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A Directional Bayesian Significance Test for Equality of Variances

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Abstract

A directional Bayesian pure significance test for the equality of variances is developed. The approach is based on the assessment of observed departure conditioned on the direction of departure in multivariate models. The resulting one-dimensional directional distribution is easily interpreted. Normality is not required. Robustness of prior selection is discussed focusing on directional properties of the multivariate prior. Several examples are considered.

Keywords: Bartlett test, Bayesian directional test, test of variances.

1. Introduction

Statistical significance tests for the equality of variation across treatment groups are often used in experimental settings where ANOVA based analysis is required. This is an issue for example in toxicological and genetic applications (Gastwirth et al. 2009). As shown in Box (1954), the robustness of the one-way ANOVA overall F-test to non-normality is dependent on the degree of inequality among the group variances. Further, in moderate size samples, the p-value calculation may not be accurate (Draper and Smith, 1998) and variance-stabilizing transformations may be required to develop a more stable result. In settings where homogeneity of variation is to be formally tested, the commonly applied Bartlett test (Bartlett, 1937) which is a slight modification of a likelihood ratio test is often used, but is sensitive to the assumption of normality. The non-parametric Levene test can provide a more robust test with the trade-off of lower power. See for example Miller (1997).

Extensions of the Bartlett and Levene tests of homogeneity have been developed, often through modification of the L_2 norm in the ANOVA setting and modifications of the assumed error distribution. The work in Gastwirth et al. (2009) reviews several such approaches where typically bootstrap sampling methods and absolute value based (L_1 norm) measures of departure are used to extend standard tests. Earlier work in this regard can be found for example in Box and Tiao (1973) where Bayesian inference is developed for variance components, tests of homogeneity, and the modeling of variation in linear models with random effects.

With regard to likelihood based approaches for the modeling and testing of variation in a linear model, likelihood and marginal likelihood approaches are given in Harville (1977) which reviews restricted maximum likelihood methods (REML) based on error contrasts. This involves considering the transformation $w = Ay$ where y is the original response and A is a selected orthogonal projection matrix. The restricted likelihood for w resulting from this projection can be used to analyze parameters in the Σ variance-covariance matrix as the mean parameter μ is orthogonalized out of the likelihood. Harville (1974) points out the usefulness of the REML approach in a Bayesian setting. If the prior selection for mean and variance component parameters is independent, the joint posterior resulting from using the restricted likelihood for inference regarding Σ is equivalent to ignoring prior information for the mean parameter and using all the data to make inferences for Σ.

In the setting of one-way ANOVA the assumption of homogeneity of variation across treatments underlies application of the standard global F-test for differences among treatment averages. To test this assumption by examining the null hypothesis $H_0 : \sigma_1^2 = \cdots = \sigma_k^2$ in the model $y_{li} \sim N(\theta_i, \sigma_i^2), i = 1, ..., k; l = 1, ..., n_i$ where $\theta_i = \mu + \tau_i$, the Bartlett test uses a statistic of the form;

$$M = -\sum_{i=1}^{k} v_i(\log s_i^2 - \log s_p^2)$$

where s_i^2 is the sample variance for each treatment group $s_p^2 = (1/v)\sum_{i=1}^{k} v_i s_i^2$, the pooled variance, $v_i = (n_i - 1)$ and $v = \sum_{i=1}^{k} v_i$. Under the null hypothesis M is distributed as a χ^2_{k-1} random variable. The related p-value calculations are standard in many statistical packages.

In this paper we develop an analogue of the standard Bartlett significance test in a directional Bayesian framework. The pure significance test developed assesses departure of the hypothesized null parameter value from the posterior mode where the posterior distribution is conditioned on the direction of departure. The approach is general and does not require normality as an assumption, though this is assumed here for comparative purposes with the Bartlett test. This measure provides an assessment of the observed discrepancy of the predicted value or posterior mode from the hypothesized null value. Its application to the multi-parameter hypothesis test $H_0 : \sigma_1^2 = \cdots = \sigma_k^2$ is examined in detail. The sensitivity of the test to prior and likelihood selection is defined. Several examples are considered.

2. Method

1. Bayesian Approach

The Bayesian approach to statistical inference is based on the posterior density function given by;

$$p(\theta|data) = c \cdot p(\theta) \cdot L(\theta|\, data)$$

where $L(\theta|\, data)$ is the observed likelihood function, $p(\theta)$ is the prior density function for θ and c denotes the constant of integration. The Bayesian approach to hypothesis testing is based on developing empirical measures of support for null parameter values typically expressed in terms of posterior odds ratios or Bayes factors (Bernardo and Smith, 1994). The posterior odds ratio for example comparing a specified value θ_0 to a general alternative θ can be written

$$\frac{p(\theta_0|data)}{1 - p(\theta_0|data)} \Big/ \frac{p(\theta|data)}{1 - p(\theta|data)} \tag{1}$$

which provides a relative measure of the support given to the competing hypotheses within the context of the joint posterior. Typically $\theta = \widehat{\theta}$,the posterior mode, is taken as a reference value to interpret the resulting odds ratio. If θ is a vector of parameters, some initial marginalisation of parameters not of interest may be necessary.

1.1 Bayesian Bartlett Test

A Bayesian test of variances that is a version of the Bartlett test is given in Box and Tiao (1973), p. 132. Assume the samples $(y_{11}, \ldots, y_{1n_1}), \ldots, (y_{k1}, \ldots, y_{kn_k})$ of $n_1, \ldots, n_k$ independent observations, respectively, are drawn from the Normal populations $N(\theta_i, \sigma_i^2)$, where $i = 1, \ldots, k$. If both the means θ_i and variances σ_i^2 are unknown and it is assumed that *a priori* θ_i and $\log \sigma_i$ are approximately independent and locally uniform it follows that the joint posterior for $\sigma_1^2, \ldots, \sigma_k^2$ is given by

$$p(\sigma_1^2, \ldots, \sigma_k^2|\mathbf{y}) = \prod_{i=1}^{k} c_i(\sigma_i^2)^{-[\nu_i/2+1]} exp(-\frac{\nu_i s_i^2}{2\sigma_i^2}) \tag{2}$$

for $\sigma_i^2 > 0,\ i = 1, \ldots, k$. Here $c_i = [\Gamma(\upsilon_i/2)]^{-1}(\nu_i s_i^2/2)^{\nu_i/2}$, $\nu_i = n_i - 1$, s_i^2 is the standard deviation for the i^{th} sample.

To compare variation across the k samples any (k - 1) linearly independent contrasts in $\log \sigma_i$ can be used. Following the development in Box and Tiao (1973) we can define

$$\psi_i = \ \log \sigma_i^2 - \log \sigma_k^2$$

for $i = 1, \ldots, (k - 1)$. Changing variables in (2), the $(k - 1)$- dimensional joint posterior for $\psi_1, \ldots, \psi_{k-1}$ can be expressed as;

$$p(\psi\,|\,\mathbf{y}) \quad = \quad c \cdot (a_1 e^{-\psi_1})^{\nu_1/2} \cdots (a_{k-1} e^{-\psi_{k-1}})^{\nu_{k-1}/2} \tag{3}$$

$$\cdot[1 + a_1 e^{-\psi_1} + \cdots + a_{k-1} e^{-\psi_{k-1}}]^{-\nu/2} \tag{4}$$

where $i = 1, \ldots, k-1$, $a_i = \nu_i s_i^2/\nu_{k-1} s_{k-1}^2$, $\nu = \sum_{i=1}^{k-1} \nu_i$ and $-\infty < \psi_i < \infty$. The mode of this distribution is given by $\psi = \widehat{\psi}$, where $\hat{\psi}_i = \log s_i^2 - \log s_k^2$ for $i = 1, \ldots, (k - 1)$.

The null hypothesis $\sigma_i^2 = \cdots = \sigma_k^2$ can be rewritten as $\psi = \mathbf{0}$. Box and Tiao (1973) use a Likelihood Ratio type pivotal quantity in a Bayesian context to develop a χ_{k-1}^2 based approximate testing procedure. The presence of $\psi = \mathbf{0}$ in the approximate 95% credible region is interpreted as evidence that the null hypothesis of homogeneity is supported by the observed data.

2. Directional Bayesian Significance Testing

The Bayesian approach to testing hypotheses is an examination of the observed posterior probability based weight observed for each potential θ value of interest. There is no assumption of a true value, rather a comparison of relative weights for potential θ values. Typically a posterior modal value for θ, $\widehat{\theta}$, can be compared to specific θ values of interest using the posterior odds ratios or Bayes factor.

The directional approach applied here emphasizes the observed magnitude of departure of the hypothesized parameter value θ_0 from a central posterior value $\hat{\theta}$, typically the mode of the posterior distribution. This is viewed in terms of two distinct elements; the direction of departure and the magnitude of departure. The direction of departure out from the central modal value $\hat{\theta}$ is defined along the unit vector $\mathbf{d} = (\theta_0 - \hat{\theta})/||\theta_0 - \hat{\theta}||$. This is formally conditioned upon as other directions are not of immediate interest in assessing $(\theta_0 - \hat{\theta})$. The observed magnitude of departure, $r_0 = ||\theta_0 - \hat{\theta}||$ is then evaluated under the resulting one-dimensional local directional distribution.

Geometrically, this involves examining the posterior distribution along the span of the unit vector $\mathbf{d}$ out from the mode, subject to an initial change of variable $\theta \to (r, \mathbf{d})$, including a Jacobian factor. The resulting conditional distribution, $p_r(r|\mathbf{d})$ provides tail areas that can be evaluated in terms of a *one-dimensional* integral using $p_r(r|\mathbf{d})$. As noted above, the joint posterior density can be written as;

$$p(\theta|\mathbf{y}) = c \cdot p(\theta) \cdot L(\theta|\mathbf{y}) \tag{5}$$

where $p(\theta)$ is the prior distribution for θ, c is a constant of integration and $L(\theta|\mathbf{y})$ is the observed likelihood function. The value for θ is taken to lie in the parameter space $\Omega \subset \Re^k$. There may be nuisance parameters involved in the initial specification of the model. We assume for simplicity of exposition that nuisance parameters have been integrated out.

Some assumptions regarding the form of the joint posterior for θ are useful; θ must be a continuously valued vector of reals defined in a parameter space Ω which is also a vector space and $p(\theta\,|\,\mathbf{y})$ is assumed to be unimodal. The support for $p(\theta|y)$, i.e., $\{\theta \in \Omega : p(\theta\,|\,\mathbf{y}) > 0\}$, is assumed here to be a convex set with the linear span of the vector $\mathbf{d} = (\theta_0 - \hat{\theta})/||\theta_0 - \hat{\theta}||$, $\mathbf{L}\{\mathbf{d}\}$ entirely within Ω. The vector $\mathbf{d}$ is a unit vector lying on S^{k-1}, the unit sphere in $(k-1)$ dimensions. In cases where the convexity is not global, the region of Ω about θ_0 should contain $\mathbf{L}\{\mathbf{d}\}$. The hypothesized parameter value $\boldsymbol{\theta_0}$ should not lie on the boundary of Ω.

With $\widehat{\theta}$ as the mode of the joint posterior, let $r = ||\theta - \hat{\theta}||$ and change variables $\theta \to (r, \mathbf{d})$. The Jacobian is proportional to $r^{k-1}\, dr\, d\mathbf{d}$. Conditioning upon $\mathbf{d}$, the resulting conditional distribution $p_r(r\,|\,\mathbf{d})$ is proportional to the initial joint distribution up to a multiplicative constant c and is given by

$$p_{r|\mathbf{d}}(r) = c \cdot p(r\mathbf{d} + \hat{\theta}|y) \cdot r^{k-1}.$$

In the specific case where $\mathbf{p}(\theta\,|\,\mathbf{y})$ is a rotationally symmetric distribution satisfying the above assumptions then $\hat{\theta}$ will be the mean of the posterior density.

To assess the global hypothesis $H_0 : \theta = \theta_0$, let $r_0 = ||\theta_0 - \hat{\theta}||$. A conditional tail area in regard to the null hypothesis can then be defined. This is given by the "Directional Posterior Level of Significance" ($DPLS$);

$$DPLS\,(\theta_0) = \frac{\int_{r_0}^{\infty} p(r\mathbf{d} + \hat{\theta}|\,\mathbf{y})\, r^{k-1}\, dr}{\int_0^{\infty} p(r\mathbf{d} + \hat{\theta}|\,\mathbf{y})\, r^{k-1}\, dr} \tag{6}$$

and the "significance" here reflects the underlying distance measure and can be assessed without reference to a specific Type I error level. This is the tail area of the conditional distribution for r given the unit directional vector $\mathbf{d}$. $DPLS\ (\theta_0)$ is the (conditional) probability of θ lying at or beyond θ_0, in the direction specified by $\mathbf{d}$. Note that this measure of departure is made in terms of r, a scalar measure of the distance from $\boldsymbol{\theta_0}$ to the central value $\hat{\theta}$. It provides a simple summary measure of the (conditional) weight to be accorded the null hypothesis $H_0 : \theta = \theta_0$.

To further interpret results in terms of the number of standard errors found in the magnitude of departure, the mean and variance for $p_{r|\mathbf{d}}(r)$ are generally available. If we define $K(m) = \int_0^{\infty} p(r\mathbf{d} + \hat{\theta} \mid \mathbf{y})\, r^{m-1} d\tau$, it follows that $E[r \mid \mathbf{d}] = K(m+1)/K(m)$ and $Var[r \mid \mathbf{d}] = Var[\tau \mid \mathbf{d}] = K(m+2)/K(m) - (K(m+1)/K(m))^2$. The approach has been applied to multi-parameter testing problems in econometrics (Brimacombe, 1996) and the development of diagnostics for asymptotic convergence in logistic regression (Brimacombe, 2016).

2.1 Rotational Symmetry

In the specific case of a rotationally symmetric posterior distribution about $\hat{\theta}$, the unit vector $\mathbf{d}$ defined above is uniformly distributed on the unit sphere S^{k-1} and the $DPLS\ (\theta_0)$ value is equivalent to an unconditional Bayes significance level, namely,

$$\int_A p(\theta|\mathbf{y})\; d\theta$$

where $A \subset \Omega$ is the set of θ values at or beyond $\boldsymbol{\theta_0}$. Rotational symmetry implies that the tail area in question is equivalent in any chosen direction. In general, if $p(\theta\,|\,\mathbf{y})$ is of the form $p(||\theta - \hat{\theta}||^2|\,\mathbf{y})$, it is trivial to show that the $DPLS\,(\theta_0)$ is independent of **d**. For example, a multivariate normal model with assumed homogeneity and independence has this property. Note that differences in the *DPLS* value for various directions out from the posterior mode can be used to assess the degree of asymmetry in the joint posterior distribution. This is examined in more detail below.

3. Results

2.2 Example 1: Box and Tiao (1973)

We discuss the application of the *DPLS* measure in the context of an example given in Box and Tiao (1973), p.136. An experiment is conducted to examine similarity in the variation of three distributions. The summary statistics are given by;

$$s_1^2 = 52.785, s_2^2 = 34.457, s_3^2 = 66.030,\ \nu_1 = \nu_2 = \nu_3 = 30,\ \nu = 90.$$

Assuming normality and locally non-informative priors, the joint posterior distribution of the two contrasts, $\psi_1 = log\ \sigma_1^2 - log\ \sigma_3^2$ and $\psi_2 = log\ \sigma_2^2 - log\ \sigma_3^2$ can be written;

$$\begin{aligned} p(\psi_1, \psi_2|\mathbf{y}) &= \frac{\Gamma(45)}{[\Gamma(15)]^3}(.79941e^{-\psi_1})^{15}(.52184e^{-\psi_2})^{15} \\ &\quad \cdot(1 + .79941e^{-\psi_1} + .52184e^{-\psi_2})^{-45} \end{aligned} \tag{7}$$

where $-\infty < \psi_i < \infty$, for i = 1,2. The mode is given by $\widehat{\psi} = (\hat{\psi_1}, \hat{\psi_2}) = (-.2239, -.6504)$. Approximate 75, 90 and 95 per cent Highest Posterior Density (H.P.D.) regions are given in Box and Tiao (1973), p. 138 with the null hypothesis $\psi_0 = (0, 0)$ lying within the 90% region, providing limited support for the null hypothesis of homogeneity. The joint posterior density is given in Figure 1.

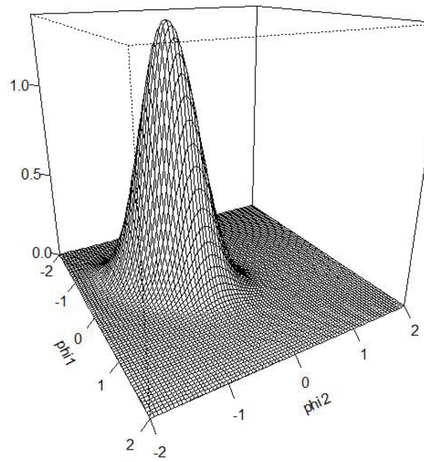

Figure 1. Joint Posterior Density

To derive a directional Bayesian Bartlett test here we initially change variables $\psi \to (r, \mathbf{d})$ with Jacobian proportional to $r^{2-1}\, dr\, d\mathbf{d}$, $r > 0$ to obtain $p_{r|\mathbf{d}}(r)$ as above. We may also further change variables with $\tau = r^2$ to obtain a chi-square related distribution. The resulting conditional distribution $p_\tau(\tau\,|\,\mathbf{d})$ in the direction defined by the null hypothesis $\psi_0 = (0, 0)$ out from the mode is given by;

$$\begin{aligned} p_{\tau|\mathbf{d}}(\tau) &= c \cdot p(\tau^{1/2}\mathbf{d} + \hat{\psi}) = \\ &= c \cdot (.79941e^{-\tau^{1/2}d_1 - \widehat{\psi_1}})^{15}(.52184e^{-\tau^{1/2}d_2 - \widehat{\psi_2}})^{15} \\ &\quad \cdot[1 + .79941e^{-\tau^{1/2}d_1 - \widehat{\psi_1}} + .52184e^{-\tau^{1/2}d_2 - \widehat{\psi_2}}]^{-45} \cdot \tau^{2/2-1} \end{aligned}$$

This is re-normed and shown in Figure 2.

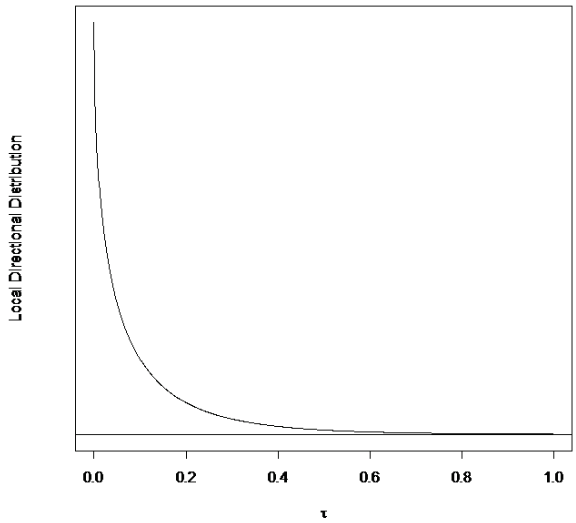

Figure 2. Directional Conditional Distribution ($H_0 : \psi_0 = (0, 0)$)

The direction of departure related to the null is given by $\mathbf{d}_0 = (\hat{\psi} - \mathbf{0})/||\hat{\psi} - \mathbf{0}|| = (0.3255, 0.9455)$ and the observed magnitude of departure $r_0 = ||\psi_0 - \widehat{\psi}|| = 0.6879$ or $\tau_0 = (0.6879)^2 = 0.4732$. The related local tail area for the null hypothesis $\psi_0 = (0, 0)$ is given by;

$$DPLS\,(\psi_0) = \frac{\int_{\tau_0}^{\infty} p_{\tau|\mathbf{d}}(\tau)\, d\tau}{\int_0^{\infty} p_{\tau|\mathbf{d}}(\tau) d\tau} = 1 - .9872 = 0.0128$$

and can be interpreted as providing little support for the null hypothesis of equal variances.

2.3 Example 2: Simulation 1

This example gives a simulated dataset where the Bartlett and Levine tests do not agree. The data and summary statistics are given in Table 1.

Table 1. Test for Equal Variances. 95% Bonferroni Confidence Intervals for Standard Deviations

Group	n	Lower Bound	Standard Deviation	Upper Bound
1	24	0.99	1.34	2.04
2	18	0.89	1.26	2.09
3	28	1.63	2.18	3.19

Group	Data
Group 1 (n=24)	5.37, 5.80, 4.70, 5.70, 3.40, 8.60, 7.48, 5.77, 7.15, 6.49, 4.09, 5.94, 6.38, 9.24, 5.66, 4.53, 6.51, 7.0, 6.20, 7.04, 4.82, 6.73, 5.26, 5.21
Group 2 (n=18)	3.96, 3.04, 5.28, 3.4, 4.1, 3.61, 6.16, 3.22, 7.48, 3.87, 4.27, 4.05, 2.40, 5.81, 4.29, 2.77, 4.4, 4.45
Group 3 (n=28)	5.37, 10.6, 5.02, 14.30, 9.9, 4.27, 5.75, 5.03, 5.74, 7.85, 6.82, 7.9, 8.36, 5.72, 6.0, 4.75, 5.83, 7.3, 7.52, 5.32, 6.05, 5.68, 7.57, 5.68, 8.91, 5.39, 4.4, 7.13

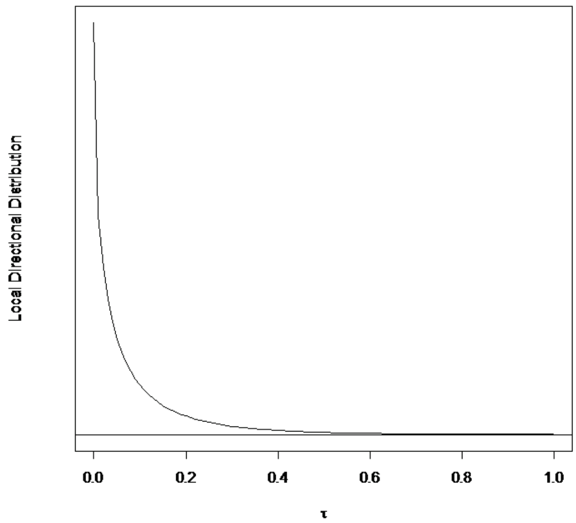

Figure 3. Directional Conditional Distribution ($H_0 : \psi_0 = (0, 0)$)

Here the Bartlett test has a p-value of 0.015 and the Levine test a p-value of 0.23, providing different inferential conclusions. The direction of departure related to the null is given by $\mathbf{d}_0 = (\hat{\psi} - \mathbf{0})/||\hat{\psi} - \mathbf{0}|| = (0.6641, 0.7476)$ and the magnitude of departure by $r_0 = ||\psi_0 - \widehat{\psi}|| = 0.6305$ or $\tau_0 = (0.6305)^2 = 0.3975$. The conditional distribution $p_\tau(\tau \,|\, \mathbf{d})$ for the null hypothesis $\psi_0 = (0, 0)$ is given by;

$$\begin{aligned} p_{\tau|\mathbf{d}}(\tau) &= c \cdot (a_1 e^{-\tau^{1/2} d_1 - \widehat{\psi}_1})^{23} (a_2 e^{-\tau^{1/2} d_2 - \widehat{\psi}_2})^{17} \\ &\quad \cdot [1 + a_1 e^{-\tau^{1/2} d_1 + \widehat{\psi}_1} + a_2 e^{-\tau^{1/2} d_2 + \widehat{\psi}_2}]^{-34} \end{aligned}$$

where c is the constant of integration. This is shown in Figure 3. The exact test tail area for this hypothesis is given by;

$$DPLS\,(\psi_0) = \frac{\int_{\tau_0}^{\infty} p_{\tau|\mathbf{d}}(\tau)\, d\tau}{\int_0^{\infty} p_{\tau|\mathbf{d}}(\tau) d\tau} = 0.03186$$

which supports mild rejection of the null hypothesis of equal variances, agreeing here with the standard Bartlett test.

2.4 Example 3: Simulation 2

The approach here extends easily into multiple treatment group settings. A simulated example examining variation across five groups ($n_i = 20, i = 1, ..., 5$) is developed with the fifth group being an increasing outlier across each of three simulations. This gives a joint posterior density for ψ in four dimensions. Summary statistics and related tail areas are given in Table 2 (data not shown). Figure 4 shows the three respective conditional densities.

Table 2. Comparison of Five Plasma Groups

	s_1 (95% interval)	**s_2 (95% interval)**	**s_3 (95% Interval)**	**s_4 (95% Interval)**	**s_5 (95% interval)**	**τ_0**	**DPLS**	**Levine**	**Bartlett**
Simulation1	1.19 (.83,1.98)	.99 (.69,1.65)	.66 (.46,1.09)	.75 (.53,1.26)	1.23 (.87, 2.06)	0.722	.202	.014	.030
Simulation2	1.19 (.83,1.98)	.99 (.69,1.65)	.66 (.46,1.09)	.75 (.53,1.26)	2.79 (1.96,4.65)	2.06	.089	.0001	.0001
Simulation3	1.19 (.83,1.98)	.99 (.69,1.65)	.66 (.46,1.09)	.75 (.53,1.26)	3.72 (2.61,6.20)	2.55	.013	.0001	.0001

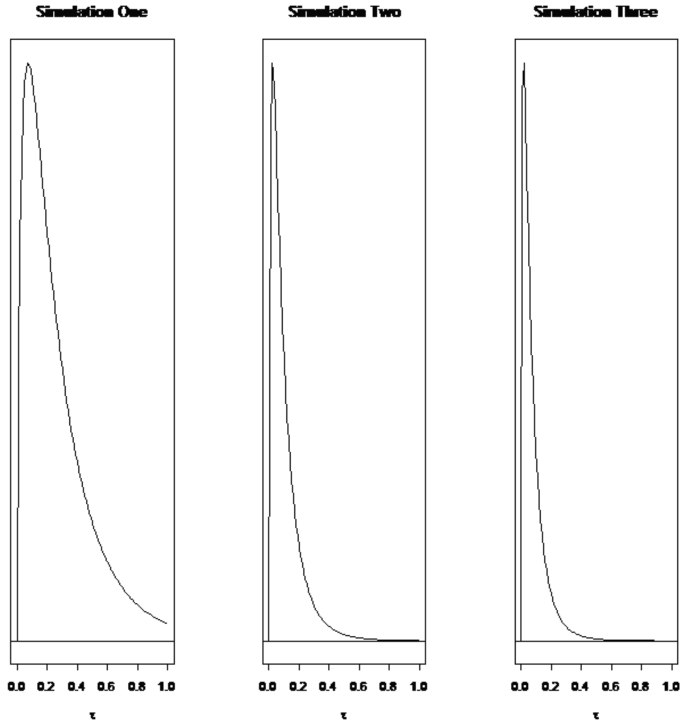

Figure 4. Directional Conditional Distributions for Three Simulations ($H_0 : \psi_0 = (0, 0, 0, 0)$)

The Bartlett and Levine tests here give significant p-values for all three datasets. Standard Bonferroni corrected confidence intervals shown in Table 2 give a large overlap of the outlier group confidence interval (group 5) with the other four group intervals in the first two datasets but none in the third. Note that while the sample size per group is relatively small, the number of groups and overall sample size lead to potential over-sampling. The conditional distribution $p_\tau(\tau \,|\, \mathbf{d})$ for the null hypothesis $\psi_0 = (0, 0, 0, 0)$ is given by;

$$\begin{aligned} p_{\tau|\mathbf{d}}(\tau) &= c \cdot (a_1 e^{-\tau^{1/2} d_1 - \widehat{\psi_1}})^{20} (a_2 e^{-\tau^{1/2} d_2 - \widehat{\psi_2}})^{20} (a_3 e^{-\tau^{1/2} d_3 - \widehat{\psi_3}})^{20} (a_4 e^{-\tau^{1/2} d_4 - \widehat{\psi_4}})^{20} \\ &\quad \cdot [1 + a_1 e^{-\tau^{1/2} d_1 + \widehat{\psi_1}} + a_2 e^{-\tau^{1/2} d_2 + \widehat{\psi_2}} + a_3 e^{-\tau^{1/2} d_3 - \widehat{\psi_3}} + a_4 e^{-\tau^{1/2} d_4 - \widehat{\psi_4}}]^{-50} \cdot \tau^{4/2-1} \end{aligned}$$

where again c is the constant of integration. See Table 2 for results. The directional approach here is relatively conservative, with clear rejection of the null supported only in the third simulated dataset where the outlier group has no overlap with the other groups in terms of standard Bonferroni corrected confidence intervals.

3. Robustness

3.1 Prior Robustness

Any application of the Bayesian perspective in relation to modeling should discuss potential effects of prior selection. As the analysis here conditions upon $\mathbf{d}$ to obtain the conditional distribution $p_r(r \,|\, \mathbf{d})$, the robustness of the analysis to choice of prior density may also be defined directionally. While examining the behavior of $p(r|\mathbf{d})$ for a variety of chosen priors is a possibility, the directional and one-dimensional nature of the *DPLS* approach allows for geometric insight. In particular, a chosen prior will have limited effect on the resulting *DPLS* value if it is relatively flat in the direction $\mathbf{d}$, directionally non-informative. This effect can be visualized by plotting the one-dimensional directionally conditioned prior component $p(r\mathbf{d} + \hat{\theta})$ as function of $r > 0$. These can be compared for a set of potential prior densities.

When selecting a multivariate prior for the covariance matrix Σ several approaches are standard. A basic approach is to use a normal-gamma distribution for the (μ, Σ) set of parameters and express this as a conditional distribution $p(\mu|\Sigma)$ times a marginal distribution $p(\Sigma)$. A simpler approach is to assume prior independence. In the *REML* setting where we assume that prior belief regarding μ and Σ are independent, it is possible to use the marginal likelihood based on error contrasts. Harville (1974) showed that the *REML* based marginal posterior for Σ is independent of the mean parameter μ, simplifying the prior selection process. Prior choices can be extended to other multivariate densities. Note that a hierarchical setting may be useful in modeling variation (Daniels, 1999) but is not considered in detail here.

If the prior density is viewed in isolation, directionalized by applying the polar transformation, then we obtain $c \cdot p(r\mathbf{d} + \hat{\theta}) \cdot$

r^{k-1} as the conditional prior density in the direction $\mathbf{d}$ where $\hat{\theta}$ is the prior mode. If a non-informative prior is interpreted as being fairly constant as a function of the parameter $r > 0$ we can plot this one-dimensional directionalized prior as a function of r and examine its behavior. We can express this more directly as;

$$\begin{aligned} c \cdot p(r\mathbf{d} + \hat{\theta}) \cdot r^{k-1} &= C \\ (c/C) \cdot p(r\mathbf{d} + \hat{\theta}) \cdot r^{k-1} &= 1 \\ \log(c/C) + \log p(r\mathbf{d} + \hat{\theta}) + \log r^{k-1} &= 0 \\ -\log p(r\mathbf{d} + \hat{\theta}) &= [\log(c/C) + (k-1)\log r] \end{aligned}$$

Plotting $-\log p(r\mathbf{d} + \hat{\theta})$ versus $\log r$ allows us to see if a straight line regression with slope $(k-1)$ is obtained. If yes, the chosen prior $p(\cdot)$ may be viewed as directionally non-informative. This can be examined in relation to other directions $\mathbf{d}$ if required and across a set of potential prior densities. A similar directional examination of the posterior density comparing actual and large sample likelihood functions can be found in Brimacombe (2016).

4. Discussion

A directional Bayesian pure significance test procedure is developed here for testing the hypothesis of equal variances. It is based on the tails areas of a conditional Bayesian directional distribution defined in relation to the specified null hypothesis and reflects the departure of the null value from the observed posterior mode. The approach does not require an assumption of normality. Where non-informative priors are employed, the approach reflects local properties of the likelihood function. Note that issues related to the power of the pure significance test are not addressed here.

The use of a directional approach allows for a unique perspective regarding prior selection and prior related robustness. The stability of the posterior can be assessed in the specific direction of the null hypothesis, examining how the prior is affecting the relevant directional properties of the posterior density. While related here to the Bartlett test, the directional approach can be applied more widely and provides a one-dimensional conditional approach to higher dimensional multiparameter significance testing problems in general. Applications to MANOVA and generalized linear models will be found elsewhere. Note that standard Markov Chain Monte Carlo based methods may be applied to determining the required tail area in more detailed settings especially where nuisance parameters must initially be averaged out.

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