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The Adjusted Log-logistic Generalized Exponential Distribution with Application to Lifetime Data

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Abstract

This paper introduces a new generator of probability distribution-the adjusted log-logistic generalized *(ALLoG)* distribution and a new extension of the standard one parameter exponential distribution called the adjusted log-logistic generalized exponential *(ALLoGExp)* distribution. The *ALLoGExp* distribution is a special case of the *ALLoG* distribution and we have provided some of its statistical and reliability properties. Notably, the failure rate could be monotonically decreasing, increasing or upside-down bathtub shaped depending on the value of the parameters δ and θ. The method of maximum likelihood estimation was proposed to estimate the model parameters. The importance and flexibility of the *ALLoGExp* distribution was demonstrated with a real and uncensored lifetime data set and its fit was compared with five other exponential related distributions. The results obtained from the model fittings shows that the *ALLoGExp* distribution provides a reasonably better fit than the one based on the other fitted distributions. The *ALLoGExp* distribution is therefore recommended for effective modelling of lifetime data sets.

Keywords: asymptotic adjustment, exponential distribution, log-logistic distribution, reliability, failure rate

1. Introduction

Due to the lack of fits that characterize the standard probability distributions in modelling various complex real data sets a lot of effort have been expended by researchers in developing new distributions as a way of circumventing the problem of inadequate fits of the already existing distributions. The new distributions often referred to as as the generalized class of distributions have consistently been shown to provide better fits than the existing (standard) ones. Almost all the available methods of generating new distributions in statistical literature depends on the cumulative distribution function of the standard distributions; for a holistic up-to-date review of these methods see; Nadarajah and Rocha (2016).

The main motivation of this paper stem from the trending literature of probability distribution construction and generalization; which in principle, entails the injection of one distribution into another distribution, in order to extend the injected distribution to a wider family of distribution with added flexibility. In the literature, the generalized distributions are often referred to as the G-distributions. For example,

1. Eugene *et al.* (2002) defined a new class of distribution called the Beta-G family of distributions as the logit of the beta random variable with cumulative density function *(cdf)* as

$$F(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^{G(x)} y^{\alpha-1}(1-y)^{\beta-1}dy;\ 0<\alpha,\ \beta<\infty,$$

   and probability density function *(pdf)* as

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}G^{\alpha-1}(x)[1-G(x)]^{\beta-1}\frac{dG(x)}{dx};\ 0<\alpha,\ \beta<\infty.$$

2. Cordeiro and de Castro (2011) introduced the Kumaraswamy-G family of distributions with *cdf*

$$F(x) = ab\int_0^{G(x)} y^{a-1}(1-y^a)^{b-1}dy; x \in (0,1),\ a,b > 0,$$

and *pdf*

$$f(x) = abg(x)G^{a-1}(x)[1-G^a(x)]^{b-1}; x \in (0,1),\ a,b > 0.$$

3. Zografos and Balakrishnan (2009) proposed the Gamma-G family of distributions based on baseline continuous distribution with reliability function $\bar{G}(x) = 1 - G(x)$. The *cdf* of the Gamma-G distributions is defined as

$$F(x) = \frac{1}{\Gamma(\delta)}\int_0^{-\log\bar{G}(x)} y^{\delta-1}e^{-y}dy;\ x \in \mathbf{R},\ \delta > 0,$$

and *pdf* as

$$f(x) = \frac{1}{\Gamma(\delta)}[-\log\bar{G}(x)]^{\delta-1}g(x);\ x \in \mathbf{R},\ \delta > 0.$$

4. An alternative version of Zografos and Balakrishnan (2009) was proposed by Ristić and Balakrishnan (2012). The *cdf* of the gamma generator due to Ristić and Balakrishnan (2012) is defined as

$$F(x) = 1 - \frac{1}{\Gamma(\delta)}\int_0^{-\log G(x)} y^{\delta-1}e^{-y}dy;\ x \in \mathbf{R},\ \delta > 0,$$

while its *pdf* is given by;

$$f(x) = \frac{1}{\Gamma(\delta)}[-\log G(x)]^{\delta-1}g(x);\ x \in \mathbf{R},\ \delta > 0.$$

5. Alzaatreh *et al.* (2013) introduced the Weibull-G distributions whose *cdf* is defined as

$$F(x) = \frac{\alpha}{\beta^\alpha}\int_0^{-\log\bar{G}(x)} y^{\alpha-1}e^{-\left(\frac{y}{\beta}\right)^\alpha}dy;\ x \in \mathbf{R},\ \alpha,\beta > 0,$$

and *pdf* as

$$f(x) = \frac{\alpha}{\beta^\alpha}\frac{g(x)}{\bar{G}(x)}\left[\frac{-\log\bar{G}(x)}{\beta}\right]^{\alpha-1} e^{-\left[\frac{-\log\bar{G}(x)}{\beta}\right]^\alpha};\ x \in \mathbf{R},\ \alpha,\beta > 0.$$

In all cases, $G(x)$ is the *cdf* of the injected distribution (or baseline distribution) with $g(x)$ as the corresponding *pdf*, $F(x)$ is the *cdf* of the new generalized version of $G(x)$ with $f(x)$ as the corresponding *pdf*, and $f(x)$ have the same support as $g(x)$.

and so on.

The aim of this paper is two-fold; first to introduce a new generator of distributions-the adjusted log-logistic generalized (*ALLoG*) distribution which as far as we know have not appeared in the literature before now and secondly, to introduce and give explicit statistical properties of the adjusted log-logistic generalized exponential (*ALLoGExp*) distribution as a sub-model of the *ALLoG* distribution which generalizes the standard one parameter exponential distribution. The *cdf G(x)* of the log-logistic distribution is given by;

$$G(x) = \frac{x^\theta}{\delta^\theta + x^\theta};\ x > 0,\ \delta,\theta > 0 \tag{1}$$

with the corresponding *pdf g(x)* defined as

$$g(x) = \frac{\frac{\theta}{\delta}\left(\frac{x}{\delta}\right)^{\theta-1}}{\left[1+\left(\frac{x}{\delta}\right)^\theta\right]^2};\ x > 0,\ \delta,\theta > 0, \tag{2}$$

where δ is the scale parameter and θ is the shape parameter.

We define the *cdf F(x)* and *pdf f(x)* of the new generator *ALLoG* with (1) and (2) as

$$F(x) = \frac{\theta(\delta^\theta + 1)}{\delta^\theta} \int_0^{G(x)} y^{\theta-1} \left[1 + \left(\frac{y}{\delta}\right)^\theta\right]^{-2} dy;\ x > 0,\ \delta, \theta > 0. \tag{3}$$

and

$$f(x) = (\delta^\theta + 1)\delta^\theta \theta g(x) G^{\theta-1}(x)[\delta^\theta + G^\theta(x)]^{-2};\ x > 0,\ \delta, \theta > 0, \tag{4}$$

respectively. Where, $\delta^\theta + 1$ in (3) is an asymptotic adjustment (normalizing constant); introduced, to ensure that $\lim_{x\to\infty} F(x) = 1$.

The quantile function is given by;

$$F^{-1}(p) = G^{-1}\left[\left(\frac{p\delta^\theta}{\delta^\theta + 1 - p}\right)^{\frac{1}{\theta}}\right];\ p \in (0, 1],\ \delta, \theta > 0, \tag{5}$$

Unlike the *cdf* and *pdf* of the Beta-G and Gamma-G family of distributions, the *cdf* and *pdf* of the new generator *ALLoG* is easy to work with analytically because, it does not contain any special function.

The remainder of this paper is organized as follows: In Section 2 we present the closed form expression for the probability density function *pdf* and the cumulative density function *cdf* and other statistical properties of the new probability distribution (*ALLoGExp*). In Section 3 the parameters of the *ALLoGExp* distribution is estimated through the method of maximum likelihood estimation. Section 4 contains some reliability characteristics and possible shapes of the *ALLoGExp* distribution. In Section 5 we illustrate the applicability of the *ALLoGExp* distribution with a real data set. Section 6 is the Monte-Carlo simulation study of the parameters and Finally, Section 7 gives the concluding remarks.

2. The *ALLoGExp* and its Properties

The standard one parameter exponential *(Exp)* distribution has its *cdf G(x)* as $G(x) = 1 - e^{-\eta x};\ x \geq 0,\ \eta > 0$ and *pdf g(x)* as $g(x) = \eta e^{-\eta x};\ x \geq 0,\ \eta > 0$; where, η is the rate parameter. Thus, it follows that the *cdf F(x)* of the *ALLoGExp* distribution is given by;

$$F(x) = \frac{(\delta^\theta + 1)(1 - e^{-\eta x})^\theta}{\delta^\theta + (1 - e^{-\eta x})^\theta};\ x \geq 0,\ \delta, \theta, \eta > 0, \tag{6}$$

with the corresponding *pdf f(x)* as

$$f(x) = (\delta^\theta + 1)\delta^\theta \theta \eta e^{-\eta x}(1 - e^{-\eta x})^{\theta-1}[\delta^\theta + (1 - e^{-\eta x})^\theta]^{-2};\ x \geq 0,\ \delta, \theta, \eta > 0, \tag{7}$$

where δ is the scale parameter, θ is the shape parameters and η is the rate parameter.

2.1 Quantile Function and Random Number Generation

By using (6), we obtain the quantile function of the *ALLoGExp* distribution as

$$F^{-1}(p) = -\frac{1}{\eta}\log\left(1 - \left[\frac{p\delta^\theta}{\delta^\theta + 1 - p}\right]^{\frac{1}{\theta}}\right);\ p \in (0, 1],\ \delta, \theta, \eta > 0. \tag{8}$$

Random samples from the *ALLoGExp* distribution can be obtained through the inverse transformation method of random number generation by simply substituting p in (8) with a Uniform $(0, 1)$ variate. Also, it is easy to obtain the median of the *ALLoGExp* distribution by simply substituting $p = 0.5$ in (8), which is given by;

$$F^{-1}(0.5) = -\frac{1}{\eta}\log\left(1 - \left[\frac{\delta^\theta}{2\delta^\theta + 1}\right]^{\frac{1}{\theta}}\right);\ \delta, \theta, \eta > 0. \tag{9}$$

2.2 Moments

In statistics and applications the moments of a random variable say X are important, they are used to characterize the underlying distribution. For example, to measure the center, spread/variation of the distribution, and to ascertain the degree of deviation from normality (skewness and kurtosis) of the distribution, etc.;

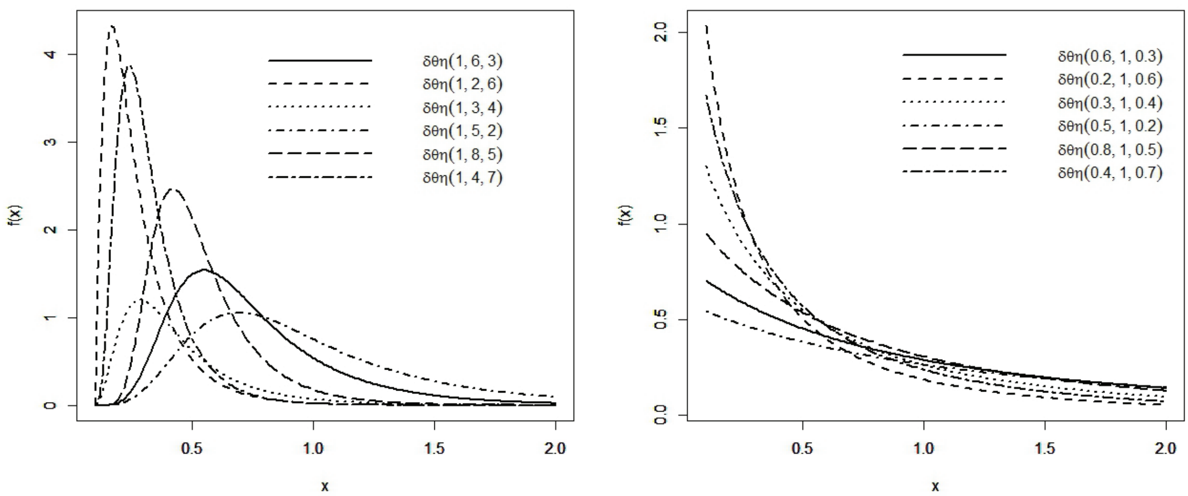


Figure 1. Possible shapes of the probability density function $pdf\ f(x)$ for some parameter values.

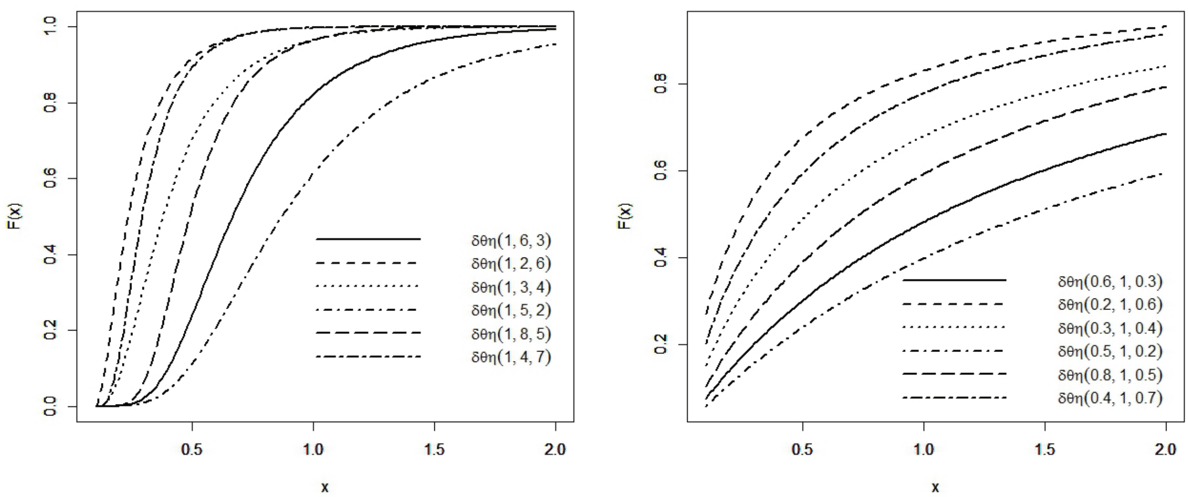


Figure 2. Possible shapes of the cumulative density function $cdf\ F(x)$ for some parameter values.

Theorem 2.1. If X follows the ALLoGExp distribution with pdf defined in (7) then its kth crude moment is given by;

$$\mu'_k = \frac{(\delta^\theta + 1)\theta}{(-\eta)^k \delta^\theta} \sum_{m=0}^{\infty} \left(\frac{-1}{\delta^\theta}\right)^m (m+1) \frac{\partial^k}{\partial \ell^k} \mathbf{B}(\ell + 1, \theta(m+1)) \Bigg|_{\ell=0},$$

where $\mathbf{B}(\cdot)$ is the beta function.

Proof. By using (7) we have

$$\mu'_k = \int_{\mathbf{R}} x^k f(x) dx. \tag{10}$$

Where (10) is computed as follows

$$\begin{aligned}\mu'_k &= \int_0^\infty x^k(\delta^\theta+1)\delta^\theta\theta\eta e^{-\eta x}(1-e^{-\eta x})^{\theta-1}[\delta^\theta+(1-e^{-\eta x})^\theta]^{-2}dx\\ &= (\delta^\theta+1)\delta^\theta\theta\eta\int_0^\infty x^k e^{-\eta x}(1-e^{-\eta x})^{\theta-1}[\delta^\theta+(1-e^{-\eta x})^\theta]^{-2}dx, \qquad (11)\end{aligned}$$

substituting $y=e^{-\eta x}$ in (11) we have

$$\begin{aligned}\mu'_k &= \frac{(\delta^\theta+1)\delta^\theta\theta}{(-\eta)^k}\int_0^1(\log(y))^k(1-y)^{\theta-1}[\delta^\theta+(1-y)^\theta]^{-2}dy\\ &= \frac{(\delta^\theta+1)\delta^\theta\theta}{(-\eta)^k}\int_0^1\frac{\partial^k}{\partial\ell^k}y^\ell(1-y)^{\theta-1}[\delta^\theta+(1-y)^\theta]^{-2}dy\Big|_{\ell=0}, \qquad (12)\end{aligned}$$

by expanding (12) we have

$$\begin{aligned}\mu'_k &= \frac{(\delta^\theta+1)\delta^\theta\theta}{(-\eta)^k}\frac{\partial^k}{\partial\ell^k}\int_0^1 y^\ell(1-y)^{\theta-1}\sum_{m=0}^\infty(-1)^m\binom{2+m-1}{m}(1-y)^{\theta m}\delta^{-\theta(2+m)}dy\Big|_{\ell=0}\\ &= \frac{(\delta^\theta+1)\theta}{(-\eta)^k\delta^\theta}\sum_{m=0}^\infty\left(\frac{-1}{\delta^\theta}\right)^m(m+1)\frac{\partial^k}{\partial\ell^k}\int_0^1 y^\ell(1-y)^{\theta(m+1)-1}dy\Big|_{\ell=0}\\ &= \frac{(\delta^\theta+1)\theta}{(-\eta)^k\delta^\theta}\sum_{m=0}^\infty\left(\frac{-1}{\delta^\theta}\right)^m(m+1)\frac{\partial^k}{\partial\ell^k}\mathbf{B}(\ell+1,\theta(m+1))\Big|_{\ell=0}. \qquad (13)\end{aligned}$$

□

Corollary 2.1.1. Evaluating (13) at k=1, 2, 3, and 4 we have the first four crude moments of the ALLoGExp distribution as follows:

$$\mu'_1=\frac{(\delta^\theta+1)\theta}{-\eta\delta^\theta}\sum_{m=0}^\infty\left(\frac{-1}{\delta^\theta}\right)^m(m+1)(\Psi(1)-\Psi(1+\theta(m+1)))\mathbf{B}(1,\theta(m+1)), \qquad (14)$$

$$\begin{aligned}\mu'_2=&\frac{(\delta^\theta+1)\theta}{(-\eta)^2\delta^\theta}\sum_{m=0}^\infty\left(\frac{-1}{\delta^\theta}\right)^m(m+1)(\Psi(1,1)-\Psi(1,1+\theta(m+1)))\mathbf{B}(1,\theta(m+1))\\ &+(\Psi(1)-\Psi(1+\theta(m+1)))^2\mathbf{B}(1,\theta(m+1)),\end{aligned} \qquad (15)$$

$$\begin{aligned}\mu'_3=&\frac{(\delta^\theta+1)\theta}{(-\eta)^3\delta^\theta}\sum_{m=0}^\infty\left(\frac{-1}{\delta^\theta}\right)^m(m+1)(\Psi(2,1)-\Psi(2,1+\theta(m+1)))\mathbf{B}(1,\theta(m+1))\\ &+3(\Psi(1,1)-\Psi(1,1+\theta(m+1)))(\Psi(1)-\Psi(1+\theta(m+1)))\mathbf{B}(1,\theta(m+1))\\ &+(\Psi(1)-\Psi(1+\theta(m+1)))^3\mathbf{B}(1,\theta(m+1)),\end{aligned} \qquad (16)$$

and

$$\begin{aligned}\mu'_4=&\frac{(\delta^\theta+1)\theta}{(-\eta)^4\delta^\theta}\sum_{m=0}^\infty\left(\frac{-1}{\delta^\theta}\right)^m(m+1)(\Psi(3,1)-\Psi(3,1+\theta(m+1)))\mathbf{B}(1,\theta(m+1))\\ &+4(\Psi(2,1)-\Psi(2,1+\theta(m+1)))(\Psi(1)-\Psi(1+\theta(m+1)))\mathbf{B}(1,\theta(m+1))\\ &+3(\Psi(1,1)-\Psi(1,1+\theta(m+1)))^2\mathbf{B}(1,\theta(m+1))+6(\Psi(1,1)\\ &-\Psi(1,1+\theta(m+1)))(\Psi(1)-\Psi(1+\theta(m+1)))^2\mathbf{B}(1,\theta(m+1))\\ &+(\Psi(1)-\Psi(1+\theta(m+1)))^4\mathbf{B}(1,\theta(m+1)),\end{aligned} \qquad (17)$$

where $\mathbf{B}(\cdot)$ is the beta function and $\Psi(\cdot)$ is the psi or digamma function.

Corollary 2.1.2. By appropriately using (14)-(17) we can obtain the variance (σ^2), coefficient of variation (CV), skewness (γ_1) and kurtosis (γ_2) of the ALLoGExp distribution as

$$\begin{aligned}\sigma^2 &= \mu_2' - \mu_1'^2,\\ CV &= \sqrt{\frac{\mu_2'}{\mu_1'^2} - 1},\\ \gamma_1 &= \frac{\mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3}{(\mu_2' - \mu_1'^2)^{\frac{3}{2}}},\end{aligned}$$

and

$$\gamma_2 = \frac{\mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4}{(\mu_2' - \mu_1'^2)^2},$$

respectively.

Theorem 2.2. If X follows the ALLoGExp distribution with pdf defined in (7) then its moment generating function (mgf) is given by;

$$M_X(t) = \theta(\delta^\theta + 1)\sum_{k,m=0}^{\infty}\frac{(-1)^m(m+1)t^k}{(-\eta)^k\delta^{\theta(m+1)}k!}\left.\frac{\partial^k}{\partial\ell^k}\mathbf{B}(\ell+1, \theta(m+1))\right|_{\ell=0}. \tag{18}$$

Proof. Using the definition of the *mgf* of the continuous random variable say X which is defined as

$$M_X(t) = E(e^{tx}) = E\left(\sum_{k=0}^{\infty}\frac{(tx)^k}{k!}\right) = \sum_{k=0}^{\infty}\frac{t^k}{k!}\mu_k', \tag{19}$$

and by substituting (13) into (19) it is clear that the *mgf* of the *ALLoGExp* distribution is as presented in (18). □

2.3 Entropy Measure

In this section we present the Rényi entropy measure of the *ALLoGExp* distribution. The Rényi entropy measure is used to quantify the uncertainty of variation in a random variable say X and the Rényi entropy measure of a continuous random variable is generally given by;

$$I_R(\varphi) = \frac{1}{1-\varphi}\log\left(\int_{\mathbf{R}} f^{\varphi}(x)dx\right);\ \varphi > 0\backslash\{1\}. \tag{20}$$

Theorem 2.3. If X follows the ALLoGExp distribution with pdf defined in (7) then its Rényi entropy measure is given by;

$$\begin{aligned}I_R(\varphi) = \frac{1}{1-\varphi}\log\Bigg[&\frac{([\delta^\theta + 1]\delta^\theta\theta)^\varphi}{\eta^{-\varphi+1}}\sum_{i,j=0}^{\infty}(-1)^{i+j}\\ &\times\frac{\delta^{-\theta(2\varphi+j)}\Gamma(\varphi(\theta-1)+1)\Gamma(2\varphi+j)\mathbf{B}(\varphi+i+1, \theta j+1)}{\Gamma(i+1)\Gamma(j+1)\Gamma(\varphi(\theta-1)-i+1)\Gamma(2\varphi)}\Bigg],\end{aligned}$$

where $\Gamma(\cdot)$ is the gamma function and $\mathbf{B}(\cdot)$ is the beta function.

Proof. Substituting (7) into (20), setting $\int_{\mathbf{R}} f^{\varphi}(x)dx$ to $\mathcal{A}_\varphi$ and evaluating the integral on the support $[0, \infty)$ gives

$$\begin{aligned}\mathcal{A}_\varphi &= \int_0^\infty f^\varphi(x)dx\\ &= \int_0^\infty([\delta^\theta+1]\delta^\theta\theta\eta)^\varphi e^{-\varphi\eta x}(1-e^{-\eta x})^{\varphi(\theta-1)}[\delta^\theta + (1-e^{-\eta x})^\theta]^{-2\varphi}dx\\ &= ([\delta^\theta+1]\delta^\theta\theta\eta)^\varphi\int_0^\infty e^{-\varphi\eta x}(1-e^{-\eta x})^{\varphi(\theta-1)}[\delta^\theta + (1-e^{-\eta x})^\theta]^{-2\varphi}dx.\end{aligned} \tag{21}$$

By expanding (21) we have

$$\mathcal{A}_{\varphi} = ([\delta^{\theta}+1]\delta^{\theta}\theta\eta)^{\varphi}\int_0^{\infty} e^{-\varphi\eta x}\sum_{i=0}^{\infty}\binom{\varphi(\theta-1)}{i}(-1)^i e^{-\eta i x} \times \sum_{j=0}^{\infty}\binom{2\varphi+j-1}{j}(-1)^j(1-e^{-\eta x})^{\theta j}\delta^{-\theta(2\varphi+j)}dx,$$

$$\mathcal{A}_{\varphi} = ([\delta^{\theta}+1]\delta^{\theta}\theta\eta)^{\varphi}\sum_{i,j=0}^{\infty}(-1)^{i+j}\binom{\varphi(\theta-1)}{i}\binom{2\varphi+j-1}{j}\delta^{-\theta(2\varphi+j)} \times \int_0^{\infty} e^{-\eta(\varphi+i)x}(1-e^{-\eta x})^{\theta j}dx, \tag{22}$$

and by substituting $y = e^{-\eta x}$ into (22) we have

$$\mathcal{A}_{\varphi} = \frac{([\delta^{\theta}+1]\delta^{\theta}\theta)^{\varphi}}{\eta^{-\varphi+1}}\sum_{i,j=0}^{\infty}(-1)^{i+j}\binom{\varphi(\theta-1)}{i}\binom{2\varphi+j-1}{j}\delta^{-\theta(2\varphi+j)} \times \int_0^{1} y^{\varphi+i}(1-y)^{\theta j}dy,$$

which further simplifies to

$$\mathcal{A}_{\varphi} = \frac{([\delta^{\theta}+1]\delta^{\theta}\theta)^{\varphi}}{\eta^{-\varphi+1}}\sum_{i,j=0}^{\infty}(-1)^{i+j}\binom{\varphi(\theta-1)}{i}\binom{2\varphi+j-1}{j}\delta^{-\theta(2\varphi+j)} \times \mathbf{B}(\varphi+i+1,\theta j+1),$$

and finally

$$\mathcal{A}_{\varphi} = \frac{([\delta^{\theta}+1]\delta^{\theta}\theta)^{\varphi}}{\eta^{-\varphi+1}}\sum_{i,j=0}^{\infty}(-1)^{i+j} \times \frac{\delta^{-\theta(2\varphi+j)}\Gamma(\varphi(\theta-1)+1)\Gamma(2\varphi+j)\mathbf{B}(\varphi+i+1,\theta j+1)}{\Gamma(i+1)\Gamma(j+1)\Gamma(\varphi(\theta-1)-i+1)\Gamma(2\varphi)}. \tag{23}$$

Thus, substituting (23) into (20) completes the proof. □

2.4 Order Statistics

Order statistics is an essential tool in reliability and life testing analysis. For instance, suppose the following n-sized random sample $X_1, X_2, \ldots, X_n$ are drawn from the *ALLoGExp* distribution with *cdf* and *pdf* corresponding to (6) and (7). Let $X_{1,n} \leq X_{2,n} \leq \ldots \leq X_{n,n}$ represent the *ith* order statistics denoted by $X_{i,n}$ then, $X_{i,n}$ could be interpreted as the lifetime of the $(n-i+1)th$ item of the total *nth* independent and identical components. The density of $X_{i,n}$ could be expressed as

$$\begin{aligned} f_{X_{(i)}}(x) &= \frac{n!}{(i-1)!(n-i)!}F^{i-1}(x)(1-F(x))^{n-i}f(x) \\ &= \frac{n!}{(i-1)!(n-i)!}\sum_{\ell=0}^{n-i}\binom{n-i}{\ell}F^{i+\ell-1}(x)f(x). \end{aligned} \tag{24}$$

The *cdf* of the *ALLoGExp* distribution in (6) to the $(i+\ell-1)th$ power is given by;

$$F^{i+\ell-1}(x) = [\delta^{\theta}+1]^{i+\ell-1}(1-e^{-\eta x})^{\theta(i+\ell-1)}[\delta^{\theta}+(1-e^{-\eta x})^{\theta}]^{-(i+\ell-1)}, \tag{25}$$

and if $\theta \in \mathbf{N}$, where $\mathbf{N}\backslash\{0\}$ is a natural number; then we have the series representation of (34) as

$$F^{i+\ell-1}(x) = \sum_{k=0}^{\theta(i+\ell-1)} \binom{\theta(i+\ell-1)}{k} (-1)^k [\delta^\theta + 1]^{i+\ell-1} e^{-\eta k x} \times \sum_{m=0}^{i+\ell-1} (-1)^m \binom{i+\ell+m-2}{m} \delta^{-\theta(i+\ell+m-1)} (1-e^{-\eta x})^{\theta m},$$

$$F^{i+\ell-1}(x) = \sum_{k=0}^{\theta(i+\ell-1)} \sum_{m=0}^{i+\ell-1} (-1)^{k+m} [\delta^\theta + 1]^{i+\ell-1} \delta^{-\theta(i+\ell+m-1)} e^{-\eta k x} \times \binom{\theta(i+\ell-1)}{k} \binom{i+\ell+m-2}{m} \sum_{n=0}^{\theta m} \binom{\theta m}{n} (-1)^n e^{-\eta n x},$$

$$F^{i+\ell-1}(x) = \sum_{k=0}^{\theta(i+\ell-1)} \sum_{m=0}^{i+\ell-1} (-1)^{k+m+n} [\delta^\theta + 1]^{i+\ell-1} \delta^{-\theta(i+\ell+m-1)} \times \binom{\theta(i+\ell-1)}{k} \binom{i+\ell+m-2}{m} \binom{\theta m}{n} e^{-\eta(k+n)x}. \tag{26}$$

Also, the series representation of (7) is given by;

$$\begin{aligned} f(x) &= [\delta^\theta + 1]\delta^\theta \theta \eta e^{-\eta x} \sum_{h=0}^{\theta-1} \binom{\theta-1}{h} (-1)^h e^{-\eta h x} \sum_{i=0}^{2} (-1)^i \binom{i+1}{i} \delta^{-\theta(2+i)} (1-e^{-\eta x})^{\theta i} \\ &= [\delta^\theta + 1]\delta^\theta \theta \eta e^{-\eta x} \sum_{h=0}^{\theta-1} \sum_{i=0}^{2} (-1)^{h+i} \binom{\theta-1}{h} \binom{i+1}{i} \delta^{-\theta(2+i)} e^{-\eta h x} \sum_{j=0}^{\theta i} \binom{\theta i}{j} (-1)^j e^{-\eta j x} \\ &= [\delta^\theta + 1]\delta^\theta \theta \eta \sum_{h=0}^{\theta-1} \sum_{i=0}^{2} \sum_{j=0}^{\theta i} (-1)^{h+i+j} \delta^{-\theta(2+i)} \binom{\theta-1}{h} \binom{i+1}{i} \binom{\theta i}{j} e^{-\eta(h+j+1)x}. \end{aligned} \tag{27}$$

Therefore substituting (26) and (27) into (24) gives the density of the *ith* order statistics of the *ALLoGExp* distribution as

$$f_{X_{(i)}}(x) = \frac{n![\delta^\theta + 1]^{i+\ell}\delta^\theta \theta \eta}{(i-1)!(n-i)!} \sum_{h=0}^{\theta-1} \sum_{i=0}^{2} \sum_{j=0}^{\theta i} \sum_{k=0}^{\theta(i+\ell-1)} \sum_{\ell=0}^{n-i} \sum_{m=0}^{i+\ell-1} \sum_{n=0}^{\theta m} (-1)^{k+\ell+m+n} \times \delta^{-\theta(2i+\ell+m+1)} \binom{\theta-1}{h} \binom{i+1}{i} \binom{\theta i}{j} \binom{\theta(i+\ell-1)}{k} \binom{n-i}{\ell} \times \binom{i+\ell+m-2}{m} \binom{\theta m}{n} e^{-\eta(h+j+k+n+1)x}.$$

The density of the smallest order statistics of the *ALLoGExp* distribution is given by;

$$f_{X_{(1)}}(x) = n[\delta^\theta + 1]^{1+\ell}\delta^\theta \theta \eta \sum_{h=0}^{\theta-1} \sum_{j=0}^{\theta} \sum_{k=0}^{\theta \ell} \sum_{\ell=0}^{n-1} \sum_{m=0}^{\ell} \sum_{n=0}^{\theta m} (-1)^{k+\ell+m+n} \delta^{-\theta(3+\ell+m)} \times \binom{\theta-1}{h} \binom{\theta}{j} \binom{\theta \ell}{k} \binom{n-1}{\ell} \binom{\ell+m-1}{m} \binom{\theta m}{n} e^{-\eta(h+j+k+n+1)x},$$

while the density of largest order statistics of the *ALLoGExp* distribution is given by;

$$f_{X_{(n)}}(x) = n[\delta^\theta + 1]^{n+\ell}\delta^\theta \theta \eta \sum_{h=0}^{\theta-1} \sum_{j=0}^{\theta n} \sum_{k=0}^{\theta(n+\ell-1)} \sum_{m=0}^{n+\ell-1} \sum_{n=0}^{\theta m} (-1)^{k+\ell+m+n} \delta^{-\theta(\ell+m+2n+1)} \binom{\theta-1}{h} \times \binom{n+1}{n} \binom{\theta n}{j} \binom{\theta(n+\ell-1)}{k} \binom{\ell+m+n-2}{m} \binom{\theta m}{n} e^{-\eta(h+j+k+n+1)x}.$$

2.4.1 Moment of the Order Statistics

Theorem 2.4. If X follows the ALLoGExp distribution with pdf of the ith order statistics $f_{X_{(i)}}(x)$ then, its pth crude moment is given by;

$$E(X^p_{i,n}) = \frac{\digamma_{x_{(i)}}\Gamma(p+1)}{[\eta(h+j+k+n+1)]^{p+1}},$$

where $\Gamma(\cdot)$ is the gamma function and $\digamma_{x_{(i)}}$ is defined below.

Proof. By the definition of moment of a continuous random variable we have,

$$E(X^p_{i,n}) = \digamma_{x_{(i)}} \int_0^\infty x^p e^{-\eta(h+j+k+n+1)x} dx, \tag{28}$$

by substituting $y = \eta(h+j+k+n+1)x$ in (28) we have

$$\begin{aligned} E(X^p_{i,n}) &= \frac{\digamma_{x_{(i)}}}{[\eta(h+j+k+n+1)]^{p+1}} \int_0^\infty y^p e^{-y} dy \\ &= \frac{\digamma_{x_{(i)}}\Gamma(p+1)}{[\eta(h+j+k+n+1)]^{p+1}}. \end{aligned}$$

Where,

$$\begin{aligned} \digamma_{x_{(i)}} = \frac{n![\delta^\theta+1]^{i+\ell}\delta^\theta\theta\eta}{(i-1)!(n-i)!} \sum_{h=0}^{\theta-1}\sum_{i=0}^{2}\sum_{j=0}^{\theta i}\sum_{k=0}^{\theta(i+\ell-1)}\sum_{\ell=0}^{n-i}\sum_{m=0}^{i+\ell-1}\sum_{n=0}^{\theta m}(-1)^{k+\ell+m+n} \\ \times \delta^{-\theta(2i+\ell+m+1)}\binom{\theta-1}{h}\binom{i+1}{i}\binom{\theta i}{j}\binom{\theta(i+\ell-1)}{k}\binom{n-i}{\ell} \\ \times \binom{i+\ell+m-2}{m}\binom{\theta m}{n}. \end{aligned}$$

□

3. Estimation

Here, we estimate the parameters of the *ALLoGExp* distribution by the method of maximum likelihood estimation. Suppose the random sample $x_1, x_2, x_3, \ldots, x_n$ of size n is drawn from the *ALLoGExp* distribution with *pdf f(x)* in (7) then the maximum likelihood estimation *(mle)* procedure for estimating its parameters is as follows:

The likelihood ($\mathcal{L}$) equation is given by;

$$\begin{aligned} \mathcal{L} &= \prod_{i=1}^{n}[\delta^\theta+1]\delta^\theta\theta\eta e^{-\eta x_i}(1-e^{-\eta x_i})^{\theta-1}[\delta^\theta+(1-e^{-\eta x_i})^\theta]^{-2} \\ &= [\delta^\theta+1]^n(\delta^\theta\theta\eta)^n e^{-\eta\sum_{i=1}^n x_i}\prod_{i=1}^{n}(1-e^{-\eta x_i})^{\theta-1}[\delta^\theta+(1-e^{-\eta x_i})^\theta]^{-2} \end{aligned} \tag{29}$$

and the log-likelihood function is given by;

$$\begin{aligned} \mathscr{L} &= n\log[\delta^\theta+1] + n\log(\delta^\theta\theta\eta) - \eta\sum_{i=1}^{n}x_i + (\theta-1)\sum_{i=1}^{n}\log(1-e^{-\eta x_i}) \\ &\quad -2\sum_{i=1}^{n}\log[\delta^\theta+(1-e^{-\eta x_i})^\theta]. \end{aligned} \tag{30}$$

Taking the partial derivatives of (30) with respect to δ, θ and η gives

$$\frac{\partial\mathscr{L}}{\partial\delta} = \frac{n\theta\delta^{\theta-1}}{\delta^\theta+1} + \frac{n\theta}{\delta} - 2\sum_{i=1}^{n}\frac{\theta\delta^{\theta-1}}{\delta^\theta+(1-e^{-\eta x_i})^\theta}, \tag{31}$$

$$\begin{aligned}\frac{\partial \mathscr{L}}{\partial \theta} &= \frac{n\delta^{\theta}\log(\delta)}{\delta^{\theta}+1} + n\log(\delta) + \frac{n}{\theta} + \sum_{i=1}^{n}\log(1-e^{-\eta x_i}) \\ &\quad - 2\sum_{i=1}^{n}\frac{\delta^{\theta}\log(\delta) + (1-e^{-\eta x_i})^{\theta}\log(1-e^{-\eta x_i})}{\delta^{\theta}+(1-e^{-\eta x_i})^{\theta}},\end{aligned} \tag{32}$$

and

$$\frac{\partial \mathscr{L}}{\partial \eta} = \frac{n}{\eta} - \sum_{i=1}^{n} x_i + (\theta-1)\sum_{i=1}^{n}\frac{x_i e^{-\eta x_i}}{1-e^{-\eta x_i}} - 2\sum_{i=1}^{n}\frac{\theta x_i e^{-\eta x_i}(1-e^{-\eta x_i})^{\theta-1}}{\delta^{\theta}+(1-e^{-\eta x_i})^{\theta}}. \tag{33}$$

Furthermore, setting (31)-(33) to zero results to a system of three equations in three unknowns which has no analytical solutions.

However, the estimates $\hat{\delta},\ \hat{\theta}$ and $\hat{\eta}$ can only be obtained by solving (31)-(33) by some non linear numerical optimization methods eg.; the Newton-Raphson or quasi-Newton-Raphson's technique.

4. Reliability

The reliability function $R(x)$ is an important tool in reliability analysis for characterizing life phenomena. The reliability function is mathematically expressed as $1 - F(x)$. Under certain predefined conditions $R(x)$ generally gives the estimate of the probability that, a system will not fail given that it has operated without failure up to time x. The reliability function of the *ALLoGExp* distribution is given by;

$$R(x) = 1 - \frac{(\delta^{\theta}+1)(1-e^{-\eta x})^{\theta}}{\delta^{\theta}+(1-e^{-\eta x})^{\theta}};\ x \geq 0,\ \delta,\theta,\eta > 0.$$

Another important reliability characteristics is the failure rate function *h(x)*. The failure rate function gives the probability of failure, for a system that has not failed up-to time x. The failure rate function is mathematically expressed as $f(x)/R(x)$. The failure rate function of the *ALLoGExp* distribution is given by;

$$h(x) = \frac{(\delta^{\theta}+1)\delta^{\theta}\theta\eta e^{-\eta x}(1-e^{-\eta x})^{\theta-1}}{[\delta^{\theta}+(1-e^{-\eta x})^{\theta}][\delta^{\theta}+(1-e^{-\eta x})^{\theta}-(\delta^{\theta}+1)[1-e^{-\eta x}]^{\theta}]};\ x \geq 0,\ \delta,\theta,\eta > 0.$$

4.1 Shapes and Asymptotics

(a) The *pdf* of the *ALLoGExp* distribution could either be a unimodal or monotonic decreasing function of x depending on the value of δ and θ, while *F(x)* is an increasing function of x for all possible values of δ and θ parameters (see; Figures 1 and 2), and the asymptotic behaviour of the *pdf* is

$$\lim_{x\to 0} f(x) = \begin{cases} 0, & \text{if } \theta > 1, \\ \eta(\delta+1)/\delta, & \text{if } \theta = 1, \\ \infty, & \text{if } \theta < 1, \end{cases}$$

and the asymptotic behaviour of the *cdf* is $\lim_{x\to\infty} F(x) = 1$ while $\lim_{x\to 0} F(x) = 0$.

(b) The reliability function *R(x)* of the *ALLoGExp* distribution is generally a monotonic decreasing function of x for all possible values of δ and θ parameters (see; Figure 3), and $\lim_{x\to\infty} R(x) = 0$ while $\lim_{x\to 0} R(x) = 1$.

(c) The failure rate function *(frf) h(x)* of the *ALLoGExp* distribution could be a decreasing, increasing or upside-down bathtub shaped function of x depending on the value of the δ and θ parameters (see; Figure 4), and $\lim_{x\to\infty} h(x) = 0$, while

$$\lim_{x\to 0} h(x) = \begin{cases} 0, & \text{if } \theta > 1, \\ \eta(\delta+1)/\delta, & \text{if } \theta = 1, \\ \infty, & \text{if } \theta < 1. \end{cases}$$

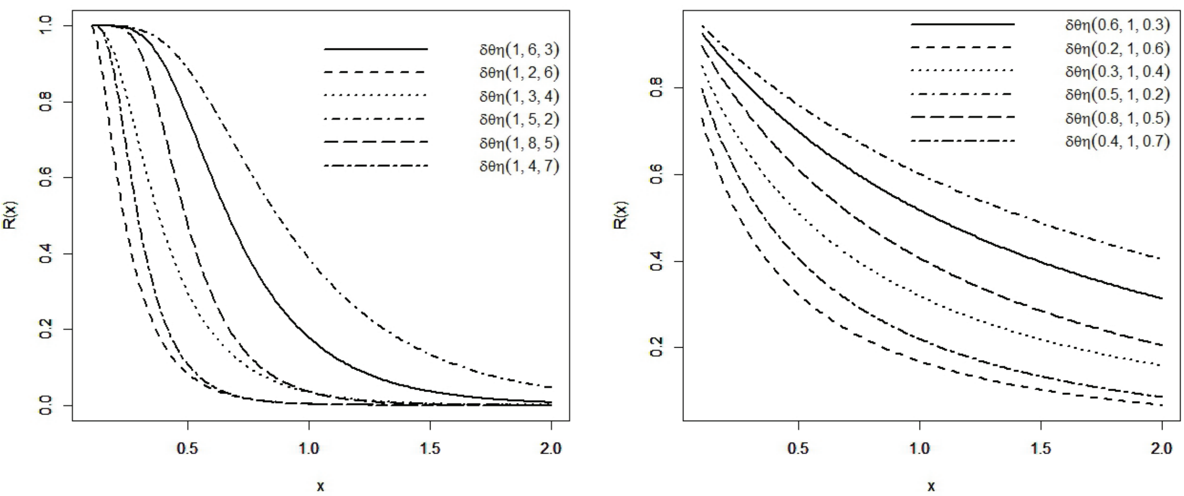


Figure 3. Possible shapes of the reliability function $R(x)$ for some parameter values.

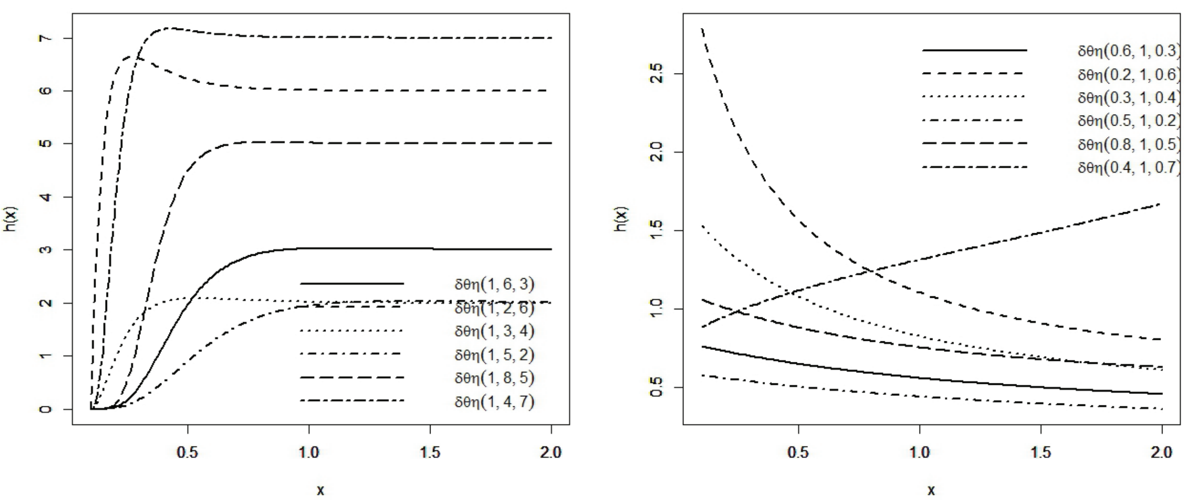


Figure 4. Possible shapes of the hazard rate function $h(x)$ for some parameter values.

The loss of memory property of the exponential distribution and the shape limitation of its failure rate are well known. The major advantage of the new distribution over the baseline model is its added tail and skewness flexibility due to the presence of δ and θ. The *ALLoGExp* is suitable for modelling lifetime data sets with increasing, decreasing, unimodal and upside-down bathtub failure rate characteristics.

5. Monte-Carlo Simulation

In this section, we investigate the consistency of the *mle* estimates of the *ALLoGExp* distribution with different sample size *(n)*, through a Monte-Carlo study. The simulation procedure as outlined below was implemented in *R (Statistical software)*:

1. simulate a random sample of size n from the *ALLoGExp* distribution with parameters $\delta = 0.5$, $\theta = 4.0$ and $\eta = 6.0$ using the inversion of the *cdf* method with Equation (8).

2. compute the *mle* of the parameters of the *ALLoGExp* distribution.

3. repeat steps 1-2 *5000* times.
4. compute the mean, standard deviation (standard error), bias and mean square error *(mse)* of the *5000* estimates of each of the parameters (δ, θ and η).
5. repeat steps 1-4 for different sample sizes *(n=150, 250,...,950.)*.

Results from the Monte-Carlo simulation study are tabulated in Tables 1 and 2.

Table 1. Simulation results of the estimates and standard errors of the *ALLoGExp* distribution parameters for different sample sizes

n	$\hat{\delta}$	$\hat{\theta}$	$\hat{\eta}$	$se_{\hat{\delta}}$	$se_{\hat{\theta}}$	$se_{\hat{\eta}}$
150	0.610892	4.067068	6.751090	1.658690	0.598297	3.510499
250	0.529417	4.054277	6.540886	0.305542	0.464951	2.948877
350	0.515020	4.031474	6.404858	0.169692	0.406658	2.484260
450	0.501594	4.003761	6.180374	0.159987	0.372123	2.285624
550	0.511001	4.027797	6.299262	0.139514	0.332690	2.060127
650	0.498788	3.997763	6.094940	0.131292	0.294331	1.894563
750	0.499263	3.996266	6.089056	0.119744	0.281376	1.757239
850	0.506678	4.013708	6.171866	0.108199	0.253175	1.602942
950	0.507347	4.011057	6.186832	0.108731	0.257343	1.612633

Table 2. Simulation results of the bias and *mse* of the *ALLoGExp* distribution parameters for different sample sizes

n	$bias_{\hat{\delta}}$	$bias_{\hat{\theta}}$	$bias_{\hat{\eta}}$	$mse_{\hat{\delta}}$	$mse_{\hat{\theta}}$	$mse_{\hat{\theta}}$
150	0.110892	0.067068	0.751090	2.760798	0.362099	12.875415
250	0.029417	0.054277	0.540886	0.094128	0.218909	8.979740
350	0.015020	0.031474	0.404858	0.028992	0.166196	6.329287
450	0.001594	0.003761	0.180374	0.025573	0.138351	5.251386
550	0.011001	0.027797	0.299262	0.019566	0.111345	4.329437
650	-0.001212	-0.002237	0.094940	0.017222	0.086549	3.594793
750	-0.000738	-0.003734	0.089056	0.014325	0.079107	3.092733
850	0.006678	0.013708	0.171866	0.011740	0.064221	2.596392
950	0.007347	0.011057	0.186832	0.011865	0.066282	2.632891

The simulation results in Table 1 indicates that the *mle* estimates of the ALLoGExp distribution is generally consistent for *n*; while the standard error, bias and *mse* approaches zero as *n* becomes large.

6. Application

This section illustrates the applicability and flexibility of the *ALLoGExp* distribution with a real data set. The goodness of fit of the new lifetime distribution would be assessed by a comparison of its performance in modelling real data with the following five distributions:

(i) The exponentiated exponential *(EE)* distribution due to Gupta and Kundu (1999),

$$f(x) = \alpha\eta(1 - e^{-\eta x})^{\alpha-1}e^{-\eta x}; \quad x, \alpha, \eta > 0.$$

(ii) The Log-logistic *(LLo)* distribution,

$$f(x) = \frac{\frac{\theta}{\delta}\left(\frac{x}{\delta}\right)^{\theta-1}}{\left[1 + \left(\frac{x}{\delta}\right)^{\theta}\right]^2}; \quad x, \theta, \delta > 0.$$

(iii) The Lindley exponential *(LE)* distribution due to Bhati *et al.* (2006),

$$f(x) = \frac{\phi^2\eta e^{-\eta x}(1 - e^{-\eta x})^{\phi-1}(1 - \log(1 - e^{-\eta x}))}{1 + \phi}; \quad x, \phi, \eta > 0.$$

(iv) The Nadarajah-Haghighi exponential *(NHE)* distribution due to Nadarajah and Haghighi (2015),

$$f(x) = \beta\eta(1+\beta x)^{\eta-1}e^{1-(1+\beta x)^{\eta}};\quad x,\beta,\eta > 0.$$

(v) The standard one parameter exponential *(Exp)* distribution,

$$f(x) = \eta e^{-\eta x};\quad x,\eta > 0.$$

Comparison of the models would be based on the following information criteria statistics:

- the Akaike information criterion (AIC) statistic, Akaike (1992),

$$AIC = -2\hat{\mathscr{L}} + 2k$$

- the AIC with a correction statistic (AICc), Sugiura (1978),

$$AICc = AIC + \frac{2k(k+1)}{n-k-1}$$

- and the Hannan-Quinn information criterion (HQC) statistic, Hannan and Quinn (1979),

$$HQC = -2\hat{\mathscr{L}} + 2k\log\log(n).$$

  Where $-\hat{\mathscr{L}},\ k,$ and n corresponds to the estimate of the model minimized log-likelihood function, number of model parameters and sample size, respectively.

We have also considered some goodness-of-fit test based on the empirical distribution function; namely, the Kolmogorov-Smirnov (K-S) statistics by Henze and Meintanis (2005) and Liao-Shimokawa statistics due to Liao and Shimokawa (1999). They are given by;

- $$K-S = \max\left(\frac{i}{n} - F(x_{(i)}), F(x_{(i)}) - \frac{i-1}{n}\right),$$

  and

- $$L-S = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\max\left(\frac{i}{n} - F(x_{(i)}), F(x_{(i)}) - \frac{i-1}{n}\right)}{\sqrt{F(x_{(i)})(1-F(x_{(i)}))}},$$

respectively. Where n is the number of observations in the data set, $1 \leq i \leq n$ and $F(x_{(i)})$ is the *cdf* of the fitted distribution under the ascending ordered data.

The real and uncensored data set in Table 3 is on the active repair times in hours for an airborne communication transceiver. The data set was originally reported by Von Alven (1964) and later Chhikara and Folks (1977) used it on the inverse Gaussian distribution.

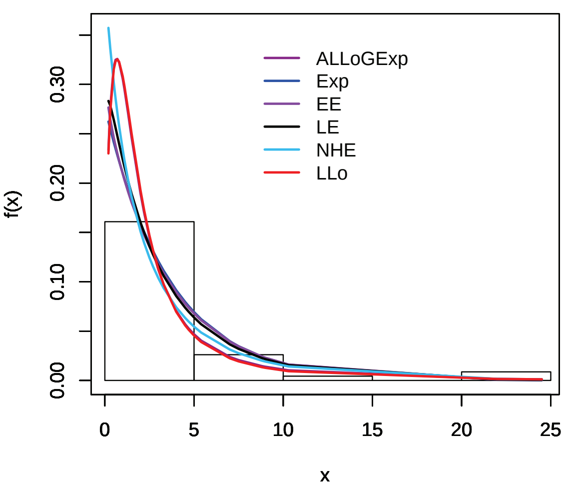
Table 3. Active repair time data

0.2	0.3	0.5	0.5	0.5	0.5	0.6	0.6	0.7	0.7	0.7	0.8	0.8	1.0	1.0	1.0	1.0
1.1	1.3	1.5	1.5	1.5	1.5	2.0	2.0	2.2	2.5	2.7	3.0	3.0	3.3	3.3	4.0	4.0
4.5	4.7	5.0	5.4	5.4	7.0	7.5	8.8	9.0	10.3	22.0	24.5					

The results of the fits are listed in Table 4. The plots in Figure 5 shows the *pdf* and *cdf* of the fitted theoretical distributions superimposed on the empirical density and distribution function, respectively.

Table 4. Results

Models	Estimates	$-\hat{\mathscr{L}}$	AIC	AICc	HQC	K-S	L-S
ALLoGExp							
$\hat{\delta}$	0.1856						
$\hat{\theta}$	1.5666	100.7382	207.4763	208.0477	209.5314	0.01157303	0.01701021
$\hat{\eta}$	0.0995						
LE							
$\hat{\phi}$	1.2803	103.7034	211.4067	211.6858	212.7768	0.01973152	0.06499466
$\hat{\eta}$	0.2401						
EE							
$\hat{\alpha}$	0.9583	104.9829	213.9658	214.2449	215.3359	0.02043486	0.08348181
$\hat{\eta}$	0.2694						
NHE							
$\hat{\beta}$	0.6364	103.2059	210.4118	210.6909	211.7819	0.01476631	0.02616426
$\hat{\eta}$	0.6348						
LLo							
$\hat{\theta}$	0.6477	101.1710	206.3421	208.3421	205.4908	0.01847885	0.02023060
$\hat{\delta}$	0.6257						
Exp							
$\hat{\eta}$	0.2773	105.0062	212.0124	212.1033	212.6974	0.0206178	0.09083219



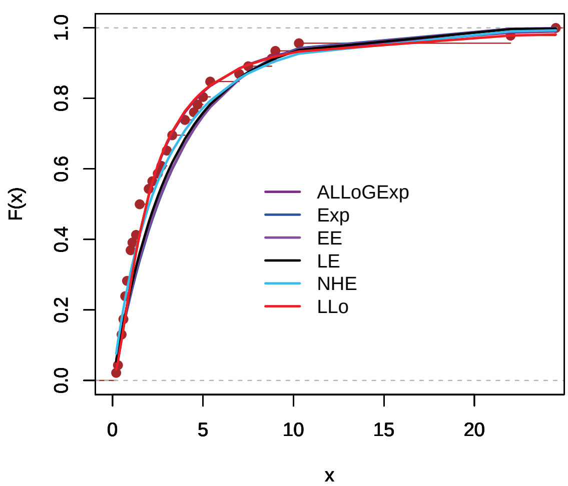


Figure 5. The estimated *pdf F(x) (left panel)* and *cdf F(x) (right panel)* plots of the fitted distributions superimposed on the empirical *pdf F(x)* and *cdf F(x)* of the active repair time data.

7. Concluding Remarks

In this paper we have introduced a new generator of distributions called the adjusted log-logistic generalized ($ALLoG$) distribution and a new lifetime distribution-the adjusted log-logistic generalized exponential ($ALLoGExp$) distribution is also introduced as a sub-model of the $ALLoG$ distribution. The new lifetime distribution generalizes the exponential (Exp) distribution. We have given explicit expressions of some of its basic statistical properties such as the probability density function, cumulative density function, *kth* raw moment, mean, variance, coefficient of variation, skewness, kurtosis, moment generating function, *pth* quantile function, the *ith* order statistics, and the Rényi's entropy measure. Also, some of its reliability characteristics like the reliability function and the failure rate function were provided; the failure rate could be monotonically decreasing, increasing or upside-down bathtub shaped depending on the value of the scale parameter δ and shape parameter θ. Estimation of the model parameters was approached through the method of maximum likelihood estimation *mle* and the stability of the *mle* estimates was verified through a Monte-Carlo simulation study. The applicability and goodness of fit of the *ALLoGExp* distribution was illustrated with the active repair times data and the results based on the AICc, K-S and L-S statistics shows that the *ALLoGExp* distribution provides a better fit than the *Exp*, *EE*, *LE*, *LLo*, and *NH* distribution, also, the density plot of the *ALLoGExp* distribution comparatively provides the best fit to the histogram of the empirical data. We strongly recommend the *ALLoGExp* distribution for effective modelling of life time data because of its flexible failure rate characteristics.

Competing Interest

The authors declare no competition of interests.

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