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Extreme Value Theory: a New Characterization of the Distribution Function for the Mixed Method

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Abstract

Consider the sample $X_1, X_2, \dots, X_N$ of N independent and identically distributed (iid) random variables with common cumulative distribution function (cdf) F, and let F_u be their conditional excess distribution function. We define the ordered sample by $X_1 \leq X_2 \leq \cdots \leq X_N$. Pickands (1975), Balkema and de Haan (1974) posed that for a large class of underlying distribution functions F, and large u, F_u is well approximated by the Generalized Pareto Distribution. The mixed method is a method for determining thresholds. This method consists in minimizing the variance of a convex combination of other thresholds.

The objective of the mixed method is to determine by which probability distribution one can approach this conditional distribution. In this article, we propose a theorem which specifies the conditional distribution of excesses when the deterministic threshold tends to the end point.

Keywords: distribution function, Generalized Pareto Distribution (GPD), Mixed Method (MM).

2000 Mathematics Subject Classifications: 60G52, 60G70, 62G20, 62G32, 60E07, 62E20

1. Introduction

Pareto distribution is traditionally used by reinsurer's excess of loss mainly because of its good mathematical properties, particularly from the simplicity of the formulas resulting from its application. The new mixed method (MM) was proposed in [1, 2, 3, 4] to determine a threshold $U = \sum_{k=1}^{p} \alpha_k U_k + \alpha_3 U_3$ with $1 \leq p \leq 2$, at which a unit is declared atypical minimizing the variance of a convex combination of thresholds obtained by the mean excess function and generalized Pareto distribution (extreme quantile were estimated with a probability of 99.9% being an extreme value for the distribution of amounts of sinister with a confidence level of 95%). This method allows a compromi between the GPD method and FME method, between a minimum strategy GPD and maximum strategy FME (Mean Excess Function). It is more correlated with the GPD method and relatively smooth.

2. Method

This article focuses on two major paragraphs. The first paragraph (see paragraph 3.1) is based on determining a threshold $U = \sum_{k=1}^{p} \alpha_k U_k + \alpha_3 U_3$ with $1 \leq p \leq 2$ by the mixed method (MM) and last paragraph (see paragraph 3.2) is to determine a distribution function of the laws of the mixed method. Let U_3: the threshold beyond which a unit is declared as extreme, obtained by the GPD function and U: the threshold beyond which a unit is declared as extreme, obtained by the mixed method (MM).Let $X_1, X_2, \dots, X_N$ N random variables (iid) common distribution function F. We are looking from the distribution F of X to define a conditional distribution F_{U_3} compared to U_3 threshold for random variables exceeding this threshold. It defines the excess over the threshold U_3 as the set of random variables y_j defined by: $y_j = X_j - U_3$ for $j \in E(U\_3) = \{j \in \{1,2, \dots, N\} / Xj > U3\}$. The function of distribution of the excess over the threshold U_3 is defined by:

$$F_{U_3}(y) = P(X - U_3 \leq y / X > U_3) = \begin{cases} \dfrac{F(U_3 + y) - F(U_3)}{1 - F(U_3)} & if \quad y \geq 0, \\ 0 & if \quad y < 0, \end{cases}$$

Thus, for large threshold U_3, the law of excess is approximated by a generalized Pareto law:

$$F_{U_3}(y) \approx F^{GPD}_{\xi,\sigma(U_3)}(y).$$

In this article, we will show that:

$$F_U(y) \approx F^{MM}_{\xi,\sigma(\mathrm{U})}(y),$$

where $F^{MM}_{\xi,\sigma(\mathrm{U})}(y)$ is the distribution function of the law of the mixed method and U is the threshold beyond which a unit is declared as extreme, obtained by the mixed method (MM).

Theorem Pickands (1975), Balkema and de Haan (1974) assures us that the law of the excess may be approaching a generalized Pareto law. In this article, we will use the theorem Pickands (1975), Balkema and de Haan (1974) to show that the law of the excess can be approached by a law of the mixed method.

3. Results

3.1. Determination of Threshold U By the Mixed Method (MM)

The new mixed method (MM) was proposed in [1, 2, 3, 4] to determine a threshold $U = \sum_{k=1}^{p} \alpha_k U_k + \alpha_3 U_3$ with $1 \leq p \leq 2$, at which a unit is declared atypical minimizing the variance of a convex combination of thresholds obtained by the mean excess function and generalized Pareto distribution (extreme quantile were estimated with a probability of 99.9% being an extreme value for the distribution of amounts of sinister with a confidence level of 95%).

Let U_1 be the threshold beyond which a unit is declared as extreme, obtained by the record values, U_2 be the threshold beyond which a unit is declared as extreme, obtained by the mean excess function and U_3 the threshold beyond which a unit is declared as extreme, obtained by the GPD function with $U_1 < U_2 < U_3$. Let $U = \alpha U_p + (1 - \alpha) U_q$ with $0 < \alpha < 1$, minimizes the variance U, p, $q = 1, 2, 3$ and $p < q$. We get.

$$\alpha = \frac{V(X_{U_q}) - Cov(X_{U_p}, X_{U_q})}{V(X_{U_p}) + V(X_{U_q}) - 2Cov(X_{U_p}, X_{U_q})}$$

For $U = \alpha U_1 + (1 + \alpha) U_2 - 2\alpha U_3$ with $\alpha \in \mathbb{R}$. We get :

$$\alpha = \frac{-V(X_{U_2}) - Cov(X_{U_1}, X_{U_2}) + 2Cov(X_{U_2}, X_{U_3})}{V(X_{U_1}) + V(X_{U_2}) + 4V(X_{U_3}) + 2Cov(X_{U_1}, X_{U_2}) - 4Cov(X_{U_1}, X_{U_3}) - 4Cov(X_{U_2}, X_{U_3})}$$

Consider the sample $X_1, X_2, \ldots, X_N$ of N independent and identically distributed (iid) random variables. We define the ordered sample by $X_1 \leq X_2 \leq \cdots \leq X_N$. Let $X_{U_j}, j = 1,2,3$ thresholds obtained by different methods. We consider a statistical series to a variable X_{U_j} ,taking the amount $X_1, X_2, \ldots, X_N$ and X_{U_j} ,which have been sorted in ascending order: $X_1 \leq X_2 \leq \cdots \leq X_k \leq U_j \leq \cdots \leq X_N$. We consider a statistical series 2 variables X and Y, taking the amount $X_1, X_2, \ldots, X_N$ and $Y_1, Y_2, \ldots, Y_N$. Which have been sorted in ascending order: $X_1 \leq X_2 \leq \cdots \leq X_N$ and $Y_1 \leq Y_2 \leq \cdots \leq Y_N$. We write:

- The means of X and Y are : $\bar{X} = \dfrac{1}{N} \sum_{i=1}^{N} X_i$ et $\bar{Y} = \dfrac{1}{N} \sum_{i=1}^{N} Y_i$

- The variances of X and Y are: $V(X)=\frac{1}{N}\sum_{i=1}^{N}(X_i-\bar{X})^2$ et $V(Y)=\frac{1}{N}\sum_{i=1}^{N}(Y_i-\bar{Y})^2$

- The covariance of X and Y is: $Cov(X,Y)=\frac{1}{N}\sum_{i=1}^{N}(X_i-\bar{X})(Y_i-\bar{Y})$

Example 1. Threshold Calculation

The data base provides a sample of 2020 observations for 4 wheel vehicle for personal use during the year 2013. The data come from a Malian insurance company and concern the amounts of claims caused by the insured of a risk class. This file contains only the amounts of claims during the insurance year. U_1 be the threshold beyond which a unit is declared as extreme, obtained by the record values. U_2 be the threshold beyond which a unit is declared as extreme, obtained by the mean excess function. U_3 the threshold beyond which a unit is declared as extreme, obtained by the GPD function and U the threshold beyond which a unit is declared as extreme, obtained by the method MM.

Let N be the number of claims and $X_1, X_2, \dots, X_N$ the realizations of X, which is the random variable representing the amounts of loss. As usual we assume mutual independence of random variables.

Table 1. Determination of threshold U by the mixed method (MM)

Record values	Mean excess function	GPD function	MM method
U_1	U_2	U_3	$U=\sum_{k=1}^{p}\alpha_k U_k+\alpha_3 U_3$ with $1\le p\le 2$
$U_1=11{,}5$		$U_3=12{,}5$	$U=\alpha U_1+(1-\alpha)U_3=11{,}88$ with $\alpha=0{,}69$
	$U_2=12$	$U_3=12{,}5$	$U=\alpha U_2+(1-\alpha)U_3=12{,}19$ with $\alpha=0{,}63$
$U_1=11{,}5$	$U_2=12$	$U_3=12{,}5$	$U=\alpha U_1+(1+\alpha)U_2-2\alpha U_3=12{,}10$ with $\alpha=-0{,}06$

3.2 Law (distribution) of The Mixed Method

In this section, we will give the main result of this paper is to write a new law of the mixed method (MM). Let U_3*:* the threshold beyond which a unit is declared as extreme, obtained by the GPD function and U: the threshold beyond which a unit is declared as extreme, obtained by the mixed method (MM). Let $X_1, X_2, \dots, X_N$ N random variables (iid) common distribution function F. We are looking from the distribution F of X to define a conditional distribution F_{U_3} compared to U_3 threshold for random variables exceeding this threshold. It defines the excess over the threshold U_3 as the set of random variables y_j defined by: $y_j=X_j-U_3$ for $j\in E_{U_3}=\{j\in\{1,2,\dots,N\}/X_j>U_3\}$. It defines the excess over the threshold U_3 as the set of random variables y_j defined by:

$$F_{U_3}(y) = P(X - U_3 \le y / X > U_3) = \begin{cases} \dfrac{F(U_3 + y) - F(U_3)}{1 - F(U_3)} & if \quad y \ge 0, \\ 0 & if \quad y < 0, \end{cases}$$

The objective of the mixed method is to determine by which probability distribution one can approach this conditional distribution. In this article, we propose the following theorem (**Theorem 2**) which specifies the conditional distribution of excesses when the deterministic threshold tends to the end point X_F.

Theorem 1 (Pickands (1975), Balkema and de Haan (1974)): Let F_{U_3} be the conditional distribution of the excess over a threshold U_3, combined with unknown distribution function F. This function F belongs to the domain of attraction of G_ξ if and only if there exist a positive function σ such

$$\lim_{U_3 \to X_F} \operatorname{Sup}_{0 < y < X_{F - U_3}} \left| F_{U_3}(y) - F^{GPD}_{\xi,\sigma(U_3)}(y) \right| = 0,$$

Where $F^{GPD}_{\xi,\sigma(U_3)}$ is the distribution function of GPD, define by:

$$F^{GPD}_{\xi,\sigma(U_3)}\,(\mathrm{y}) = \begin{cases} 1 - \left(1 + \dfrac{y\xi}{\sigma}\right)^{\frac{-1}{\xi}} & if\ \xi \ne 0 \\ 1 - e^{\frac{-y}{\sigma}} & if\ \xi = 0 \end{cases}$$

for $y \in [0, (X_F - U_3)]$ if $\xi \ge 0$ and $y \in \left[0, Min\left(\frac{-\xi}{\rho}, X_F - U_3\right)\right]$ if $\xi < 0$ with $X_F = sup\{X \in \mathbb{R}, F(X) < 1\}$

Theorem 2: Let F_U be the conditional distribution of the excess over a threshold $U = \sum_{k=1}^{p} \alpha_k\, U_k + \alpha_3 U_3$ with $1 \le p \le 2$, combined with unknown distribution function F. This function F belongs to the domain of attraction of G_ξ if and only if there exists a positive function σ such

$$\lim_{U \to X_F} \operatorname{Sup}_{0 < y < X_{F - U}} \left| F_U(y) - F^{MM}_{\xi,\sigma(\mathrm{U})}(y) \right| = 0,$$

Where $F^{MM}_{\xi,\sigma(\mathrm{U})}$ is the distribution function of mixed method (MM), define by:

$$F^{MM}_{\xi,\sigma(\mathrm{U})}\,(\mathrm{y}) = \begin{cases} 1 - \left(1 + \dfrac{y\xi}{\sigma}\right)^{\frac{-1}{\xi}} & if\ \xi \ne 0 \\ 1 - e^{\frac{-y}{\sigma}} & if\ \xi = 0 \end{cases}$$

for $y \in [0, (X_F - U)]$ if $\xi \ge 0$ and $y \in \left[0, Min\left(\frac{-\xi}{\rho}, X_F - U\right)\right]$ if $\xi < 0$ with $X_F = sup\{X \in \mathbb{R}, F(X) < 1\}$.

Proof: The conditional distribution F_U of the excesses above the threshold U with is defined by:

$F_U(y) = P(X - U \le y / X > U) = \frac{F(U+y) - F(U)}{1 - F(U)}$ for $0 \le y \le X_F - U$.

This is equivalent to:

$F_U(x) = P(X < x / X > U) = \frac{F(x) - F(U)}{1 - F(U)}$ for $x \ge U$.

The proof of $F_U(y) \approx F^{MM}_{\xi,\sigma(\mathrm{U})}(y)$ results directly from the evidence of $F_{U_3}(y) \approx F^{GPD}_{\xi,\sigma(U_3)}(y)$ (theorem 1) and $U = \sum_{k=1}^{p} \alpha_k\, U_k + \alpha_3 U_3$ with $1 \le p \le 2$.

Example 2: Winting $\mathrm{F}^{\mathrm{MM}}_{\xi,\sigma(\mathrm{U})}$ (y).

U_1: be the threshold beyond which a unit is declared as extreme, obtained by the record values, *U_2*: be the threshold beyond which a unit is declared as extreme, obtained by the mean excess function, *U_3*: the threshold beyond which a unit

is declared as extreme, obtained by the GPD function and U: the threshold beyond which a unit is declared as extreme, obtained by the MM function. Let N be the number of claims and $X_1, X_2, \dots, X_N$ the realizations of X, which is the random variable representing the amounts of sinister.

Table 2. Knowing the parameters (σ, ξ) and the thresholds, we can write the distribution functions of GPD and MM.

Parameters GPD $\xi,\ \sigma$	Threshold GPD U_3	Threshold MM U	distribution function of GPD $F^{GPD}_{\xi,\sigma(U_3)}(y) = 1 - \left(1 + \frac{y\xi}{\sigma}\right)^{\frac{-1}{\xi}}$ With $y = X - U_3$	distribution function of MM $F^{MM}_{\xi,\sigma(U)}(y) = 1 - \left(1 + \frac{y\xi}{\sigma}\right)^{\frac{-1}{\xi}}$ With $y = X - U$
$\xi = -0{,}3293$ $\sigma = 1{,}5576$	$U_3 = 12{,}5$	$U = 11{,}88$	$1 - \left(1 - \frac{0{,}3293y}{1{,}5576}\right)^{\frac{1}{0{,}3293}}$ $y = X - 12{,}5$	$1 - \left(1 - \frac{0{,}3293y}{1{,}5576}\right)^{\frac{1}{0{,}3293}}$ $y = X - 11{,}88$
$\xi = -0{,}3293$ $\sigma = 1{,}5576$	$U_3 = 12{,}5$	$U = 12{,}19$	$1 - \left(1 - \frac{0{,}3293y}{1{,}5576}\right)^{\frac{1}{0{,}3293}}$ $y = X - 12{,}5$	$1 - \left(1 - \frac{0{,}3293y}{1{,}5576}\right)^{\frac{1}{0{,}3293}}$ $y = X - 12{,}19$
$\xi = -0{,}3293$ $\sigma = 1{,}5576$	$U_3 = 12{,}5$	$U = 12{,}10$	$1 - \left(1 - \frac{0{,}3293y}{1{,}5576}\right)^{\frac{1}{0{,}3293}}$ $y = X - 12{,}5$	$1 - \left(1 - \frac{0{,}3293y}{1{,}5576}\right)^{\frac{1}{0{,}3293}}$ $y = X - 12{,}10$

Example 3: Threshold Calculation By the Graphical Method.

Knowledge of parameters (σ, ξ) allows to determine graphically the threshold U_3 by the GPD method and U by MM method (mixed method). To do this, we will write a program on the MAPLE software to determine these thresholds.

Distribution Function $F^{MM}(\xi, \sigma, \mathrm{u})$:

$$Distribution := \mathbf{proc}(\zeta, \sigma, u)\ \mathbf{local}\ i, j; plot\left(1 - \left(1 + \frac{\zeta.(U-u)}{\sigma}\right)^{\frac{-1}{\zeta}}, U = 0..20, thickness = 2, color = black\right) \mathbf{end};$$

$$Density := \mathbf{proc}(\zeta, \sigma, u)\ \mathbf{local}\ i, j; plot\left(\frac{1}{\sigma} \cdot \left(1 + \frac{\zeta.(x-u)}{\sigma}\right)^{\frac{-1-\zeta}{\zeta}}, = 0..20, thickness = 2, color = black\right) \mathbf{end};$$

Table 3. Knowing the parameters (σ, ξ) and the thresholds, we can graphically read the thresholds

<table>
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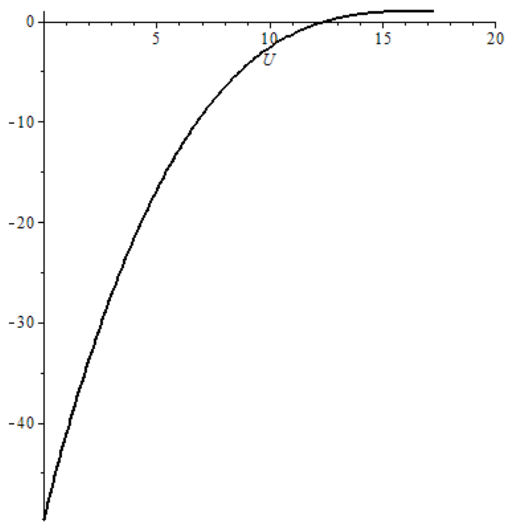
GPD method:

$\xi = -0{,}3293,\ \sigma = 1{,}5576$ with $U_3 = 12{,}5$

$F^{GPD}_{\xi,\sigma(U_3)}(y) = 1 - \left(1 - \frac{0{,}3293y}{1{,}5576}\right)^{\frac{1}{0{,}3293}}$



The threshold can be read graphically $U_3 = 12{,}5$

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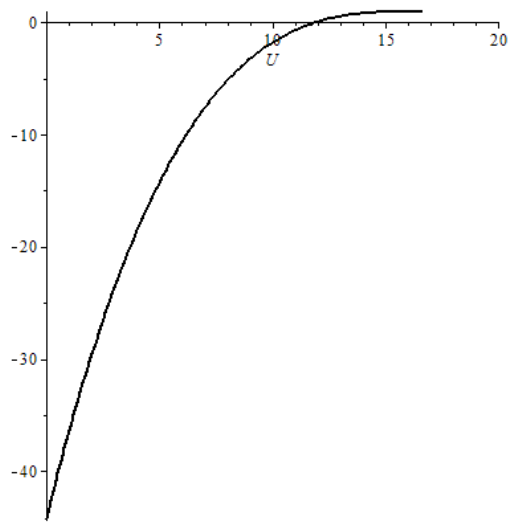
MM method:

$\xi = -0{,}3293,\ \sigma = 1{,}5576$ with $U = 11{,}88$

$F^{MM}_{\xi,\sigma(U)}(y) = 1 - \left(1 - \frac{0{,}3293y}{1{,}5576}\right)^{\frac{1}{0{,}3293}}$



The threshold can be read graphically $U = 11{,}88$

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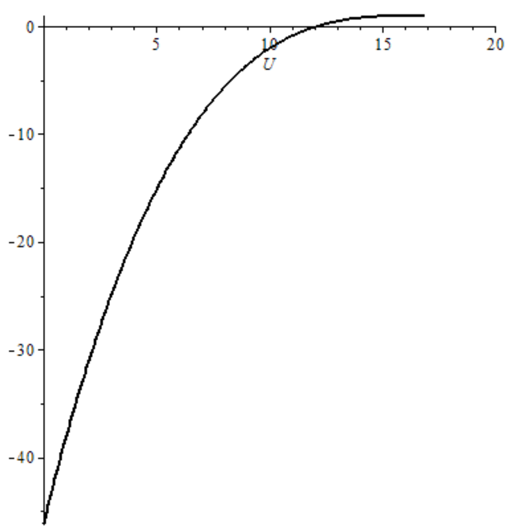
MM method:

$\xi = -0{,}3293,\ \sigma = 1{,}5576$ with $U = 12{,}10$

$F^{MM}_{\xi,\sigma(U)}(y) = 1 - \left(1 - \frac{0{,}3293y}{1{,}5576}\right)^{\frac{1}{0{,}3293}}$



The threshold can be read graphically $U = 12{,}10$

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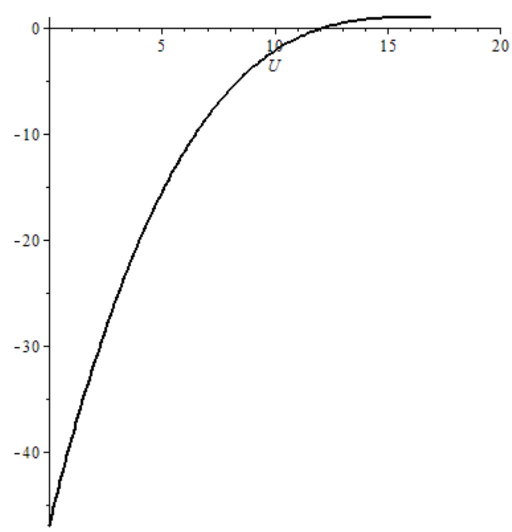
MM method:

$\xi = -0{,}3293,\ \sigma = 1{,}5576$ with $U = 12{,}19$

$F^{MM}_{\xi,\sigma(U)}(y) = 1 - \left(1 - \frac{0{,}3293y}{1{,}5576}\right)^{\frac{1}{0{,}3293}}$



The threshold can be read graphically $U = 12{,}19$

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</tr>
</table>

4. Conclusions

In the literature, various methods have been proposed to estimate the parameters (σ, ξ) of the GDP law: the method of maximum likelihood, method of moments, the Bayesian method, the estimator Pickands (1975) and the Hill estimator (1975). Note that these last two estimators can only be used for the index of the tail of the distribution ξ. Knowledge of parameters (σ, ξ) by the first two methods allows to determine graphically the threshold U_3 by the GPD method and U by MM method (mixed method). Therefore, we must carefully determine the parameters by different methods.

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