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Estimating the Common Mean of k Normal Populations with Known Variance

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Abstract

Consider the problem of estimating the common mean of k normal populations with known variances. We study the admisibility of the Best linear Risk Unbiased Equivariant (BLRUE) estimator of the common mean of k normal populations under the squared error and LINEX loss function when the variances are known.

Keywords: Admissibility, asymmetric loss, best Equivariant estimator, common mean, LINEX loss, risk unbiased

1. Introduction

Suppose we have $\boldsymbol{k}$ independent populations where the ith population follows from N(θ,$\sigma_i{}^2$), $i = 1, \dots, k.$ The parameter θ is unknown and $\sigma_i{}^2 > 0, i = 1, \dots, k$ are all assumed to be known. Let $X_{ij}, j = 1, \dots, n_i$ be i.i.d observations from the ith population, $1 \le i \le k$. Define $\overline{X}_i$ as

$$\overline{X}_i = \frac{1}{n}\sum_{j=1}^{n_i} X_{ij} \quad i = 1,2, \dots, k$$

and note that $\overline{X}_i \sim N(\theta, {}^{\sigma^2}/_{n_i})$.

Combining two or more unbiased estimators of an unknown parameter θ in order to obtain a batter unbiased estimators (in the sense of smaller risk) is a problem that often arises in statistics; for example when k independent sets of measurements of the same quantity are available. The problem of estimating the common mean of two or more independent normal populations has received attention from several authors in the past. For some references in this regard see Graybill and Deal(1959),Sinha and Mouqadem(1989) and Pal and Sinha(1996)for a complete bibliography. See also Lehmann and Casella (1998) pp 95-96, Sanjari Farsipour (1999), Sanjari Farsipour and Asgharzadeh (2002), for further references and comments. In section 2, a class of risk unbiased estimators which combines the means of the samples i.e., $\overline{X}_i$'s, is developed and in section 3 the rejoin of admissibility of the estiomators of the from $\sum_{i=1}^{k} c_i \overline{X}_i + d$ is derived under the squared error loss function.

$$L_1(\delta, \theta) = (\delta - \theta)^2 \tag{1.1}$$

Which is a symmetric loss function. In section 4, the inadmissibility of the estimators of the from $\sum_{j=1}^{k} c_i \overline{X}_i + d$ are studied under the Loss function. In practice, the real loss function is often not symmetric and overestimation can lead to more or less severe consequences than underestimation. Varian(1975) employed an asymmetric loss function, which is known as LINEX loss, and was extensively used by Zellner(1986), Rojo(1987), Sadooghi-Alvandi and Nematollahi (1989) and others. In this Regard, our next loss function is

$$L_2(\delta, \theta) = b\{e^{a(\delta-\theta)} - a(\delta - \theta) - 1\} \tag{1.2}$$

where $a \neq 0$ and $b > 0$. The region of the admissibility and inadmissibility of the estimators of the form $\sum_{i=1}^{k} c_i \overline{X}_i + d$ under the loss function(1.2) are derived in section 5 and 6.

2. Risk Unbiased Equivariant Estimation

From decision theoretic approach when symmetries are present in a problem, It is natural to require a corresponding symmetry to hold for the estimators. The location parameter estimation problem is an important example. It is symmetric, or, to use the usual terminology, equivariant with respect to translation of the sample space, that is

$$\delta(X+a)=\delta(X)+a \text{ for all a} \tag{2.1}$$

Where $=(X_1, X_2, \ldots, X_n)$. An estimator satisfying (2.1) will called equivariant under translation. An alternative impartiality restriction which is applicable to our problem is the condition of unbiasedness.following Lehman and Casella (1998), an estimator δ of θ is said to be risk-unbiased if it satisfies

$$E_\theta[L(\theta,\delta)] \le E_\theta[L(\theta',\delta)] \text{ for all } \theta' = \theta \tag{2.2}$$

and if the loss is as in (1.2),(2.2) reduces to

$$E_\theta\left[e^{a\hat{\theta}}\right]=e^{a\theta} \tag{2.3}$$

now note that

$$E_\theta\left[e^{a\overline{X}_i-\frac{a^2\sigma^2{}_i}{2n_i}}\right]=e^{a\theta}$$

so $a\overline{X}_i-\frac{a^2\sigma^2{}_i}{2n_i}, i=1,\ldots,k$ are all risk unbiased estimators of θ. Now, consider a combined estimator of the from

$$\hat{\theta}_\alpha=\sum_{i=1}^k \alpha_i(\overline{X}_i-\frac{a^2\sigma^2{}_i}{2n_i})$$

Where $0<\alpha_i<1$, $i=1,\ldots,k,$ are real numbers and $\sum_{i=1}^k \alpha_i$=1. We can verify that $\hat{\theta}_\alpha$ is a translation-equivariant estimator. Also note that

$$E\left[e^{a\hat{\theta}_\alpha+\frac{a^2}{2}\sum_{i=1}^k\frac{a_i(1-\sigma_i)\sigma^2{}_i}{n_i}}\right]=e^{a\theta}$$

and hence

$$\theta_{\widehat{R.U}}(\alpha)=\sum_{i=1}^k \alpha_i\left(\overline{X}_i-\frac{a^2\sigma^2{}_i}{2n_i}\right)+\frac{a}{2}\sum_{i=1}^k\frac{a_i(1-\sigma_i)\sigma^2{}_i}{n_i}$$

Is a risk unbiased estimator of θ on the basis of $\overline{X}_i\,'s$ under the LINEX loss function. The risk function of $\theta_{\widehat{R.U}}(\alpha)$ with respect to the loss (1.2) is easily computed as

$$R(\theta,\theta_{\widehat{R.U}})=\frac{ba^2}{2}\sum_{i=1}^k\frac{\alpha^2{}_i\sigma^2{}_i}{n_i} \tag{2.4}$$

The risk (2.4) in minimized under $\sum_{i=1}^k \alpha_i$=1, when

$$\alpha_i=\frac{n_i/\sigma^2{}_i}{\sum_{i=1}^k n_i/\sigma^2{}_i},\ i=1,\ldots,k$$

and hence the Best Linear Risk Unbiased Equivariant (BLURE) estimator of θ under the LINEX loss is

$$\sum_{i=1}^k(\frac{n_i/\sigma^2{}_i}{\sum_{i=1}^k n_i/\sigma^2{}_i})\overline{X}_i-\frac{a}{2\sum_{i=1}^k n_i/\sigma^2{}_i} \tag{2.5}$$

With the same approach, the BLRUE estimator of θ under the squared error (1.1) is

$$\sum_{i=1}^k(\frac{n_i/\sigma^2{}_i}{\sum_{i=1}^k n_i/\sigma^2{}_i})\overline{X}_i \tag{2.6}$$

The estimator (2.6) is also the unique minimum variance unbiased estimator (UMVUE) as well as the best linear unbiased estimator (BLUE) (without normality) for estimating θ. Both estimators (2.5) and (2.6) are special cases of the more general class of linear estimators of the form $\sum_{i=1}^k c_i\,\overline{X}_i+d$. To study admissibility of the estimators (2.5) and (2.6), we study admissibility of the class of linear estimators of the form $\sum_{i=1}^k c_i\,\overline{X}_i+d$.

It should be mentioned here that in (2.5) and (2.6), the BLURE estimators are seen to depend on $\sigma_i{}^2(i=1,\ldots,k)$.when $\sigma_i{}^2(i=1,\ldots,k)$ are completely unknown, they can be replaced by $\sum_{i=1}^k\left(X_i-\overline{X}_i\right)^2/(n_i-1)$ $(i=1,\ldots,k)$.

In this case reasonable estimators of θ are provided by

$$\sum_{i=1}^{k}(\frac{n_i/s^2{}_i}{\sum_{i=1}^{k} n_i/s^2{}_i})\overline{X}_i - \frac{a}{2\sum_{i=1}^{k} n_i/s^2{}_i} \tag{2.7}$$

And

$$\sum_{i=1}^{k}(\frac{n_i/s^2{}_i}{\sum_{i=1}^{k} n_i/s^2{}_i})\overline{X}_i \tag{2.8}$$

Obviously the estimators (2.7) and (2.8) are location equivariant (see(2.1)) but their risks are complicated.

3. Admissibility Results under Loss(1.1)

Consider the admissibility of an arbitrary linear function $\sum_{i=1}^{k} c_i\ \overline{X}_i + d$ under the loss (1.1). The risk function$\sum_{i=1}^{k} c_i\ \overline{X}_i + d$ with respect to the squared error loss (1.1) is

$$\rho(c_1, \dots, c_k, d) = E[\sum_{i=1}^{k} c_i\ \overline{X}_i + d - \theta]^2$$

$$= \sum_{i=1}^{k} \frac{c_i{}^2\sigma^2{}_i}{n_i} + [(\sum_{i=1}^{k} c_i - 1)\theta + d]^2 \tag{3.1}$$

So, we have the following theorem.

Theorem 3.1: The estimator $\sum_{i=1}^{k} c_i\ \overline{X}_i + d$ is admissible for θ whenever

$$0 \le c_i \le 1, i = 1, \dots, k \text{ , and } 0 \le \sum_{i=1}^{k} c_i < 1 \text{ or } c_i = \frac{n_i/\sigma^2{}_i}{\sum_{i=1}^{k} n_i/\sigma^2{}_i},\ i = 1, \dots, k, \text{and } d = 0.$$

Proof: The notation $\delta(c_1, \dots, c_k, d)$ is used for $\sum_{i=1}^{k} c_i\ \overline{X}_i + d$.

(i)The case $0 \le c_i \le 1,\ i = 1, \dots, k,$ and $0 \le \sum_{i=1}^{k} c_i < 1$ is considered first. If $c_i = 0,\ i = 1, \dots, k,$ then $\delta(0, \dots, 0, d)$ is admissible since it is the only estimator with zero risk at $\theta = d$. For finding the Bayes estimator of θ, consider the normal prior with mean μ and variance τ [2.] The posterior distribution is then normal with mean and variance given by

$$\frac{\sum_{i=1}^{k}\frac{n_i}{\sigma^2{}_i}\overline{X}_i + \frac{\mu}{\tau 2}}{\sum_{i=1}^{k}\frac{n_i}{\sigma^2{}_i} + \frac{1}{\tau 2}} \text{ and } \frac{1}{\sum_{i=1}^{k}\frac{n_i}{\sigma^2{}_i} + \frac{1}{\tau 2}}$$

Respectively.it can be seen that the unique Bayes estimator is

$$\sum_{i=1}^{k}(\frac{\frac{n_i}{\sigma^2{}_i}}{\sum_{i=1}^{k}\frac{n_i}{\sigma^2{}_i} + \frac{1}{\tau 2}})\ \overline{X}_i + \frac{\frac{\mu}{\tau 2}}{\sum_{i=1}^{k}\frac{n_i}{\sigma^2{}_i} + \frac{1}{\tau 2}} \tag{3.2}$$

and that the associated Bayes risk is finite and hence admissible. It follows that $\delta(c_1, \dots, c_k, d)$ is admissible whenever $0 \le c_i \le 1,, i = 1, \dots, k$, and $0 \le \sum_{i=1}^{k} c_i < 1$.

(ii) If $c_i = \frac{\frac{n_i}{\sigma^2{}_i}}{\sum_{i=1}^{k}\frac{n_i}{\sigma^2{}_i}} = c_i'$ (say), $i = 1, \dots, k,$ and $d = 0$, the risk of $\delta(c_1', \dots, c_k', 0)$ as seen from (3.1) is given by

$$\rho(c_1', \dots, c_k', 0) = \frac{1}{\sum_{i=1}^{k} n_i/\sigma^2{}_i}$$

Note that if $\sum_{i=1}^{k} c_i = 1$ and $= 0$, then we have

$$\rho(c_1, \dots, c_k, d) = \sum_{i=1}^{k} \frac{c_i{}^2\sigma^2{}_i}{n_i} \tag{3.3}$$

It can be shown that the risk (3.3) is minimized under $\sum_{i=1}^{k} c_i = 1$, when $c_i = c_i'$, and hence $\sum_{i=1}^{k} c_i\ \overline{X}_i$ is inadmissible when $c_i \neq c_i'$. To show that δ(c$_1$′, …, c$_k$′,0) is admissible, the limiting Bayes method due to Blyth (1951)may be used. Suppose that δ(c$_1$′, …, c$_k$′,0) is not admissible. Then, there is an estimator δ* such that

$$R(\theta, \delta^*) \leq R(\theta, \sum_{i=1}^{k} c_i \, \overline{X}_i)$$

$$= \frac{1}{\sum_{i=1}^{k} n_i/\sigma^2{}_i}$$

For all θ, and with strict inequality for at least some θ. Now, $R(\theta, \delta)$ is a continuous function of θ for every δ so that there exists $\epsilon > 0$ and $\theta_0 < \theta_1$ such that

$$R(\theta, \delta^*) < \frac{1}{\sum_{i=1}^{k} n_i/\sigma^2{}_i} - \epsilon$$

For all $\theta_0 < \theta < \theta_1$. Let r_τ^* be the average risk of δ^* with respect to the prior distribution $N(0, \tau^2)$, and let r_τ be the Bayes risk of the Bayes estimator (3.2) with respect to $N(0, \tau^2)$. Then it follows that

$$r_\tau = \frac{1}{\sum_{i=1}^{k} \frac{n_i}{\sigma^2{}_i} + \frac{1}{\tau^2}}$$

Hence

$$\frac{\frac{1}{\sum_{i=1}^{k} \frac{n_i}{\sigma^2{}_i}} - r_\tau^*}{\frac{1}{\sum_{i=1}^{k} \frac{n_i}{\sigma^2{}_i}} - r_\tau} = \frac{\frac{1}{\sqrt{2\pi}\tau}\int_{-\infty}^{+\infty}\left[\frac{1}{\sum_{i=1}^{k} \frac{n_i}{\sigma^2{}_i}} - R(\theta, \delta^*)\right] e^{\frac{-\theta^2}{2\tau^2}} d\theta}{\frac{1}{\sum_{i=1}^{k} \frac{n_i}{\sigma^2{}_i}} - \frac{1}{\sum_{i=1}^{k} \frac{n_i}{\sigma^2{}_i} + \frac{1}{\tau^2}}}$$

$$\geq \frac{\tau\left(\sum_{i=1}^{k} \frac{n_i}{\sigma^2{}_i}\right)\left(\sum_{i=1}^{k} \frac{n_i}{\sigma^2{}_i} + \frac{1}{\tau^2}\right)\epsilon}{\sqrt{2\pi}} \int_{\theta_0}^{\theta_1} e^{\frac{-\theta^2}{2\tau^2}} d\theta.$$

The integrand converges monotonically to 1 as $\tau \to \infty$ and hence by the Lebesgue monotone convergence theorem, the integral converges to $\theta_1 - \theta_0$ and hence the ratio converges to ∞. Thus, there exists $\tau_0 < \infty$ such that $r_{\tau_0^*} < r_{\tau_0}$, which contradicts the fact that r_{τ_0} is the Bayes risk for $N(0, \tau_0^2)$. It follows that δ(c$_1$′, …, c$_k$′,0) is admissible.

4. The Inadmissibility Results under Loss (1.1)

To see what can be said about the other values of c_i's, $i = 1, \ldots, k,$ we shall now prove an inadmissibility result for linear estimators $\sum_{i=1}^{k} c_i \, \overline{X}_i + d$, which is quite general and in particular does not require the assumption of normality.

Theorem 4.1: The estimator $\sum_{i=1}^{k} c_i \, \overline{X}_i + d$ is inadmissible under squared error loss whenever one of the following conditions hold.

(i) $c_i > 1$, for some $i = 1,2, \ldots, k$

(ii) $c_i \leq 1$, $c_i + c_j > 1$, for some $i, j = 1,2, \ldots, k$

(iii) $c_i < 0$, $\sum_{j \neq i} c_j \leq 1$, for some $i = 1,2, \ldots, k$

(iv) $c_i \leq 1$, $\sum_{j \neq i} c_j < 0$, for some $i = 1,2, \ldots, k$

(v) $\sum_{i=1}^{k} c_i \leq 0$.

Proof : (i) If $c_i > 1$, for some $i = 1,2, \ldots, k$, then it follows from (3.1) that

$$\rho(c_1, \ldots, c_k, d) \geq \frac{{c_i}^2 {\sigma^2}_i}{n_i} > \frac{{\sigma^2}_i}{n_i} = \rho(0, \ldots, 0,1,0, \ldots, 0)$$

So that $\sum_{i=1}^{k} c_i \, \overline{X}_i + d$ is dominated by $\overline{X}_i$ and hence is inadmissible.

(ii) If $c_i \leq 1$, $c_i + c_j > 1$, for some $i, j = 1,2, \ldots, k$, then $c_j^2 > (1 - c_i)^2$ and hence

$$\rho(c_1, \dots, c_k, d) \geq \frac{{c_i}^2 {\sigma^2}_i}{n_i} + \frac{{c_j}^2 {\sigma^2}_j}{n_j}$$

$$> \frac{{c_i}^2 {\sigma^2}_i}{n_i} + \frac{(1-c_i)^2 {\sigma^2}_j}{n_j}.$$

But the function $g(c_i) = \frac{{c_i}^2 {\sigma^2}_i}{n_i} + \frac{(1-c_i)^2 {\sigma^2}_j}{n_j}$ is minimized when $c_i = \frac{\sigma^2{}_j/n_j}{\sigma^2{}_i/n_i + \sigma^2{}_j/n_j} = c_{min}$ (say), also

$$g(c_{min}) = \frac{\sigma^2{}_i \sigma^2{}_j / n_i n_j}{\sigma^2{}_i/n_i + \sigma^2{}_j/n_j} = R(\theta, c_{min}\overline{X}_i + (1 - c_{min})\overline{X}_j).$$

Thus, $\sum_{i=1}^{k} c_i \ \overline{X}_i + d$ is dominated by $c_{min}\overline{X}_i + (1 - c_{min})\overline{X}_j$ and hence is inadmissible.

(iii) If $c_i < 0$, $\sum_{j \neq i} c_j \leq 1$, for some $i = 1, 2, \dots, k$, then $(\sum_{i=1}^{k} c_i \ - 1)^2 > (\sum_{j \neq i} c_j - 1)^2$ and hence

$$\rho(c_1, \dots, c_k, d) > \sum_{j \neq i} \frac{{c_j}^2 {\sigma^2}_j}{n_j} + \left[\left(\sum_{i=1}^{k} c_i \ - 1\right)\theta + d\right]^2$$

$$= \sum_{j \neq i} \frac{{c_j}^2 {\sigma^2}_j}{n_j} + \left(\sum_{i=1}^{k} c_i \ - 1\right)^2 \left[\theta + \frac{d}{\sum_{i=1}^{k} c_i \ - 1}\right]^2$$

$$> \sum_{j \neq i} \frac{{c_j}^2 {\sigma^2}_j}{n_j} + (\sum_{j \neq i} c_j - 1)^2 \left[\theta + \frac{d}{\sum_{i=1}^{k} c_i \ - 1}\right]^2$$

$$= \sum_{j \neq i} \frac{{c_j}^2 {\sigma^2}_j}{n_j} + \left[\left(\sum_{j \neq i} c_j - 1\right)\theta + \frac{(\sum_{j \neq i} \ c_j - 1)d}{\sum_{j \neq i} \ c_i - 1}\right]^2$$

$$= \rho\left(c_1, \dots, c_{i-1}, 0, c_{i+1}, \dots, c_k, \frac{(\sum_{j \neq i} \ c_j - 1)d}{\sum_{j \neq i} \ c_i - 1}\right).$$

Thus in this case, $\sum_{i=1}^{k} c_i \ \overline{X}_i + d$ is dominated by the estimator

$$\sum_{j \neq i} \ c_j \ \overline{X}_j + \frac{(\sum_{j \neq i} \ c_j - 1)d}{\sum_{j \neq i} \ c_i - 1}.$$

(iv) If $c_i \leq 1$, $\sum_{j \neq i} c_j < 0$, for some $i = 1, 2, \dots, k$, then $(\sum_{i=1}^{k} c_i \ - 1)^2 > (c_i - 1)^2$ and hence

$$\rho(c_1, \dots, c_k, d) \geq \frac{{c_i}^2 {\sigma^2}_i}{n_i} + \left[\left(\sum_{i=1}^{k} c_i \ - 1\right)\theta + d\right]^2$$

$$= \frac{{c_i}^2 {\sigma^2}_i}{n_i} + (\sum_{i=1}^{k} c_i \ - 1)^2 \left[\theta + \frac{d}{\sum_{i=1}^{k} c_i \ - 1}\right]^2$$

$$> \frac{{c_i}^2 {\sigma^2}_i}{n_i} + (c_i - 1)^2 \left[\theta + \frac{d}{\sum_{i=1}^{k} c_i \ - 1}\right]^2$$

$$= \frac{{c_i}^2 {\sigma^2}_i}{n_i} + \left[(c_i - 1)\theta + \frac{(c_i - 1)d}{\sum_{i=1}^{k} c_i \ - 1}\right]^2$$

$$= \rho\left(0, \dots, 0, c_i, 0, \dots, 0, \frac{(c_i - 1)d}{\sum_{i=1}^{k} c_i - 1}\right).$$

Thus, $\sum_{i=1}^{k} c_i \ \overline{X}_i + d$ is dominated by $c_i \overline{X}_i + \frac{(c_i-1)d}{\sum_{i=1}^{k} c_i - 1}$.

(v) If $\sum_{i=1}^{k} c_i \leq 0$, then($\sum_{i=1}^{k} c_i - 1)^2 > 1$ and hence

$$\rho(c_1, \dots, c_k, d) \geq \left[\left(\sum_{i=1}^{k} c_i - 1\right)\theta + d\right]^2$$

$$= \left(\sum_{i=1}^{k} c_i - 1\right)^2 \left[\theta + \frac{d}{\sum_{i=1}^{k} c_i - 1}\right]^2$$

$$> \left[\theta + \frac{d}{\sum_{i=1}^{k} c_i - 1}\right]^2$$

$$= \rho\left(0, \dots, 0, \frac{-d}{\sum_{i=1}^{k} c_i - 1}\right).$$

Thus, $\sum_{i=1}^{k} c_i \ \overline{X}_i + d$ is dominated by the constant estimator $\delta = \frac{-d}{\sum_{i=1}^{k} c_i - 1}$.

5. The Admissibility Results Under Loss (1.2)

Consider the question of admissibility of the estimators of the from $\sum_{i=1}^{k} c_i \ \overline{X}_i + d$ under the loss (1.2). Since the parameter b does not have any influences on our results so without loss of generality we can take $b = 1$. The risk function of the estimator $\sum_{i=1}^{k} c_i \ \overline{X}_i + d$ with respect to the loss (1.2) is easily computed as

$$\gamma(c_1, \dots, c_k, d) = E\left[e^{a(\sum_{i=1}^{k} c_i \overline{X}_i + d - \theta)} - a\left(\sum_{i=1}^{k} c_i \ \overline{X}_i + d - \theta\right) - 1\right]$$

$$= e^{\frac{a^2}{2}\left(\sum_{i=1}^{k} \frac{{c_i}^2 {\sigma^2}_i}{n_i}\right) + a\left(\sum_{i=1}^{k} c_i - 1\right)\theta + ad}$$

$$-a\left(\sum_{i=1}^{k} c_i - 1\right)\theta - ad - 1. \tag{5.1}$$

So, we have the following theorem.

Theorem 5.1: The estimator $\sum_{i=1}^{k} c_i \ \overline{X}_i + d$ is admissible for θ whenever $0 \leq c_i < 1$, $i = 1, \dots, k$ and $0 \leq \sum_{i=1}^{k} c_i < 1$, or $c_i = \frac{n_i/\sigma_i^2}{\sum_{i=1}^{k} n_i/\sigma_i^2}$, $i = 1, \dots, k$ and $d = \frac{-a}{2\sum_{i=1}^{k} n_i/\sigma_i^2}$.

Proof :

(i) The case $0 \leq c_i < 1$, $i = 1, \dots, k$ and $0 \leq \sum_{i=1}^{k} c_i < 1$, is considered first. If $c_i = 0$, $i = 1, \dots, k$, then $\delta(0, \dots, 0, d)$ is admissible since it is the only estimator with zero risk at $\theta = d$.

Now consider the Bayes estimator when the prior distribution on θ is normal with mean μ and variance τ^2. Then, using (3.2) in Zellner (1986), it follows that the unique Bayes estimator is

$$\sum_{i=1}^{k}\left(\frac{n_i/\sigma_i^2}{\sum_{i=1}^{k} n_i/\sigma_i^2 + 1/\tau^2}\right)\overline{X}_i - \frac{1}{\sum_{i=1}^{k} n_i/\sigma_i^2 + 1/\tau^2}\left(\frac{a}{2} - \frac{\mu}{\tau^2}\right). \tag{5.2}$$

and that the associated Bayes risk is finite and hence admissible. It follows that $\sum_{i=1}^{k} c_i \ \overline{X}_i + d$ is admissible whenever $0 \leq c_i < 1$, $i = 1, \dots, k$ and $0 < \sum_{i=1}^{k} c_i < 1$.

(ii) If $c_i = \frac{n_i/\sigma_i^2}{\sum_{i=1}^{k} n_i/\sigma_i^2}$ (the same c_i') and $d = \frac{-a}{2\sum_{i=1}^{k} n_i/\sigma_i^2} = d'$(say), then the risk of $\delta(c_1', \dots, c_k', d')$ as is seen from

(5.1) is given by

$$\gamma(c_1', \ldots, c_k', d') = \frac{a}{2\sum_{i=1}^{k} n_i/\sigma_i^2}.$$

Note that if $\sum_{i=1}^{k} c_i = 1$, then we have

$$\gamma(c_1, \ldots, c_k, d) = e^{\frac{a^2}{2}\left(\sum_{i=1}^{k}\frac{c_i{}^2\sigma^2{}_i}{n_i}\right)+ad} - ad - 1. \quad (5.3)$$

It can be shown that the risk (5.3) is minimized when $c_i = c_i'$ and $= d'$, and hence in this case $\delta(c_1, \ldots, c_k, d)$ is inadmissible when $c_i \neq c_i'$ and $d \neq d'$. To show that $\delta(c_1', \ldots, c_k', d')$ is admissible, again the limiting Bayes method may be used. Suppose that $\delta(c_1', \ldots, c_k', d')$ is not admissible, then there exists an estimator δ^* such that

$$R(\theta, \delta^*) \leq R(\theta, \sum_{i=1}^{k} c_i' \overline{X}_i + d')$$

$$= \frac{a^2}{2\sum_{i=1}^{k} n_i/\sigma_i^2}$$

for all θ, and with strict inequality for at least some θ. By the continuity of $R(\theta, \delta)$, there exists $\epsilon > 0$ and $\theta_0 < \theta_1$ such that

$$R(\theta, \delta^*) < \frac{a^2}{2\sum_{i=1}^{k} n_i/\sigma_i^2} - \epsilon$$

for all $\theta_0 < \theta < \theta_1$. Let r_τ^* be the average risk of δ^* with respect to the prior distribution $N(0, \tau^2)$. Then it can be shown that

$$r_\tau = \frac{a^2}{2(\sum_{i=1}^{k} n_i/\sigma_i^2 + 1/\tau^2)}$$

Hence

$$\frac{\frac{a^2}{2\sum_{i=1}^{k} n_i/\sigma_i^2} - r_\tau^*}{\frac{a^2}{2\sum_{i=1}^{k} n_i/\sigma_i^2} - r_\tau} = \frac{\frac{1}{\sqrt{2\pi}\tau}\int_{-\infty}^{+\infty}\left[\frac{a^2}{\sum_{i=1}^{k} n_i/\sigma^2{}_i} - R(\theta, \delta^*)\right] e^{\frac{-\theta^2}{2\tau^2}} d\theta}{\frac{a^2}{2\sum_{i=1}^{k} n_i/\sigma^2{}_i} - \frac{a^2}{2(\sum_{i=1}^{k} n_i/\sigma_i^2 + 1/\tau^2)}}$$

$$> \frac{2\tau\left(\sum_{i=1}^{k} n_i/\sigma^2{}_i\right)\left(\sum_{i=1}^{k} n_i/\sigma^2{}_i + 1/\tau^2\right)\epsilon}{\sqrt{2\pi}a^2} \int_{\theta_0}^{\theta_1} e^{\frac{-\theta^2}{2\tau^2}} d\theta.$$

The integrand converges monotonically to 1 as $\tau \to \infty$ and hence by the Lebesgue monotone convergence theorem, the integral converges to $\theta_1 - \theta_0$ and hence the ratio converges to ∞. Thus, there exists $\tau_0 < \infty$ such that $r_{\tau_0^*} < r_{\tau_0}$, which contradicts the fact that r_{τ_0} is the Bayes risk for $N(0, \tau_0^2)$. It follows that $\sum_{i=1}^{k} c_i \overline{X}_i + d$ is admissible when $c_i = c_i'$ for $= 1, \ldots, k$, and $d = d'$.

6. The Inadmissibility Results Under Loss (1.2)

We shall now prove an inadmissibility result for linear estimators under the loss (1.2).

Theorem 6.1: The linear estimator $\sum_{i=1}^{k} c_i \overline{X}_i + d$ is inadmissible under LINEX loss whenever one of the following conditions hold

(i) $c_i > 1$ for some $i = 1, \ldots, k$

(ii) $c_i \leq 1$, $c_i + c_j > 1$, for some $i, j = 1,2, \ldots, k$

(iii) $c_i < 0$ for some $i = 1, \ldots, k$.

Proof: (i) If $c_i > 1$ for some $i = 1, \ldots, k$, then

$$\gamma(c_1,\dots,c_k,d)=e^{\frac{a^2}{2}\left(\sum_{i=1}^{k}\frac{c_i{}^2\sigma^2{}_i}{n_i}\right)+a\left(\sum_{i=1}^{k}c_i-1\right)\theta+ad}-a\left(\sum_{i=1}^{k}c_i-1\right)\theta-ad-1$$

$$\geq\frac{a^2}{2}\left(\sum_{j=1}^{k}\frac{c_j{}^2\sigma^2{}_j}{n_j}\right)\text{ (since } e^x\geq 1+x)$$

$$\geq\quad\frac{a^2c_i{}^2\sigma^2{}_i}{2n_i}$$

$$>\quad\frac{a^2\sigma^2{}_i}{2n_i}$$

$$=\gamma\left(0,\dots,0,1,0,\dots,0,\frac{-a\sigma^2{}_i}{2n_i}\right).$$

So that $\sum_{i=1}^{k}c_i\ \overline{X}_i+d$ is dominated by $\overline{X}_i-\frac{a\sigma^2{}_i}{2n_i}$.

(ii) If $c_i\leq 1$, $c_i+c_j>1$, for some $i,j=1,2,\dots,k$, then $c_j^2>(1-c_i)^2$ and hence

$$\gamma(c_1,\dots,c_k,d)\geq\frac{a^2c_i{}^2\sigma_i^2}{2n_i}+\frac{a^2c_j{}^2\sigma_j^2}{2n_j}$$

$$>\frac{a^2c_i{}^2\sigma_i^2}{2n_i}+\frac{a^2(1-c_i)^2\sigma_j^2}{2n_j}.$$

But the function $g(c_i)=\frac{a^2c_i{}^2\sigma_i^2}{2n_i}+\frac{a^2(1-c_i)^2\sigma_j^2}{2n_j}$ is minimized at $c_i=\frac{\sigma_j^2/n_j}{\sigma_i^2/n_i+\sigma_j^2/n_j}$ (the same c_{min}), and

$$g(c_{min})=\frac{a^2\sigma_i^2\sigma_j^2/n_in_j}{2(\sigma_i^2/n_i+\sigma_j^2/n_j)}=R(\theta,c_{min}\overline{X}_i+(1-c_{min})\overline{X}_j+d_0)$$

Where

$$d_0=\frac{-a^2\sigma_i^2\sigma_j^2/n_in_j}{2(\sigma_i^2/n_i+\sigma_j^2/n_j)}.$$

Thus in this case, $\sum_{i=1}^{k}c_i\ \overline{X}_i+d$ is dominated by $c_{min}\overline{X}_i+(1-c_{min})\overline{X}_j+d_0)$.

(iii) If $c_i<0$ for some $i=1,\dots,k$, then it will be shown that $\sum_{i=1}^{k}c_i\ \overline{X}_i+d$ is dominated by $\sum_{j\neq i}c_j^*\,\overline{X}_j+d^*$ where $c_j^*=\frac{c_j}{1-c_i}$ for $j\neq i$ and $d^*=\frac{d}{1-c_i}+\frac{ac_i}{2(1-c_i)^3}\sum_{j\neq i}\frac{c_j{}^2\sigma_j^2}{n_j}$. since $e^{\frac{a^2c_i{}^2\sigma_i^2}{2n_i}}>1$, we have from (5.1)

$$\gamma(c_1,\dots,c_k,d)-\gamma(c_1^*,\dots,c_{i-1}^*,0,c_{i+1}^*,\dots,c_k^*,d^*)$$

$$>e^{\frac{a^2}{2}\sum_{j\neq i}\frac{c_j{}^2\sigma_j^2}{n_j}+a(\sum_{j=1}^{k}c_j-1)\theta+ad}$$

$$-e^{\frac{a^2}{2}\sum_{j\neq i}\frac{c_j^{*2}\sigma_j^2}{n_j}+a(\sum_{j\neq i}c_j^*-1)\theta+ad^*}$$

$$-ac_i\theta-a\sum_{j\neq i}(c_j-c_j^*)\theta-a(d-d^*)\ .$$

Now, using the inequality

$$e^x-e^y\geq(x-y)e^y\ \text{ for all } x,y,$$

and noting that $c_j^2-c_j^{*2}=\frac{c_i(c_i-2)c_j^2}{(1-c_i)^2}\geq 0$, for all $j=1,\dots,k$, and $j\neq i$,it follows that

$$\gamma(c_1,\dots,c_k,d)-\gamma(c_1^*,\dots,c_{i-1}^*,0,c_{i+1}^*,\dots,c_k^*,d^*)$$
$$>\left[ac_i\theta+a\sum\nolimits_{j\neq i}(c_j-c_j^*)\theta+a(d-d^*)\right]$$
$$\times\ e^{\frac{a^2}{2}\sum_{j\neq i}\frac{c_j^{*2}\sigma_j^2}{n_j}+a(\sum_{j\neq i}c_j^*)\theta+ad^*-a\theta}$$
$$-ac_i\theta-a\sum\nolimits_{j\neq i}(c_j-c_j^*)\theta-a(d-d^*).$$

But $c_j-c_j^*=-c_ic_j^*$ for $j\neq i$ and $-d^*=-c_id^*-\frac{ac_i}{2}\sum_{j\neq i}\frac{c_j^{*2}\sigma_j^2}{n_j}$, hence

$$\gamma(c_1,\dots,c_k,d)-\gamma(c_1^*,\dots,c_{i-1}^*,0,c_{i+1}^*,\dots,c_k^*,d^*)$$
$$>\left[ac_i\theta-ac_i(\sum\nolimits_{j\neq i}c_j^*)\theta-ac_id^*-\frac{a^2c_i}{2}\sum\nolimits_{j\neq i}\frac{c_j^{*2}\sigma_j^2}{n_j}\right]$$
$$\times\ e^{\frac{a^2}{2}\sum_{j\neq i}\frac{c_j^{*2}\sigma_j^2}{n_j}+a(\sum_{j\neq i}c_j^*)\theta+ad^*-a\theta}$$
$$-ac_i\theta+ac_i(\sum\nolimits_{j\neq i}c_j^*)\theta+ac_id^*+\frac{a^2c_i}{2}\sum\nolimits_{j\neq i}\frac{c_j^{*2}\sigma_j^2}{n_j}$$
$$=-c_i\left[\frac{a^2}{2}\sum\nolimits_{j\neq i}\frac{c_j^{*2}\sigma_j^2}{n_j}+a(\sum\nolimits_{j\neq i}c_j^*)\theta+ad^*-a\theta\right]$$
$$\times\left[e^{\frac{a^2}{2}\sum_{j\neq i}\frac{c_j^{*2}\sigma_j^2}{n_j}+a(\sum_{j\neq i}c_j^*)\theta+ad^*-a\theta}-1\right]\geq 0.$$

Since $c_i<0$ and $(e^y-1)\geq 0$, for all y.

Remark (6.1): The BLRUE estimators given in (2.5) and (2.6) are admissible and minimax.

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