

Characterizations Based on Cumulative Entropy of the Last-order Statistics

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Abstract

Cumulative Entropy (CE) as a measure of uncertainty alternative to Shannon entropy is proposed by Di Crescenzo and Longobardi (2009). In this paper, some properties of the cumulative entropy are derived and under conditions are showed the cumulative entropy of the last order statistics can determine the distribution function uniquely. Weibull family is characterized by ratio of the cumulative entropy of the last order statistics to its expectation. Also, some inequalities are presented for the cumulative entropy of reversed residual lifetime of a parallel system.

Keywords: cumulative entropy, shannon entropy, last order statistics, weibull family, reversed residual lifetime

1. Introduction

Shannon (1948) introduced entropy, known Shannon entropy, as a measure of uncertainty for a random variable. Suppose X be a non-negative continuous random variable having probability density function f and cumulative distribution function F . Hence, Shannon entropy $h(X)$ of a continuous non-negative random variable X with probability density function $f(x)$ is defined as

$$h(X) = - \int_0^{+\infty} f(x) \log f(x) dx$$

Di Crescenzo and Longobardi (2002) introduced the entropy of the reversed residual lifetime ${}_tX = [t - X | X < t]$ as a dynamic measure of uncertainty as follows

$$H(X; [t]) = - \int_0^t \frac{f(x)}{F(t)} \log \frac{f(x)}{F(t)} dx$$

It's clear that $H(X; \infty) = h(X)$.

Rao et al. (2004) defined Cumulative Residual Entropy (CRE) as an alternative measure of uncertainty to Shannon entropy in that the probability density function is replaced by survival (reliability) function and obtained some properties and applications of that (2005). Rao showed CRE overcomes some problems with Shannon entropy. Some properties of CRE are: CRE is more general rather than the Shannon entropy, it has more mathematical properties rather than Shannon entropy, and it is easily computed from sample data. CRE has applications in reliability engineering and image processing, for more details see Rao (2004, 2005). CRE for a non-negative univariate random variable is as follows:

$$CRE(X) = - \int_0^{\infty} \bar{F}(x) \log \bar{F}(x) dx.$$

That $\bar{F}(x) = P(X > x) = 1 - F(x)$ is the survival function of X .

Crescenzo and Longobardi (2009) introduced a new information measure that turns out to be useful to measure information on the reversed residual lifetime of a system. The reversed residual lifetime is a concept in reliability that is convenient to describe the time elapsing between the failure of a system and the time when it is down. This measure is defined as follows:

$$CE(X) = - \int_0^{\infty} F(x) \log F(x) dx \quad (1)$$

where $F(x)$ is the cumulative distribution function of the non-negative random variable X . Also, they proposed dynamic form of cumulative entropy that called dynamic cumulative entropy and obtained some of its properties.

Cumulative entropy for reversed residual lifetime with distribution function $F_t(x) = P[t - X|X < t]$ is

$$CE(X; t) = - \int_0^t F_t(x) \log F_t(x) dx = - \int_0^t \frac{F(x)}{F(t)} \log \frac{F(x)}{F(t)} dx$$

It's clear $CE(X; \infty) = CE(X)$.

By using probability integral transformation $F(x) = U$ that U have uniform distribution function on $(0,1)$, then we obtain

$$CE(X) = - \int_0^1 \frac{u \log u}{f(F^{-1}(u))} du$$

That F^{-1} is inverse function of F .

Suppose X_1, X_2, \dots, X_n be a random sample, the order statistics of the sample is defined by the arrangement X_1, X_2, \dots, X_n from the minimum to the maximum by $X_{1:n}, X_{2:n}, \dots, X_{n:n}$. Order statistics are used in entropy estimation, quality control, reliability, insurance, etc. For more details see Arnold et al. (1992), David and Nagaraja (2003).

Baratpour (2010) have derived characterizations result based on Cumulative residual entropy of first order statistics. Also Thapliyal et al. (2013) studied cumulative entropy and dynamic cumulative entropy using order statistics. They showed a characterization property that dynamic cumulative entropy of the i th order statistics determines the distribution function. The purpose of this paper is characterization result the parent distribution based on Cumulative Entropy of the last order statistics.

The paper is organized as follows: In section 2, characterizing the parent distributions based on the CE of the last order statistics is presented. Also, Weibull distribution is characterized based on the ratio of the CE of the last order statistics to the expectation of it. Last section includes some Characterizations based on CE of reversed residual lifetime of parallel systems.

2. Characterizations Based on the Last-order Statistics

Last order statistics is an important special case of order statistics. Suppose $X_{n:n}$ be the last order statistics in a random sample of size n , cumulative distribution function of $X_{n:n}$ is $F_{X_{n:n}}(x) = (F(x))^n$, Thus CE of $X_{n:n}$ is defined as follow:

$$CE(X_{n:n}) = -n \int_0^\infty (F(x))^n \log F(x) dx$$

By changing variable $F(x) = U$, we have

$$CE(X_{n:n}) = -n \int_0^1 \frac{u^n \log(u)}{f(F^{-1}(u))} dx \tag{2}$$

In the following theorem, we show that only in Weibull family the ratio of the CE of the last order statistics to the expectation of it is constant. First we need a lemma taken from Baratpour (2010).

Lemma 2.1: For any increasing sequence of positive integers $\{n_j, j \geq 1\}$, the sequence of polynomials $\{x^{n_j}\}$ is complete on $L(0,1)$, if and only if $\sum_{j=1}^\infty \frac{1}{n_j}$ be infinitely.

Theorem 2.1: Let X_1, X_2, \dots, X_n are positive, independent and identically(iid) distributed observations from an absolutely continuous cumulative distribution function $F(x)$ and probability density function $f(x)$. Then $F(x)$ belongs to Weibull family if and only if $\frac{CE(X_{n:n})}{E(X_{n:n})} = c(> 0)$ for all $n = n_j, j \geq 1$ such that $\sum_{j=1}^\infty \frac{1}{n_j}$ be infinitely.

Proof: The ‘if’ part of the theorem is trivial. Hence we prove ‘only if’ part, using integral transformation $F(x) = U$ in

$$E(X_{n:n}) = \int_0^\infty nxf(x)[F(x)]^{n-1} dx = n \int_0^1 [F(u)]^{-1} u^{n-1} du \tag{3}$$

according to relations (2) and (3) we obtain

$$\frac{CE(X_{n:n})}{E(X_{n:n})} = - \frac{\int_0^1 \frac{u^n \log(u)}{f(F^{-1}(u))} du}{\int_0^1 [F(u)]^{-1} u^{n-1} du} \tag{4}$$

Relation (4) coincides c, so we have

$$\int_0^1 \frac{u^n \log(u)}{f(F^{-1}(u))} du + c \int_0^1 [F(u)]^{-1} u^{n-1} du = 0$$

$$\int_0^1 u^{n-1} \left[\frac{u \log u}{f(F^{-1}(u))} + cF^{-1}(u) \right] du = 0 \tag{5}$$

If relation (5) holds for $n = n_j, j \geq 1$ such that $\sum_{j=1}^\infty \frac{1}{n_j} = \infty$, then from lemma 2.1 we have

$$\frac{(1-v) \log(1-v)}{f(F^{-1}(1-v))} + cF^{-1}(1-v) = 0$$

Because $\frac{dF^{-1}(1-v)}{dv} = \frac{1}{f(F^{-1}(1-v))}$ then it follows:

$$(1-v) \log(1-v) \frac{dF^{-1}(1-v)}{dv} + cF^{-1}(1-v) = 0 \quad v \in (0,1)$$

By solving this differential equation, we have

$$\frac{\frac{dF^{-1}(1-v)}{dv}}{F^{-1}(1-v)} = - \frac{c}{(1-v) \log(1-v)}$$

$F^{-1}(1-v) = c_1[-\log(1-v)]^c, v \in (0,1)$ thus $F(x) = 1 - \exp\left\{-\left(\frac{x}{c_1}\right)^{\frac{1}{c}}\right\}, x > 0$ we conclude that $F(x)$ belongs to the Weibull family. □

Theorem 2.2: Suppose X, Y be two positive random variables with probability density functions $f(x)$ and $g(x)$ and absolutely continuous distribution functions $F(x)$ and $G(x)$, respectively. Then F and G belong to the same family of distributions, but for a location shift, if and only if

$$CE(X_{n:n}) = CE(Y_{n:n})$$

for $n = n_j, j \geq 1$ such that $\sum_{j=1}^\infty \frac{1}{n_j}$ be infinitely.

Proof: The ‘if’ part of the theorem is trivial. Hence we prove ‘only if’ part, according to relation (2) if $CE(X_{n:n}) = CE(Y_{n:n})$, then

$$\int_0^1 u^n \log u \left[\frac{1}{f(F^{-1}(u))} - \frac{1}{g(G^{-1}(u))} \right] du = 0 \tag{6}$$

if for all $n = n_j, j \geq 1$ such that $\sum_{j=1}^\infty \frac{1}{n_j} = \infty$, then from Lemma 2.1 we can conclude that

$$f(F^{-1}(t)) = g(G^{-1}(t)), \quad 0 < t < 1.$$

Because $\frac{dF^{-1}(t)}{dt} = \frac{1}{f(F^{-1}(t))}$, we have $\frac{dF^{-1}(t)}{dt} = \frac{dG^{-1}(t)}{dt}$, $0 < t < 1$, then $F^{-1}(t) = G^{-1}(t) + d$, $0 < t < 1$. this means for a change of location parameter, F and G belong to the same family. □

3. Characterizations Based on Cumulative Entropy of Reversed Residual Lifetime of Parallel Systems

Suppose X_1, X_2, \dots, X_n be continuous and iid random variables with common cumulative distribution function F denote the lifetimes of n components of a parallel system. Also $X_{1:n}, X_{2:n}, \dots, X_{n:n}$ be the ordered lifetimes of the components. Then $X_{n:n}$ represents the lifetime of that parallel system with cumulative distribution function $F_{X_{n:n}}(x) = (F(x))^n, x > 0$. The distribution function of $[t - X_{n:n} | X_{n:n} < t]$ is $F_{X_{n:n}}(x) = (\frac{F(t-x)}{F(t)})^n$ that $[t - X_{n:n}]$ is called reversed residual lifetime of system. Now cumulative entropy for reversed residual lifetime of parallel system with distribution function $F_{X_{n:n}}(x)$ is as follows:

$$\begin{aligned} CE(X_{n:n,t}) &= - \int_0^\infty X_{n:n,t} \log X_{n:n,t} dx = - \int_0^t \left(\frac{F(x)}{F(t)}\right)^n \log \left(\frac{F(x)}{F(t)}\right)^n dx \\ &= - \frac{1}{(F(t))^n} \int_0^t (F(x))^n \log(F(x))^n dx + n \log F(t) \int_0^t \left(\frac{F(x)}{F(t)}\right)^n dx \\ &= - \frac{1}{(F(t))^n} \int_0^t (F(x))^n \log(F(x))^n dx + n \log F(t) K_{X_{n:n}}(t) \end{aligned} \tag{7}$$

That $K_{X_{n:n}}(t) = E(t - X_{n:n} | X_{n:n} < t)$ is the mean reversed residual lifetime of the system.

Now according to relation(7) and that $\log F(t) \leq 0$, we obtain some lower bounds for $CE(X_{n:n})$ as follows:

$$CE(X_{n:n,t}) \leq - \frac{1}{(F(t))^n} \int_0^t (F(x))^n \log(F(x))^n dx \leq - \frac{1}{(F(t))^n} \int_0^\infty (F(x))^n \log(F(x))^n dx = \frac{1}{(F(t))^n} CE(X_{n:n})$$

hence, for all t we have

$$CE(X_{n:n}) \geq (F(t))^n CE(X_{n:n,t})$$

According to non-negativity of CE and relation (7) we obtain

$$\begin{aligned} K_{X_{n:n}}(t) &\leq - \frac{1}{n \log|F(t)|(F(t))^n} \int_0^t (F(x))^n \log(F(x))^n dx \leq - \frac{1}{n \log|F(t)|(F(t))^n} \int_0^\infty (F(x))^n \log(F(x))^n dx \\ &\leq - \frac{1}{n \log|F(t)|(F(t))^n} CE(X_{n:n}) \end{aligned}$$

Finally

$$CE(X_{n:n}) \geq n \log|F(t)|(F(t))^n K_{X_{n:n}}(t)$$

Theorem 3.1: Suppose X and Y be two positive random variable with density functions $f(x)$ and $g(x)$ and absolutely continuous cumulative distribution functions $F(x)$ and $G(x)$, respectively. Then F and G belong to the same family of distributions, but for a change in location and scale parameters, if and only if for $t > 0$

$$CE(X_{n:n,t}) = CE(Y_{n:n,t})$$

for $n = n_j, j \geq 1$ such that $\sum_{j=1}^\infty \frac{1}{n_j}$ be infinitely.

Proof: The ‘if’ part of the theorem is obvious. So to prove ‘only if’ part, if for all $n = n_j, j \geq 1, \sum_{j=1}^{\infty} \frac{1}{n_j} = \infty$, $CE(X_{n:n,t}) = CE(Y_{n:n,t})$, then according to theorem (2.2) $[X|X < t]$ and $[Y|Y < t]$ have the same distribution but for a location shift, means $f_t(x) = g_t(x + c)$ that f_t, g_t are respectively density functions of $[X|X < t]$ and $[Y|Y < t]$. Hence $f(x) = \frac{F(t)}{G(t)}g(x + c)$ that is F and G belong to the same family of distributions but for a change in location and scale parameters. □

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