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Characterizations of Extreme Value Extended Marshall-Olkin Models with Exponential Marginals

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Abstract

We construct and characterize bivariate extreme value distributions with exponential marginals generated by the stochastic representation $(X_1, X_2) = (\min(T_1, T_3), \min(T_2, T_3))$ where the random variable T_3 is independent of random variables T_1 and T_2 which are assumed to be dependent. A building procedure is suggested when the joint distribution of (T_1, T_2) is absolutely continuous and T_i's are not necessarily exponentially distributed, $i = 1, 2, 3$. The Pickands representation of the vector (X_1, X_2) is computed. We illustrate the general relations by examples.

Keywords: bivariate extreme value distribution, extended Marshall-Olkin model, Pickands measure and dependence function.

1. Introduction

Let us consider the fatal shock model defined by the stochastic representation

$$(X_1, X_2) = (\min(T_1, T_3), \min(T_2, T_3)), \tag{1}$$

where non-negative continuous random variables T_1 and T_2 identify the occurrence of independent "individual shocks" affecting two devices and T_3 is their "common shock". The random vector (X_1, X_2) presents the joint distribution of both lifetimes.

Denote by $S_{X_1,X_2}(x_1, x_2) = P(X_1 > x_1, X_2 > x_2)$ the joint survival function of the vector (X_1, X_2) for all $x_1, x_2 \geq 0$. If the shocks are governed by independent homogeneous Poisson processes, then T_i's in (1) are exponentially distributed with parameters $\lambda_i > 0,\ i = 1, 2, 3$, and we obtain the classical Marshall-Olkin's (MO) bivariate exponential distribution

$$S_{X_1,X_2}(x_1, x_2) = \exp\{-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 \max(x_1, x_2)\}, \quad x_1, x_2 \geq 0, \tag{2}$$

see Marshall and Olkin (1967).

To give a probability interpretation of model (1), consider a system composed by two items, to be denoted by 1 and 2. We associate with each item $j,\ j = 1, 2$, a Bernoulli random variable Z_j, indicating whether the item is operational ($Z_j = 1$) or failed ($Z_j = 0$). The bivariate Bernoulli random vector (Z_1, Z_2) represents the state of the system. It is specified in terms of MO construction (1). The vector (X_1, X_2) exhibits the latent state of the system, since the MO model (2) is defined in terms of vector (T_1, T_2, T_3) of latent variables that identify independent exponential shock times. Each shock takes down a given subset of items ({1}, {2} or {both 1 and 2}) and occurs at an exponential time with constant rates λ_1, λ_2 and λ_3, respectively.

The stochastic relation (1) is widely used in literature. For example, Li and Pellerey (2011) launched the Generalized MO (GMO) model considering non exponential independent random variables T_i in (1), $i = 1, 2, 3$. The corresponding joint distributions do not possess bivariate lack of memory property, i.e., are "aging". As a further step, Pinto and Kolev (2015) introduced the Extended MO (EMO) model assuming dependence between latent variables T_1 and T_2, but keeping T_3 independent of them. The motivation is that the individual shocks might be dependent if the items share a common environment. In this case however, the EMO distributions can be "aging" or "non-aging" depending on the parameters of joint distribution of (T_1, T_2) and the distribution of T_3.

We will assume further that T_1 and T_2 are no more independent, but defined by their joint survival function $S_{T_1,T_2}(x_1, x_2) = P(T_1 > x_1, T_2 > x_2)$ and the random variable T_3 is independent of T_1 and T_2. Let $S_{T_i}(x) = P(T_i > x)$ be the survival functions of $T_i,\ i = 1, 2, 3$ for $x \geq 0$. Thus, the joint survival function of EMO model generated by (1) can be written as

$$S_{X_1,X_2}(x_1, x_2) = S_{T_1,T_2}(x_1, x_2) S_{T_3}(\max\{x_1, x_2\}). \tag{3}$$

In Section 2 we establish the extreme value representation and characterizations of a subclass of EMO distributions defined by (3) whose marginals X_1 and X_2 are exponentially distributed, see Theorem 1 and the most general Theorem 3. We suggest a procedure to construct EMO distributions with exponential marginals even if T_i's in (1) are not exponentially distributed under the corresponding additional restriction, e.g., Theorem 2. We obtain in Section 3 the Pickands dependence function and Pickands measure corresponding to (3) if the joint distribution of (T_1, T_2) is absolutely continuous. We illustrate the general relationships with two examples. As a particular case one can find the corresponding representations associated to the MO's bivariate exponential distribution (2) obtained by Mai and Scherer (2010). We finish the article with a short discussion.

2. Extreme Value EMO Distributions

There is a number of mathematical results characterizing multivariate extreme value distributions and extreme value copulas, see Chapter 6 in Joe (1997) for example. The marginals of any multivariate extreme value distribution must be members of the class of univariate generalized extreme value distributions, i.e., location-scale families of distributions based on Weibull, Fréchet and Gumbel distributions, see Theorem 2.4.1 in Galambos (1978).

Without loss of generality, we will assume hereafter marginal exponential distributions in (3), i.e., $X_i \sim Exp(\lambda_{X_i}),\ i = 1, 2$. Such a choice does not have influence on the corresponding survival copula to be obtained, being invariant on strict monotone transformations. It is also well known that these transformations can be used to move from one member to the other in the class of univariate generalized extreme value distributions, see Galambos (1978) and Beirlant et al. (2005) for a related discussion.

The joint survival function of the EMO distributions specified by (3) can be equivalently represented by

$$S_{X_1,X_2}(x_1, x_2) = S_{T_1,T_2}(x_1, x_2) \min\{S_{T_3}(x_1), S_{T_3}(x_2)\}. \tag{4}$$

Note that the right hand side in (4) is a product of two bivariate distributions. The first one is defined by $S_{T_1,T_2}(x_1, x_2)$, and the second one refers to a bivariate random vector with comonotonic components sharing the same marginal distribution as T_3. Such a product construction technique in terms of copula has been discussed by Genest et al. (1998), see their Proposition 2. Consult Liebscher (2008) for a general power function based approach as well.

Starting from the latent random vector (T_1, T_2) we "modify" it via (4) to get (X_1, X_2), which can be interpreted as follows: the "modified" joint distribution (X_1, X_2) can be used to model a complementary amount of bivariate asymmetry induced by (T_1, T_2). Note that, in general, this additional asymmetry does not necessarily imply an increase of upper tail dependence (if exists) governed by (T_1, T_2), see supporting comments in Joe (2015), page 184.

The simplest way to ensure exponentially distributed marginals X_1 and X_2 in (4) is to advocate that $T_i \sim Exp(\lambda_i)$, $i = 1, 2, 3$. This distributional choice has nice properties and will be justified in Theorem 1.

Other distributional possibilities for $T_i,\ i = 1, 2, 3$, do exist in order to construct EMO model with exponentially distributed marginals X_j with parameter λ_{X_j}, $j = 1, 2$. To believe, denote by $r_{T_i}(x) = \frac{d}{dx}[-\ln S_{T_i}(x)]$ the failure rates of $T_i,\ i = 1, 2, 3$. Note that the marginal survival functions in (4) are given by $S_{X_i}(x_i) = S_{T_i}(x_i)S_{T_3}(x_i)$ and the only condition in terms of failure rate functions to get exponential marginals in (4) is $r_{T_i}(x_i) + r_{T_3}(x_i) = \lambda_{X_i}$ for all $x_i \geq 0,\ i = 1, 2$. For example, consider $r_{T_1}(x) = r_{T_2}(x) = 2 + \sin(x)$ and $r_{T_3}(x) = 1 - \sin(x)$ for $x \geq 0$, to obtain $\lambda_{X_i} = 3$, $i = 1, 2$.

Denote by BEV_E the set of bivariate extreme value distributions with exponential marginals and let $BEV_E - EMO$ be a subclass of BEV_E satisfying (4), where $(T_1, T_2) \subset BEV_E$ and $T_3 \sim Exp(\lambda_3)$. This means that the following functional equation is fulfilled

$$S_{T_1,T_2}(tx_1, tx_2) = [S_{T_1,T_2}(x_1, x_2)]^t \quad \text{for all} \quad t > 0. \tag{5}$$

Joe (1997) observed that when multivariate copulas take on univariate generalized extreme value distributions one obtains multivariate extreme value distributions for maxima, but when the copulas take on generalized extreme value survival margins for minima, multivariate extreme value survival functions for minima results, see page 176. A general related discussion in terms of functional equations involving joint survival functions can be found in Marshall and Olkin (1991) as well.

Relation (5) implies that the survival copula $\overline{C}_{T_1,T_2}(u, v)$ associated to $S_{T_1,T_2}(x_1, x_2)$ is also an extreme value copula for all $t > 0$, i.e.,

$$\overline{C}_{T_1,T_2}(u^t, v^t) = [\overline{C}_{T_1,T_2}(u, v)]^t, \quad u, v \in [0, 1). \tag{6}$$

It follows our first characterization statement.

Theorem 1 *Let $T_3 \sim Exp(\lambda_3)$ in (4). Then $(X_1, X_2) \subset BEV_E - EMO$ if and only if $(T_1, T_2) \subset BEV_E$.*

Proof. Let $(X_1, X_2) \subset BEV_E - EMO$. Taking into account that X_i and T_3 are exponentially distributed and relations $S_{X_i}(x_i) = S_{T_i}(x_i)S_{T_3}(x_i)$, we conclude that T_i are exponentially distributed, $i = 1, 2$. Since $S_{X_1,X_2}(x_1, x_2)$ is an extreme value survival function with exponential marginals, then $S_{X_1,X_2}(tx_1, tx_2) = [S_{X_1,X_2}(x_1, x_2)]^t$ and applying representation (3) we get

$$\begin{aligned} S_{X_1,X_2}(tx_1, tx_2) &= S_{T_1,T_2}(tx_1, tx_2)S_{T_3}(\max\{tx_1, tx_2\}) \\ &= [S_{T_1,T_2}(x_1, x_2)S_{T_3}(\max\{x_1, x_2\})]^t. \end{aligned}$$

But T_3 is exponentially distributed, so that $[S_{T_3}(\max\{x_1, x_2\})]^t = S_{T_3}(\max\{tx_1, tx_2\})$ and we arrive to (5).

Now assume that $(T_1, T_2) \subset BEV_E$ and $T_3 \sim Exp(\lambda_3)$ in (4). Since the comonotonic copula $M(u, v) = \min(u, v)$ is an extreme value copula, applying Sklar's theorem in (4) and taking into account (6) we obtain

$$\begin{aligned} S_{X_1,X_2}(tx_1, tx_2) &= \overline{C}_{T_1,T_2}(\exp(-\lambda_1 tx_1), \exp(-\lambda_2 tx_2)) \min\{\exp(-\lambda_3 tx_1), \exp(-\lambda_3 tx_2)\} \\ &= \{\overline{C}_{T_1,T_2}[\exp(-\lambda_1 x_1), \exp(-\lambda_2 x_2)]\}^t[\min\{\exp(-\lambda_3 x_1), \exp(-\lambda_3 x_2)\}]^t \\ &= [S_{X_1,X_2}(x_1, x_2)]^t. \end{aligned}$$

Thus, the functional equation (5) for the vector (X_1, X_2) is satisfied and (X_1, X_2) belongs to the class $BEV_E - EMO$.

Obviously, if both $(T_1, T_2) \subset BEV_E$ and $(X_1, X_2) \subset BEV_E - EMO$ in (4) then $T_3 \sim Exp(\lambda_3)$.

To proceed, we will need to recall that any bivariate survival function $S_{T_1,T_2}(x_1, x_2)$ can be represented as

$$S_{T_1,T_2}(x_1, x_2) = \exp\{-H_{T_1}(x_1) - H_{T_2}(x_2) + D_{T_1,T_2}(x_1, x_2)\}, \tag{7}$$

where $H_{T_i}(x_i) = -\ln[S_{T_i}(x_i)]$ are the cumulative hazard functions of random variables T_i, $i = 1, 2$, and $D_{T_1,T_2}(x_1, x_2) = \ln\left[\frac{S_{T_1,T_2}(x_1,x_2)}{S_{T_1}(x_1)S_{T_2}(x_2)}\right]$ is the dependence function introduced by Sibuya (1960), to be referred as *Sibuya's dependence function*. It exhibits interesting relationships with dependence phenomena, see Kolev (2016) or Pinto and Kolev (2015a) for a deep discussion.

With an additional assumption of absolute continuity of the joint distribution of (T_1, T_2), we justify in the next Lemma 1 the choice of exponential T_i's in order to get a member of $BEV_E - EMO$ via (4), $i = 1, 2, 3$. Let us denote by $BEV_E - EMO^{AC}$ this subclass.

Lemma 1 *If $(X_1, X_2) \subset BEV_E - EMO^{AC}$ in (4) and $S_{T_1,T_2}(x_1, x_2)$ is absolutely continuous, then $(T_1, T_2) \subset BEV_E$ and $T_3 \sim Exp(\lambda_3)$.*

Proof. Suppose $x_1 > x_2 \geq 0$ and $t > 0$. Since $S_{X_1,X_2}(x_1, x_2)$ is an extreme value survival function we have $S_{X_1,X_2}(tx_1, tx_2) = [S_{X_1,X_2}(x_1, x_2)]^t$. Applying the exponential representation for bivariate survival functions (7) in (3) we obtain

$$\begin{aligned} &\exp\{-H_{T_1}(tx_1) - H_{T_3}(tx_1) - H_{T_2}(tx_2) + D_{T_1,T_2}(tx_1, tx_2)\} \\ &= \exp\{-tH_{T_1}(x_1) - tH_{T_3}(x_1) - tH_{T_2}(x_2) + tD_{T_1,T_2}(x_1, x_2)\}. \end{aligned}$$

Taking logarithms in both sides of former equation and calculating the mixed partial derivatives, we get

$$t\frac{\partial^2}{\partial x_1 \partial x_2} D_{T_1,T_2}(tx_1, tx_2) = \frac{\partial^2}{\partial x_1 \partial x_2} D_{T_1,T_2}(x_1, x_2).$$

Integrating we have

$$\int_0^{x_1}\int_0^{x_2} t\frac{\partial^2}{\partial u \partial v} D_{T_1,T_2}(tu, tv)\mathrm{d}v\mathrm{d}u = \int_0^{x_1}\int_0^{x_2} \frac{\partial^2}{\partial u \partial v} D_{T_1,T_2}(u, v)\mathrm{d}v\mathrm{d}u.$$

The boundary conditions $D_{T_1,T_2}(x_1, 0) = D_{T_1,T_2}(0, x_2) = 0$ imply equalities

$$\frac{\partial}{\partial x_1} D_{T_1,T_2}(x_1, 0) = \frac{\partial}{\partial x_2} D_{T_1,T_2}(0, x_2) = 0.$$

Thus, we arrive to the functional equation

$$D_{T_1,T_2}(tx_1, tx_2) = tD_{T_1,T_2}(x_1, x_2) \quad \text{for all} \quad t > 0. \tag{8}$$

Since $S_{X_1,X_2}(x_1,x_2)$ is an extreme value survival function with exponential marginals, we deduce that

$$\left[\frac{S_{X_1,X_2}(x_1,x_2)}{S_{X_1}(x_1)S_{X_2}(x_2)}\right]^t = \frac{S_{X_1,X_2}(tx_1,tx_2)}{S_{X_1}(tx_1)S_{X_2}(tx_2)}. \tag{9}$$

Applying the exponential representation (7) in (9) and taking into account (8), we obtain the homogeneous of order 1 functional equation $H_{T_3}(tx_2) = tH_{T_3}(x_2)$, for all $x_2, t > 0$, where $H_{T_3}(x_2)$ is the cumulative hazard of T_3. Substitute $\lambda_3 = H_{T_3}(1)$ to get the general solution $H_{T_3}(x_2) = \lambda_3 x_2$ with $\lambda_3 > 0$, since $H_{T_3}(x_2)$ is nonnegative for all $x_2 \geq 0$, see Aczel (1966) as well.

The conclusion is the same when $x_2 \geq x_1 \geq 0$. Thus, $T_3 \sim Exp(\lambda_3)$. Applying the "only if" branch of Theorem 1 we finish the proof.

Linking Theorem 1 and Lemma 1 we obtain the following characterization.

Theorem 2 *Let the joint distribution of* (T_1,T_2) *be absolutely continuous. Then* $(X_1,X_2) \subset BEV_E - EMO^{AC}$ *if and only if* $(T_1,T_2) \subset BEV_E$ *and* $T_3 \sim Exp(\lambda_3)$.

The absolute continuity assumption for (T_1,T_2) in Theorem 2 is an important condition. In the next we suggest a building procedure showing that it is possible to construct a member of $BEV_E - EMO$ class where the random variables T_i in (3) are not necessarily exponentially distributed, $i = 1,2,3$, but as a compensation we have to relax the assumption of absolutely continuity of the survival function $S_{T_1,T_2}(x_1,x_2)$.

Example 1 (Building procedure). Let us consider three nonnegative absolutely continuous and independent random variables defined by $S_{Y_1}(x) = S_{Y_2}(x) = \exp\{-x\}$ and $S_{Y_3}(x) = \exp\{-bx + f(x) - a\}$, where $a, b \geq 0$ and the continuous function $f(x)$ is such that $a - bx < f(x) < a + bx$ for all $x > 0$ with $f(0) = a$.

The following two-step procedure generates a vector $(X_1,X_2) \subset BEV_E - EMO$ with non-exponentially distributed T_i's:

1. Construct a GMO distribution $(T_1,T_2) = (\min(Y_1,Y_3), \min(Y_2,Y_3))$. Its joint survival function is $S_{T_1,T_2}(x_1,x_2) = \exp\{-x_1 - x_2 - H_{Y_3}(\max(x_1,x_2))\}$;
2. Select a random variable T_3, independent of (T_1,T_2), with a survival function $S_{T_3}(x) = \exp\{-bx - f(x) + a\}$. Apply (3) to get the corresponding EMO survival function $S_{X_1,X_2}(x_1,x_2) = \exp\{-x_1 - x_2 - 2b\max(x_1,x_2)\}$ with exponentially distributed marginals.

Notice that $S_{T_1,T_2}(x_1,x_2)$ obtained in the first procedure step is neither an extreme value survival function nor absolutely continuous, since has a singular component with support on the set $\{(x_1,x_2) \in [0,\infty)^2 \mid x_1 = x_2 = x\}$. In addition, T_3 defined in the second step is not exponentially distributed. Finally, inequalities $a - bx < f(x) < a + bx$ and $f(0) = a$ guarantee that $S_{Y_3}(x)$ and $S_{T_3}(x)$ are proper survival functions.

Now observe that the functional equation (8) involving Sibuya's dependence function $D_{T_1,T_2}(x_1,x_2)$ is homogeneous of order 1 for all $x_1,x_2 \geq 0$ and $t > 0$. As a consequence of Theorem 6.2 in Joe (1997), equation (8) can serve as a characterization of bivariate extreme value survival functions $S_{T_1,T_2}(x_1,x_2)$ with exponential marginals even when absolute continuity for (T_1,T_2) does not hold true. Hence, we deduce the following characterization.

Lemma 2 *A bivariate distribution is* BEV_E *if an only if its Sibuya's dependence function is homogeneous of order 1.*

To characterize distributions belonging to the subclass $BEV_E - EMO$ when the joint distribution of (T_1,T_2) is not absolutely continuous, we will need an additional assumption since $BEV_E - EMO \subset BEV_E$. It is given below.

Theorem 3 $(X_1,X_2) \subset BEV_E - EMO$ *if and only if the functional equation*

$$D_{T_1,T_2}(tx_1,tx_2) + H_{T_3}(t\min(x_1,x_2)) = tD_{T_1,T_2}(x_1,x_2) + tH_{T_3}(\min(x_1,x_2))$$

is satisfied for all $x_1,x_2 \geq 0$ *and* $t > 0$.

Proof. Under conditions in the theorem, the Sibuya's dependence function for the EMO model (3) is given by

$$D_{X_1,X_2}(x_1,x_2) = D_{T_1,T_2}(x_1,x_2) + H_{T_3}(\min(x_1,x_2)).$$

To finalize the proof, just apply Lemma 2.

Example 2 ($BEV_E - EMO$ distributions when (T_1,T_2) is not absolutely continuous). The Marshall-Olkin survival function

$$S_{X_1,X_2}(x_1,x_2) = \exp\{-x_1 - x_2 - 2\max(x_1,x_2)\}$$

is of EMO-type, being an example of extreme value survival function as well. We use relation (4) and the two step procedure from Example 1: first selecting random variables (T_1, T_2) with survival function $S_{T_1,T_2}(x_1, x_2) = \exp\{-x_1 - x_2 - H_{Y_3}(\max(x_1, x_2))\}$, where $H_{Y_3}(x) = x - \cos(x) + 1$, and second, choosing T_3 independent of (T_1, T_2) with cumulative failure rate $H_{T_3}(x) = x + \cos(x) - 1$. Notice that $S_{T_1,T_2}(x_1, x_2)$ is neither an extreme value survival function nor absolutely continuous, as well as T_3 is not exponentially distributed. In this case we have $D_{T_1,T_2}(x_1, x_2) = H_{Y_3}(\min(x_1, x_2))$, so that the functional equation given in Theorem 3 is satisfied.

3. Pickands Representation and Examples

Given the characterization established in Theorem 2, in the sequel we assume absolutely continuous distributions for (T_1, T_2) to ensure uniqueness in our construction of extreme value EMO distributions. In the next theorem we obtain the general form of the corresponding survival function and related copula. We will need to remind basic facts related to Pickands measure involved first.

Pickands (1981) proves that each multivariate extreme value distribution is uniquely characterized by a finite measure satisfying boundary conditions and related dependence function. In the bivariate case, the bivariate extreme value (survival) copulas can be completely characterized by the relation

$$\overline{C}(u, v) = \exp\left\{(\ln uv)\mathcal{A}\left(\frac{\ln u}{\ln uv}\right)\right\}, \tag{10}$$

where $\mathcal{A}(w) = \int_0^1 \max((1-x)w, x(1-w))d\mathbb{H}(x)$, for a positive finite measure $\mathbb{H}$ on $[0, 1]$, denominated Pickands measure, see Joe (1997).

The so-called Pickands dependence function $\mathcal{A}(w) : [0, 1] \to [\frac{1}{2}, 1]$ must be convex and should satisfy $\max(w, 1-w) \leq \mathcal{A}(w) \leq 1$. The lower bound of $\mathcal{A}(w)$ corresponds to the comonotonic copula and related Pickands measure puts mass 2 at $w = \frac{1}{2}$; the upper bound of $\mathcal{A}(w)$ is associated to the independence copula with Pickands measure assigning mass 1 to both $w = 0$ and $w = 1$.

The Pickands dependence function can be recovered from the copula $\overline{C}$ by setting

$$\mathcal{A}(w) = -\ln \overline{C}(\exp(-w), \exp(-(1-w))), \quad w \in [0, 1] \tag{11}$$

and is uniquely related to the measure $\mathbb{H}$ via equation

$$\mathbb{H}([0, w]) = \begin{cases} 1 + \frac{d}{dw}\mathcal{A}(w), & \text{if } w \in [0, 1), \\ 2, & \text{if } w = 1, \end{cases} \tag{12}$$

where $\frac{d}{dw}\mathcal{A}(w)$ is the right-hand derivative of $\mathcal{A}(w)$, see Theorem 3.1 in Pickands (1981) and section 8.5.3 in Beirlant et al. (2005). Moreover, the point masses of $\mathbb{H}$ at 0 and 1 are given by

$$\mathbb{H}(\{0\}) = 1 + \frac{d}{dw}\mathcal{A}(0) \quad \text{and} \quad \mathbb{H}(\{1\}) = 1 - \frac{d}{dw}\mathcal{A}(1),$$

where $\frac{d}{dw}\mathcal{A}(1) = \sup_{0 \leq w < 1} \frac{d}{dw}\mathcal{A}(w)$.

Theorem 4 *Suppose $(X_1, X_2) \in BEV_E - EMO^{AC}$. Then, the survival function of (X_1, X_2) has the form*

$$S_{X_1,X_2}(x_1, x_2) = \exp\left\{-(\lambda_1 x_1 + \lambda_2 x_2)\mathcal{A}_{T_1,T_2}\left(\frac{\lambda_1 x_1}{\lambda_1 x_1 + \lambda_2 x_2}\right) - \lambda_3 \max(x_1, x_2)\right\}, \tag{13}$$

where $\mathcal{A}_{T_1,T_2}(w)$ is the Pickands dependence function corresponding to $\overline{C}_{T_1,T_2}(u, v)$ and $\lambda_i > 0$, $i = 1, 2, 3$.

The associated survival copula writes as

$$\begin{aligned} &\overline{C}_{X_1,X_2}(u, v) = \\ &= \exp\left\{(\ln uv)\left[(\Lambda_1(w) + \Lambda_2(w))\,\mathcal{A}_{T_1,T_2}\left(\tfrac{\Lambda_1(w)}{\Lambda_1(w)+\Lambda_2(w)}\right) + \max\left(\Lambda_1(w), \Lambda_2(w)\right)\right]\right\}, \end{aligned} \tag{14}$$

where $u, v \in [0, 1)$, $w = \frac{\ln u}{\ln uv}$, $\Lambda_1(w) = \frac{\lambda_1 w}{\lambda_1+\lambda_3}$ and $\Lambda_2(w) = \frac{\lambda_2(1-w)}{\lambda_2+\lambda_3}$.

The Pickands measure $\mathbb{H}$ is given by

$$\mathbb{H}([0, w]) = \begin{cases} 1 + \left(\frac{\lambda_1}{\lambda_1+\lambda_3} - \frac{\lambda_2}{\lambda_2+\lambda_3}\right)\mathcal{A}_{T_1,T_2}\left(\frac{\Lambda_1(w)}{\Lambda_1(w)+\Lambda_2(w)}\right) & \\ \quad + \left(\frac{\lambda_1 w}{\lambda_1+\lambda_3} + \frac{\lambda_2(1-w)}{\lambda_2+\lambda_3}\right)\frac{d}{dw}\mathcal{A}_{T_1,T_2}\left(\frac{\Lambda_1(w)}{\Lambda_1(w)+\Lambda_2(w)}\right) - \frac{\lambda_3}{\lambda_2+\lambda_3}, & \text{if } 0 \leq w < \frac{\lambda_1+\lambda_3}{\lambda_1+\lambda_2+2\lambda_3}, \\ 1 + \left(\frac{\lambda_1}{\lambda_1+\lambda_3} - \frac{\lambda_2}{\lambda_2+\lambda_3}\right)\mathcal{A}_{T_1,T_2}\left(\frac{\Lambda_1(w)}{\Lambda_1(w)+\Lambda_2(w)}\right) & \\ \quad + \left(\frac{\lambda_1 w}{\lambda_1+\lambda_3} + \frac{\lambda_2(1-w)}{\lambda_2+\lambda_3}\right)\frac{d}{dw}\mathcal{A}_{T_1,T_2}\left(\frac{\Lambda_1(w)}{\Lambda_1(w)+\Lambda_2(w)}\right) + \frac{\lambda_3}{\lambda_1+\lambda_3}, & \text{if } \frac{\lambda_1+\lambda_3}{\lambda_1+\lambda_2+2\lambda_3} \leq w < 1, \\ 2 & \text{if } w = 1, \end{cases} \tag{15}$$

where $\frac{d}{dw}\mathcal{A}_{T_1,T_2}(.)$ *refers to the right-hand derivative.*

Proof. Since $(X_1, X_2) \subset BEV_E - EMO^{AC}$, according to Lemma 1 we have $(T_1, T_2) \subset BEV_E$ and $T_3 \sim Exp(\lambda_3)$. From Sklar's theorem and the expression for bivariate copulas written in terms of Pickands dependence function (11) we have

$$\begin{aligned} S_{T_1,T_2}(x_1, x_2) &= \overline{C}_{T_1,T_2}(S_{T_1}(x_1), S_{T_2}(x_2)) = \overline{C}_{T_1,T_2}(\exp\{-\lambda_1 x_1\}, \exp\{-\lambda_2 x_2\}) \\ &= \exp\left\{-(\lambda_1 x_1 + \lambda_2 x_2)\mathcal{A}_{T_1,T_2}\left(\frac{\lambda_1 x_1}{\lambda_1 x_1 + \lambda_2 x_2}\right)\right\}. \end{aligned}$$

Considering the expression of EMO survival functions (3) we obtain relation (13), i.e., the survival function of (X_1, X_2).

Let $u = S_{X_1}(x_1) = \exp\{-(\lambda_1 + \lambda_3)x_1\}$ and $v = S_{X_2}(x_2) = \exp\{-(\lambda_2 + \lambda_3)x_2\}$. From Sklar's theorem, $\overline{C}_{X_1,X_2}(u, v) = S_{X_1,X_2}(\frac{-\ln u}{\lambda_1+\lambda_3}, \frac{-\ln v}{\lambda_2+\lambda_3})$. Applying (11) we have $\mathcal{A}_{X_1,X_2}(w) = -\ln S_{X_1,X_2}\left(\frac{w}{\lambda_1+\lambda_3}, \frac{1-w}{\lambda_2+\lambda_3}\right)$. Substituting $S_{X_1,X_2}(x_1, x_2)$ by its expression given by (13) we obtain

$$\mathcal{A}_{X_1,X_2}(w) = (\Lambda_1(w) + \Lambda_2(w))\,\mathcal{A}_{T_1,T_2}\left(\frac{\Lambda_1(w)}{\Lambda_1(w) + \Lambda_2(w)}\right) + \max\left(\Lambda_1(w), \Lambda_2(w)\right). \tag{16}$$

Considering again expression (11) and relation (16) we get (14), i.e., the survival copula of the bivariate extreme value EMO distribution. By its turn, from the expression of Pickands measure in terms of the derivative of Pickands dependence function (12) and from (16) we obtain (15), the Pickands measure $\mathbb{H}$.

Whenever the derivative of $\mathcal{A}_{T_1,T_2}(w)$ is continuous from (15) we obtain

$$\mathbb{H}\left(\left\{\frac{\lambda_1 + \lambda_3}{\lambda_1 + \lambda_2 + 2\lambda_3}\right\}\right) = \frac{\lambda_3}{\lambda_1 + \lambda_3} + \frac{\lambda_3}{\lambda_2 + \lambda_3}. \tag{17}$$

We will apply relations established in Theorem 4 in the next two examples.

Example 3 (MO bivariate exponential distribution). Consider the MO distribution (2). In this case T_1 and T_2 are independent and hence $\overline{C}_{T_1,T_2}(u, v) = uv$. Therefore, $\mathcal{A}_{T_1,T_2}(w) = 1$ when $w \in [0, 1]$. Substituting $w = \frac{\ln u}{\ln uv}$ for $u, v \in [0, 1)$ in (16), the Pickands dependence function $\mathcal{A}_{X_1,X_2}(w)$ can be presented as

$$\mathcal{A}_{X_1,X_2}(u, v) = \begin{cases} 1 - \frac{\lambda_3}{\lambda_1+\lambda_3}\frac{\ln u}{\ln uv}, & \text{if} \quad 1 > u^{\frac{\lambda_3}{\lambda_1+\lambda_3}} > v^{\frac{\lambda_3}{\lambda_2+\lambda_3}} > 0, \\ \frac{\lambda_2}{\lambda_2+\lambda_3} + \frac{\lambda_3}{\lambda_2+\lambda_3}\frac{\ln u}{\ln uv}, & \text{if} \quad 0 < u^{\frac{\lambda_3}{\lambda_1+\lambda_3}} \leq v^{\frac{\lambda_3}{\lambda_2+\lambda_3}} < 1. \end{cases}$$

Applying (15) one can get the corresponding Pickands measure, confirming the representation obtained by Mai and Scherer (2010).

Example 4 (EMO extreme value distribution where the dependence structure of (T_1, T_2) is represented by Gumbel-Hougaard survival copula). Let T_i be exponentially distributed with parameter λ_i, $i = 1, 2$ and consider the Gumbel's Type III bivariate exponential survival function for (T_1, T_2) given by

$$S_{T_1,T_2}(x_1, x_2) = \exp\{-[(\lambda_1 x_1)^m + (\lambda_2 x_2)^m]^{\frac{1}{m}}\}, \quad m > 0,$$

see Gumbel (1960). The corresponding survival copula is

$$\overline{C}(u, v) = \exp\{-[(-\ln u)^m + (-\ln v)^m]^{\frac{1}{m}}\},$$

being an example of bivariate extreme value copula. Select T_3 exponentially distributed with parameter λ_3, independent of (T_1, T_2) and consider EMO survival function (3). We obtain the following bivariate extreme value EMO distribution

$$S_{X_1,X_2}(x_1, x_2) = \exp\{-[(\lambda_1 x_1)^m + (\lambda_2 x_2)^m]^{\frac{1}{m}}\}\exp\{-\lambda_3\max(x_1, x_2)\}.$$

From (11), the Pickands dependence function for (T_1, T_2) is

$$\mathcal{A}_{T_1,T_2}(w) = \left[w^m + (1 - w)^m\right]^{\frac{1}{m}},$$

and from (16) we get the Pickands dependence function

$$\mathcal{A}_{X_1,X_2}(w) = \left[\left(\frac{\lambda_1 w}{\lambda_1 + \lambda_3}\right)^m + \left(\frac{\lambda_2(1 - w)}{\lambda_2 + \lambda_3}\right)^m\right]^{\frac{1}{m}} + \max\left(\frac{\lambda_3 w}{\lambda_1 + \lambda_3}, \frac{\lambda_3(1 - w)}{\lambda_2 + \lambda_3}\right).$$

Applying relation (15), we obtain the Pickands measure

$$\mathbb{H}([0,w]) = \begin{cases} 1 - \frac{\lambda_3}{\lambda_2+\lambda_3} + \left[\left(\frac{\lambda_1 w}{\lambda_1+\lambda_3}\right)^m + \left(\frac{\lambda_2(1-w)}{\lambda_2+\lambda_3}\right)^m\right]^{\frac{1}{m}-1} & \\ \times\left[\left(\frac{\lambda_1}{\lambda_1+\lambda_3}\right)^m w^{m-1} - \left(\frac{\lambda_2}{\lambda_2+\lambda_3}\right)^m (1-w)^{m-1}\right], & \text{if } 0 \le w < \frac{\lambda_1+\lambda_3}{\lambda_1+\lambda_2+2\lambda_3}, \\ 1 + \frac{\lambda_3}{\lambda_1+\lambda_3} + \left[\left(\frac{\lambda_1 w}{\lambda_1+\lambda_3}\right)^m + \left(\frac{\lambda_2(1-w)}{\lambda_2+\lambda_3}\right)^m\right]^{\frac{1}{m}-1} & \\ \times\left[\left(\frac{\lambda_1}{\lambda_1+\lambda_3}\right)^m w^{m-1} - \left(\frac{\lambda_2}{\lambda_2+\lambda_3}\right)^m (1-w)^{m-1}\right], & \text{if } \frac{\lambda_1+\lambda_3}{\lambda_1+\lambda_2+2\lambda_3} \le w < 1, \\ 2, & \text{if } w = 1. \end{cases}$$

Finally, using (17) we get

$$\mathbb{H}\left(\left\{\frac{\lambda_1+\lambda_3}{\lambda_1+\lambda_2+2\lambda_3}\right\}\right) = \frac{\lambda_3}{\lambda_1+\lambda_3} + \frac{\lambda_3}{\lambda_2+\lambda_3}.$$

Notice that when $m = 1$, the Gumbel-Hougaard survival copula becomes the independence copula and we repeat the results of Example 3.

For $1 < m < \infty$, the remaining mass $2 - \left(\frac{\lambda_3}{\lambda_1+\lambda_3} + \frac{\lambda_3}{\lambda_2+\lambda_3}\right)$ is spread over the interval $[0, \frac{\lambda_1+\lambda_3}{\lambda_1+\lambda_2+2\lambda_3}) \bigcup (\frac{\lambda_1+\lambda_3}{\lambda_1+\lambda_2+2\lambda_3}, 1]$ and, in particular, $\mathbb{H}(\{0\}) = \mathbb{H}(\{1\}) = 0$.

4. Conclusions

In this note, we found representation of extreme value EMO distributions with exponential marginals. We characterize it when the vector (T_1, T_2) in (3) is absolutely continuous or not, see Theorem 2 and Theorem 3, respectively. The corresponding Pickands representation is obtained in Theorem 4, having as a particular case earlier conclusions of Mai and Scherer (2010) regarding the MO's bivariate exponential distribution (2).

We would like to call attention that Theorem 2 shows the way to generate extreme value EMO distributions with exponential marginals when the vector (T_1, T_2) in (3) is absolutely continuous, see the Example 2. If (T_1, T_2) in (3) has a singular component, one may follow a procedure given after Theorem 2 by using GMO models as a first step. The only restriction in the choice of the failure rate of T_3 in the second step is that $r_{T_i}(x_i) + r_{T_3}(x_i) = \lambda_{X_i}$, $i = 1, 2$, being the constant failure rates of the marginals of extreme value EMO distribution of (X_1, X_2). Another general option is to select dependence function $H_{T_1,T_2}(x_1, x_2)$ and cumulative hazard $H_{T_3}(x)$, satisfying the functional equation in Theorem 3, since it is valid even when the joint distribution of (T_1, T_2) is not absolutely continuous and T_i' in (3) do not need to be exponentially distributed.

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