactory accuracy.

6. A Numerical Example

Suppose that there is an agreement between a consumer and a producer to determine the acceptability of a lot. In particular, if the nonconforming fraction of a lot is smaller than 0.00041, then the lot should be accepted with a probability of at least 0.95; while if the nonconforming fraction of a lot is larger than 0.01840, then it should be rejected with a probability of at least 0.90. Assume that an ALT reliability sampling plan with two over-stress levels is used to determine the acceptability of the lot. The probabilities for a unit to fail at use condition and high stress level are estimated to be 0.01 and 0.1, respectively. A progressive Type II interval censoring scheme is employed, and the censoring fractions on two stress levels are 0.8. The proportions of the inspection length to the corresponding distribution mean on both stresses are set to be 0.1. Besides, based on prior information, it is assumed that units' lifetimes are Weibull distributed with shape parameter $\delta = 1$ and a unit is likely to be removed at each inspection with probability 0.1. The problem is to determine the number of units used in this ALT sampling plan and to determine the low stress level and the allocation proportions to two stresses so that (1) both the consumer's risk and the producer's risk can be satisfied and (2) the maximum amount of information on units' lifetime distribution can be collected.

The optimal ALT sampling plan is obtained using the proposed method. The required sample size is 17, with 7 and 10 units allocated to the low and high stress levels, respectively. The low stress level should be settled at 0.02 multiplied by the actual high stress. Besides, the acceptability constant which is required to make the decision is 5.6560.

7. Conclusion

The design of ALT sampling plans under progressive Type II interval censoring with random removals was discussed in this paper. For ALT sampling plans with two over-stress levels, the optimal stress levels and the corresponding allocation proportions, which minimize the generalized asymptotic variance of the MLE of model parameters, were found. The sample size and the acceptability constant required to judge the acceptability of the lot were calculated.

The properties of the derived ALT sampling plans were examined by a numerical study. It is shown that generally the removal probability is helpful in reducing the required sample size. More importantly, when there exists random removal, short inspection interval doesn't always yield small required sample size, which is different from the case of no random removal. These interesting patterns would provide useful insights to experimenter in designing similar ALT sampling plans. The accuracy of the proposed sampling plans was evaluated by a Monte Carlo simulation. The results show the simulated acceptance probabilities are close to their nominal values in most cases, which indicates that the performance of the derived ALT sampling plans is satisfactory.

Table 1. Simulated acceptance probabilities for two over-stress levels ALT sampling plans under progressive Type II interval censoring with random removals **(** $m = 2$; $\delta = 1$; $p_{\partial} = 0.00041$; $1-\partial = 0.95$; $p_{\beta} = 0.01840$; $\beta = 0.10$ **)**

τ	n	$f_{c1} = f_{c2} = 0.5$ Selected points on OC curve		Simulated probability		n	$f_{c1} = f_{c2} = 0.8$ Selected points on OC curve		Simulated probability	
		99%	95%	99%	95%		99%	95%	99%	95%
					$p = 0.0$					
0.02	25	0.00016	0.00039	0.987	0.973	17	0.00016	0.00037	0.986	0.967
0.05	29	0.00018	0.00042	0.984	0.957	19	0.00017	0.00040	0.985	0.963
0.1	29	0.00016	0.00039	0.982	0.949	19	0.00015	0.00038	0.991	0.970
0.3	45	0.00017	0.00042	0.966	0.923	24	0.00017	0.00040	0.989	0.953
					$p = 0.05$					
0.02	21	0.00018	0.00041	0.992	0.978	17	0.00017	0.00043	0.995	0.972
0.05	23	0.00016	0.00038	0.991	0.973	17	0.00015	0.00039	0.988	0.965
0.1	29	0.00017	0.00038	0.989	0.966	17	0.00015	0.00040	0.982	0.972
0.3	44	0.00017	0.00041	0.972	0.949	24	0.00017	0.00042	0.985	0.933
					$p = 0.1$					
0.02	21	0.00019	0.00044	0.991	0.975	17	0.00018	0.00044	0.995	0.974
0.05	22	0.00016	0.00039	0.994	0.973	17	0.00017	0.00041	0.985	0.971
0.1	25	0.00016	0.00039	0.985	0.969	17	0.00015	0.00039	0.989	0.957
0.3	41	0.00016	0.00038	0.968	0.948	24	0.00018	0.00044	0.982	0.962
					$p = 0.3$					
0.02	21	0.00016	0.00042	0.992	0.979	17	0.00016	0.00040	0.992	0.970
0.05	20	0.00016	0.00039	0.997	0.976	15	0.00014	0.00037	0.984	0.968
0.1	20	0.00014	0.00035	0.992	0.967	17	0.00017	0.00038	0.988	0.978
0.3	37	0.00015	0.00039	0.973	0.945	22	0.00017	0.00042	0.982	0.955

τ	n	$f_{c1} = 0.5, f_{c2} = 0.7$ Selected points on OC curve	Simulated probability	n	$f_{c1} = 0.5, f_{c2} = 0.9$ Selected points on OC curve	Simulated probability

		99%	95%	99%	95%		99%	95%	99%	95%
					$p = 0.0$					
0.02	21	0.00015	0.00040	0.991	0.974	18	0.00017	0.00042	0.989	0.956
0.05	22	0.00016	0.00037	0.989	0.969	17	0.00015	0.00036	0.992	0.967
0.1	24	0.00016	0.00039	0.984	0.961	19	0.00017	0.00042	0.990	0.948
0.3	31	0.00016	0.00038	0.977	0.954	24	0.00017	0.00042	0.985	0.963
					$p = 0.05$					
0.02	18	0.00017	0.00039	0.994	0.976	17	0.00019	0.00041	0.994	0.965
0.05	21	0.00018	0.00043	0.984	0.967	17	0.00016	0.00041	0.992	0.962
0.1	21	0.00016	0.00038	0.981	0.977	19	0.00018	0.00042	0.987	0.969
0.3	31	0.00016	0.00040	0.979	0.947	22	0.00016	0.00038	0.981	0.964
					$p = 0.1$					
0.02	17	0.00015	0.00037	0.994	0.975	17	0.00015	0.00042	0.999	0.972
0.05	19	0.00017	0.00041	0.997	0.973	17	0.00016	0.00042	0.994	0.968
0.1	19	0.00014	0.00038	0.990	0.973	17	0.00015	0.00039	0.994	0.969
0.3	30	0.00017	0.00042	0.979	0.950	22	0.00017	0.00042	0.982	0.951
					$p = 0.3$					
0.02	19	0.00017	0.00041	0.995	0.971	17	0.00017	0.00040	0.993	0.967
0.05	18	0.00015	0.00038	0.995	0.978	17	0.00016	0.00038	0.993	0.975
0.1	19	0.00016	0.00040	0.988	0.980	17	0.00016	0.00040	0.990	0.970
0.3	27	0.00016	0.00038	0.985	0.961	21	0.00017	0.00040	0.983	0.967

Table 2. Simulated acceptance probabilities for three over-stress levels ALT sampling plans under progressive Type II interval censoring with random removals **(** $m = 3$; $\delta = 1$; $p_{\partial} = 0.00041$; $1-\partial = 0.95$; $p_{\beta} = 0.01840$; $\beta = 0.10$ **)**

Case I. $(\alpha_1 : \alpha_2 : \alpha_3) = (1/3,\ 1/3,\ 1/3)$

τ	n	$f_{c1} = f_{c2} = f_{c3} = 0.5$ Selected points		Simulated probabilities		n	$f_{c1} = f_{c2} = f_{c3} = 0.8$ Selected points		Simulated probabilities		n	$f_{c1} = 0.5, f_{c2} = 0.7, f_{c3} = 0.9$ Selected points		Simulated probabilities	
		99%	95%	99%	95%		99%	95%	99%	95%		99%	95%	99%	95%
							$p = 0.0$								
0.02	28	0.00016	0.00041	0.991	0.972	19	0.00019	0.00043	0.994	0.964	20	0.00017	0.00043	0.996	0.972
0.05	29	0.00016	0.00041	0.987	0.947	19	0.00017	0.00042	0.988	0.973	21	0.00018	0.00044	0.984	0.959
0.1	30	0.00015	0.00040	0.991	0.947	18	0.00015	0.00038	0.988	0.952	21	0.00016	0.00042	0.987	0.963
0.3	42	0.00016	0.00038	0.961	0.936	27	0.00018	0.00041	0.983	0.969	27	0.00017	0.00040	0.980	0.953
							$p = 0.05$								
0.02	21	0.00019	0.00045	0.992	0.971	17	0.00017	0.00042	0.993	0.970	18	0.00017	0.00042	0.998	0.969
0.05	25	0.00016	0.00038	0.985	0.961	17	0.00017	0.00042	0.986	0.968	19	0.00018	0.00042	0.997	0.964
0.1	29	0.00016	0.00039	0.989	0.960	18	0.00016	0.00039	0.988	0.973	20	0.00017	0.00042	0.989	0.978
0.3	42	0.00016	0.00042	0.976	0.938	27	0.00019	0.00042	0.980	0.967	27	0.00017	0.00043	0.983	0.953
							$p = 0.1$								
0.02	19	0.00016	0.00037	0.994	0.975	17	0.00017	0.00041	0.993	0.966	17	0.00015	0.00038	0.981	0.969
0.05	22	0.00017	0.00041	0.990	0.972	17	0.00017	0.00042	0.991	0.973	18	0.00018	0.00040	0.988	0.967
0.1	29	0.00018	0.00043	0.990	0.963	17	0.00016	0.00039	0.990	0.964	19	0.00016	0.00039	0.996	0.951
0.3	42	0.00016	0.00040	0.989	0.946	25	0.00016	0.00041	0.983	0.952	27	0.00018	0.00042	0.977	0.956
							$p = 0.3$								
0.02	21	0.00016	0.00040	0.997	0.978	17	0.00017	0.00041	0.995	0.974	19	0.00017	0.00039	0.990	0.968
0.05	21	0.00018	0.00042	0.994	0.971	17	0.00018	0.00041	0.992	0.970	18	0.00016	0.00041	0.995	0.963
0.1	21	0.00017	0.00038	0.985	0.967	17	0.00016	0.00040	0.993	0.954	19	0.00018	0.00042	0.991	0.977
0.3	33	0.00016	0.00039	0.976	0.964	24	0.00016	0.00042	0.989	0.974	23	0.00016	0.00039	0.987	0.956

Table 2.(Cont'd) Simulated acceptance probabilities for three over-stress levels ALT sampling plans under progressive Type II interval censoring with random removals **(** $m=3$; $\delta=1$; $p_{\partial}=0.00041$; $1-\partial=0.95$; $p_{\beta}=0.01840$; $\beta=0.10$ **)**
Case II. $(\alpha_1:\alpha_2:\alpha_3)=(0.5,\ 0.3,\ 0.2)$

	$f_{c1}=f_{c2}=f_{c3}=0.5$					$f_{c1}=f_{c2}=f_{c3}=0.8$					$f_{c1}=0.5, f_{c2}=0.7, f_{c3}=0.9$				
τ	n	Selected points		Simulated probabilities		n	Selected points		Simulated probabilities		n	Selected points		Simulated probabilities	
		99%	95%	99%	95%		99%	95%	99%	95%		99%	95%	99%	95%
$p=0.0$															
0.02	27	0.00017	0.00039	0.995	0.970	19	0.00018	0.00044	0.990	0.974	20	0.00018	0.00044	0.995	0.971
0.05	29	0.00017	0.00043	0.984	0.962	19	0.00016	0.00042	0.991	0.961	20	0.00017	0.00042	0.993	0.966
0.1	31	0.00017	0.00041	0.988	0.965	20	0.00018	0.00041	0.989	0.969	22	0.00019	0.00044	0.982	0.969
0.3	45	0.00016	0.00041	0.964	0.942	27	0.00018	0.00044	0.983	0.962	31	0.00017	0.00040	0.973	0.969
$p=0.05$															
0.02	19	0.00016	0.00040	0.988	0.979	17	0.00018	0.00041	0.995	0.972	17	0.00018	0.00040	0.991	0.975
0.05	22	0.00016	0.00041	0.989	0.966	17	0.00015	0.00039	0.989	0.974	17	0.00016	0.00040	0.990	0.965
0.1	28	0.00017	0.00040	0.987	0.962	19	0.00018	0.00042	0.988	0.964	20	0.00016	0.00040	0.993	0.959
0.3	45	0.00017	0.00042	0.981	0.955	22	0.00015	0.00039	0.984	0.951	27	0.00015	0.00036	0.982	0.965
$p=0.1$															
0.02	19	0.00017	0.00041	0.998	0.974	16	0.00017	0.00042	0.988	0.975	17	0.00018	0.00040	0.988	0.977
0.05	22	0.00018	0.00044	0.991	0.971	17	0.00017	0.00041	0.987	0.974	17	0.00017	0.00039	0.991	0.966
0.1	27	0.00017	0.00042	0.990	0.955	18	0.00017	0.00040	0.989	0.966	20	0.00017	0.00042	0.991	0.976
0.3	42	0.00015	0.00038	0.970	0.941	22	0.00016	0.00038	0.985	0.962	27	0.00015	0.00038	0.981	0.953
$p=0.3$															
0.02	21	0.00016	0.00038	0.993	0.976	17	0.00016	0.00041	0.992	0.966	19	0.00017	0.00043	0.996	0.982
0.05	21	0.00015	0.00041	0.994	0.974	17	0.00017	0.00041	0.989	0.969	17	0.00016	0.00039	0.991	0.972
0.1	21	0.00015	0.00037	0.990	0.969	17	0.00015	0.00039	0.989	0.968	19	0.00017	0.00042	0.986	0.977
0.3	35	0.00016	0.00041	0.980	0.960	21	0.00016	0.00040	0.984	0.967	27	0.00017	0.00042	0.985	0.952

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