International Journal of Statistics and Probability; Vol. 6, No. 1; January 2017
ISSN 1927-7032 E-ISSN 1927-7040
Published by Canadian Center of Science and Education

Limit Distribution of a Generalized Ornstein – Uhlenbeck Process

Andriy Yurachkivsky[1]

[1] Taras Shevchenko National University, Kyiv, Ukraine

Correspondence: Correspondence: Andriy Yurachkivsky, Taras Shevchenko National University, 64 Volodymyrska St., Kyiv, 01601, Ukraine. E-mail: andriy.yurachkivsky@gmail.com

Received: October 11, 2016 Accepted: November 4, 2016 Online Published: December 13, 2016

doi:10.5539/ijsp.v6n1p24 URL: https://doi.org/10.5539/ijsp.v6n1p24

Abstract

Let an $\mathbb{R}^d$-valued random process ξ be the solution of an equation of the kind $\xi(t) = \xi(0) + \int_0^t A(u)\xi(u)\mathrm{d}\iota(u) + S(t)$, where $\xi(0)$ is a random variable measurable w. r. t. some σ-algebra $\mathcal{F}(0)$, S is a random process with $\mathcal{F}(0)$-conditionally independent increments, ι is a continuous numeral random process of locally bounded variation, and A is a matrix-valued random process such that for any $t > 0$ $\int_0^t \|A(s)\|\,|\mathrm{d}\iota(s)| < \infty$. Conditions guaranteing existence of the limiting, as $t \to \infty$, distribution of $\xi(t)$ are found. The characteristic function of this distribution is written explicitly.

Keywords: limit distribution, generalized Ornstein – Uhlenbeck process

Mathematics Subject Classification: 60F05, 60H10

1. Introduction

The random processes under consideration are assumed given on a common probability space $(\Omega, \mathcal{F}, \mathsf{P})$. It is assumed that a a filtration $\mathbb{F} = (\mathcal{F}(t),\ t \in \mathbb{R}_+)$ on $\mathcal{F}$ is given. We consider, without loss generality, that it is right-continuous and each $\mathcal{F}(t)$ contains all P-negligible sets from $\mathcal{F}$ (these are so called usual conditions – see (Gikhman & Skorokhod, 1982; Jacod & Shiryaev, 1987; Liptser & Shiryaev, 1989)). We will consider also the trivial filtration $\mathbb{F}^0 = (\mathcal{F}^0(t),\ t \in \mathbb{R}_+)$, where $\mathcal{F}^0(t) = \mathcal{F}(0)$ for all t. Thus a random process is $\mathbb{F}^0$-adapted iff its value at any nonrandom time is an $\mathcal{F}(0)$-measurable random variable. We introduce the notation: $\mathsf{E}^0 = \mathsf{E}(\cdots|\mathcal{F}(0))$; $\mathcal{V}_0^c$ is the class of all starting from zero $\mathbb{F}^0$-adapted continuous random processes; $\mathfrak{S}_+$ is the class of all nonnegative (in the spectral sense) symmetric $d \times d$ matrices with real entries; $\mathbb{M}_+(\mathfrak{C})$ is the class of all σ-finite measures on a σ-algebra $\mathfrak{C}$; l.i.p. signifies the limit in probability. In integrals with a discontinuous integrator, $\int_a^b$ means $\int_{]a,b]}$. The indicator of a set $\{\cdots\}$ is denoted by $I\{\cdots\}$.

All vectors are thought of, unless otherwise stated, as columns; all matrices are meant of size $d \times d$, with real entries. The unit matrix is denoted by $\mathbf{I}$, and the space of all d-dimensional row vectors with real components by $\mathbb{R}^{d*}$. We use the Euclidean norm $|\cdot|$ of vectors and the operator norm $\|\cdot\|$ of matrices. For symmetric matrices A and B, the inequality $A \le B$ means that $B - A \in \mathfrak{S}_+$ (so that one may speak about increasing $\mathfrak{S}_+$-valued functions).

The quadratic characteristic of a locally square-integrable $\mathbb{R}^d$-valued martingale Z will be denoted by $\langle Z \rangle$. This is an increasing $\mathfrak{S}_+$-valued random process.

The words "almost surely" are tacitly implied in relations between random variables, including the convergence relation unless it is explicitly written as the convergence in probability (denoted by $\xrightarrow{\mathsf{P}}$) or in distribution (denoted by $\xrightarrow{\mathrm{d}}$).

The reference books for the notions and results of stochastic analysis used in this paper are (Gikhman & Skorokhod, 1982, 2009; Jacod & Shiryaev, 1987; Liptser & Shiryaev, 1989). A number of more specific definitions and statements relevant to the topic can be found in (Yurachkivsky, 2013).

The goal of the article is to find the limit, as $t \to \infty$ (which will be tacitly meant in all asymptotic relations), of the one-dimensional distribution of the solution of the stochastic equation

$$\xi(t) = \xi(0) + \int_0^t A(u)\xi(u)\mathrm{d}\iota(u) + S(t), \tag{1}$$

where S is a random process with $\mathcal{F}(0)$-conditionally independent increments, $\iota \in \mathcal{V}_0^c$, and A is a matrix-valued $\mathbb{F}^0$-adapted random process such that for any $t > 0$

$$\int_0^t \|A(s)\|\,|\mathrm{d}\iota(s)| < \infty. \tag{2}$$

We call thus defined ξ a *generalized Ornstein – Uhlenbeck process*. Recall that a classical Ornstein – Uhlenbeck process is the solution of (1) with $d = 1, U = 0, \iota(t) = t$, constant A and a homogeneous Wiener process as S. For them, the problem is easy and solved long ago (the limit distribution exists iff $A < 0$). But even in a seemingly simple case when S is a homogeneous generalized (i. e., with random jumps) Poisson process the problem is nontrivial. It was solved first by Zakusilo in 1981 (this result is contained in (Anisimov, Zakusilo & Donchenko, 1987)). A more general theorem, but also only for the case of homogeneous S and $\iota(t) = t$, was proved in (Sato & Yamazato, 1984). Unlike these authors, we focus on the described above model. This level of generality requires quite different technique that will be demonstrated below. The only known to the author work with $\iota(t)$ possibly other than t and nonhomogeneous S is (Ivanenko, 2009). But the proof therein relies on very specific assumptions (for example, $A = -c\mathbf{I}$) and does not carry over to a more general situation.

The structure of the article is clear from the titles of its sections: General preliminaries, Special preliminaries and The main result.

2. General Preliminaries

The following statement is immediate from Lebesgue's dominated convergence theorem and Lévy's continuity theorem.

Proposition 2.1. *Let ξ be an $\mathbb{R}^d$-valued random process. Suppose that there exists a $\mathbb{C}$-valued random function J on $\mathbb{R}^{d*}$ such that: $\operatorname*{l.i.p.}_{z\to 0} J(z) = 0$; for any $z \in \mathbb{R}^{d*}$ $\mathsf{E}^0 e^{iz\xi(t)} \xrightarrow{\mathrm{d}} e^{J(z)}$. Then there exists a random variable ξ_∞ such that $\mathsf{E}e^{iz\xi_\infty} = e^{J(z)}$ and $\xi(t) \xrightarrow{\mathrm{d}} \xi_\infty$.*

Let us consider the equation

$$K(t, s) = \mathbf{I} + \int_s^t A(\tau)K(\tau, s)\mathrm{d}\iota(\tau). \tag{3}$$

The standard convention $\int_s^t = -\int_t^s$ entitles us to consider that the variables s and t independently range over $\mathbb{R}_+$. The integral on the r. h. s. of (22) being pathwise, the variable $\omega \in \Omega$ performs in it as a parameter. Thus we may consider this equation deterministic. Its solution will be an important tool in our study.

Lemma 2.2 (Yurachkivsky, 2013; Lemma 3.10)**.** *Let ι be a numeral function of locally bounded variation, and A be a Borel matrix-valued function satisfying condition* (2). *Then the solution of equation* (3) *exists, is unique and has locally bounded variation in each argument.*

In what follows, $\mathfrak{C}$ is a σ-algebra of subsets of some set Θ.

We denote by $\mathcal{K}$ the class of $\mathbb{F}$-adapted $\mathbb{R}^d$-valued random processes M such that, firstly, $\mathsf{E}^0|M(t)|^2 < \infty$ for all $t \in \mathbb{R}_+$, secondly, $\mathsf{E}(M(t) - M(s)|\mathcal{F}(s)) = M(s)$ for all $t > s \geq 0$ and, thirdly, the process $\mathsf{E}^0|M|^2$ is continuous (here E^0 is the extended conditional expectation, so we need not assume that $\mathsf{E}|M(t)| < \infty$). This class is contained in the class of locally square integrable martingales (see, e. g., (Yurachkivsky, 2013, 2014)).

From this time on we deal with the following particular case of (1):

$$\xi(t) = \xi(0) + \int_0^t A(s)\xi(s)\mathrm{d}\iota(s) + U(t) + \int_0^t \int_\Theta f(s,\theta)\nu(\mathrm{d}s\,\mathrm{d}\theta) + \int_0^t \int_\Theta g(s,\theta)\,(\upsilon - \widetilde{\upsilon})\,(\mathrm{d}s\,\mathrm{d}\theta) + W(t) \tag{4}$$

which may be written shortly in the notation of stochastic analysis (see (Gikhman & Skorokhod, 1982; Jacod & Shiryaev, 1987; Liptser & Shiryaev, 1989)) as

$$\xi = \xi(0) + (A\xi) \circ \iota + U + f * \nu + g * (\upsilon - \widetilde{\upsilon}) + W.$$

To make it quite definite we impose the assumptions:

1. $\xi(0)$ is an $\mathbb{R}^d$-valued $\mathcal{F}(0)$-measurable random variable.

2. ι is an $\mathbb{R}$-valued random process of class $\mathcal{V}_0^c$.

3. A is a matrix-valued $\mathcal{F}(0) \otimes \mathcal{B}_+$-measurable in (ω, t) random process satisfying, for all t, condition (2).

4. U is an $\mathbb{R}^d$-valued random process of class $\mathcal{V}_0^c$.

5. ν and υ are $\mathbb{F}$-adapted quasicontinuous integer-valued random measures on $\mathcal{B}_+ \otimes \mathfrak{C}$ with $\mathbb{F}^0$-adapted compensators $\widetilde{\nu}$ and $\widetilde{\upsilon}$, respectively.

6. For any random $\mathcal{F}(0) \otimes \mathcal{B}_+ \otimes \mathfrak{C}$-measurable functions q_1 and q_2 on $\mathbb{R}_+ \times \Theta$ such that $|q_1| * \widetilde{\nu}(t) + |q_2|^2 * \widetilde{\upsilon}(t) < \infty$ for all t, the processes $q_1 * \nu$ and $q_2 * (\upsilon - \widetilde{\upsilon})$ have no simultaneous jumps.

7. W is a starting from zero $\mathbb{R}^d$-valued continuous random process of class $\mathcal{K}$ with $\mathbb{F}^0$-adapted quadratic characteristic.

8. f and g are $\mathbb{R}^d$-valued $\mathcal{F}(0) \otimes \mathcal{B}_+ \otimes \mathfrak{C}$-measurable random functions on $\mathbb{R}_+ \times \Theta$.

9. $(|g|^2 \wedge |g|) * \widetilde{\upsilon}(t) < \infty$ for all t.

10. $(|f| \wedge 1) * \widetilde{\nu}(t) < \infty$ for all t.

11. $|f| * \nu(t) < \infty$ for all t.

Theorem 2.3 (Yurachkivsky, 2013; Corollary 6.2)**.** *Let assumptions **1 – 11** be satisfied. Then for any $t \in \mathbb{R}_+$ and $z \in \mathbb{R}^{d*}$*

$$\mathsf{E}^0 e^{iz\xi(t)} = e^{izK(t,0)\xi(0)+izQ_t-zR_tz^\top/2+F_t(z)+G_t(z)},$$

where

$$Q_t = \int_0^t K(t,s)\mathrm{d}U(s), \tag{5}$$

$$R_t = \int_0^t K(t,s)\mathrm{d}\langle W\rangle(s)K(t,s)^\top, \tag{6}$$

$$F_t(z) = \int_0^t \int_\Theta \left(e^{izK(t,s)f(s,\theta)} - 1\right)\widetilde{\nu}(\mathrm{d}s\,\mathrm{d}\theta), \tag{7}$$

$$G_t(z) = \int_0^t \int_\Theta \left(e^{izK(t,s)g(s,\theta)} - 1 - izK(t,s)g(s,\theta)\right)\widetilde{\upsilon}(\mathrm{d}s\,\mathrm{d}\theta). \tag{8}$$

Corollary 2.4. *Let assumptions **1 – 11** be satisfied, and $Q_t, R_t, F_t(z)$ and $G_t(z)$ be defined by* (5) – (8)*. Suppose also that*

$$K(t,0) \xrightarrow{\mathsf{P}} 0 \tag{9}$$

and there exist an $\mathbb{R}^d$-valued random variable ϑ, a matrix-valued random variable Υ and a $\mathbb{C}$-valued random function Φ and Γ on $\mathbb{R}^d$, all given on a common probability space and such that

$$\operatorname*{l.i.p.}_{z\to 0}\Phi(z) = 0, \tag{10}$$

$$\operatorname*{l.i.p.}_{z\to 0}\Gamma(z) = 0 \tag{11}$$

and for all $z \in \mathbb{R}^{d*}$

$$(Q_t, R_t, F_t(z), G_t(z)) \xrightarrow{\mathrm{d}} (\vartheta, \Upsilon, \Phi(z), \Gamma(z)). \tag{12}$$

Then the distribution of $\xi(t)$ weakly converges, as $t \to \infty$, to the distribution with characteristic function $\mathsf{E}e^{iz\vartheta-z\Upsilon z^\top/2+\Phi(z)+\Gamma(z)}$.

Lemma 2.5. *Let μ be a finite measure on some σ-algebra $\mathcal{X}$ of subsets of a set X, and let for each $t > 0$ ζ_t be a nonnegative measurable random function on X. Suppose the following: for any $x \in X$*

$$\zeta_t(x) \xrightarrow{\mathsf{P}} 0; \tag{13}$$

there exists a measurable random function Z on X such that

$$\int_X Z\mathrm{d}\mu < \infty \tag{14}$$

and for all $t > 0$ and $x \in X$

$$\zeta_t(x) \le Z(x). \tag{15}$$

Then

$$\int_X \zeta_t\mathrm{d}\mu \xrightarrow{\mathsf{P}} 0. \tag{16}$$

Proof. In this and the subsequent proofs, $\int$ stands for $\int_X$.

Denote $Z^L = ZI\{Z > L\}$. The relations

$$\zeta_t = \zeta_t I\{\zeta_t > L\} + \zeta_t I\{\zeta_t \le L\} \le Z^L + \zeta_t \wedge L,$$

of which the inequality follows from (15), imply that for any positive ε and L

$$\mathsf{P}\left\{\int \zeta_t \mathrm{d}\mu > 2\varepsilon\right\} \le \mathsf{P}\left\{\int Z^L \mathrm{d}\mu > \varepsilon\right\} + \mathsf{P}\left\{\int \zeta_t(u) \wedge L \mathrm{d}\mu > \varepsilon\right\}. \tag{17}$$

By construction $Z^L \to 0$ as $L \to$ for all $x \in X$ and $\omega \in \Omega$. Hence and from (14) we get by the dominated convergence theorem (DCT) applied at those $\omega \in \Omega$ where (14) holds

$$\lim_{L\to\infty} \int Z^L \mathrm{d}\mu = 0. \tag{18}$$

Condition (13) implies by the DCT that $\mathsf{E}(\zeta_t(x) \wedge L) \to 0$ for all $L > 0$ and $x \in X$. Hence by the DCT and due to finiteness of μ

$$\int \mathsf{E}(\zeta_t \wedge L)\mathrm{d}\mu \to 0.$$

Herein

$$\int \mathsf{E}(\zeta_t \wedge L)\mathrm{d}\mu = \mathsf{E}\int \zeta_t \wedge L\mathrm{d}\mu$$

by the Fubini – Tonelli theorem. This together with the preceding relation and (17) implies that for any positive ε and L

$$\varlimsup_{t\to\infty} \mathsf{P}\left\{\int \zeta_t \mathrm{d}\mu > 2\varepsilon\right\} \le \mathsf{P}\left\{\int Z^L \mathrm{d}\mu > \varepsilon\right\}.$$

It remains to make use of (18). □

Lemma 2.6. *Let μ be a σ-finite measure on some σ-algebra $\mathcal{X}$ of subsets of a set X, and let for each $t > 0$ ζ_t be a nonnegative measurable random function on X. Suppose the following: for any $x \in X$ relation* (13) *holds; there exists a nonrandom measurable function Z on X satisfying conditions* (14), (15) *and, for any $r > 0$, the condition*

$$\mu\{x : Z(x) > r\} < \infty. \tag{19}$$

Then relation (16) *holds.*

Proof. The set $\{x \in X : Z(x) > r\}$ is nonrandom (since in this lemma so is Z), belongs to $\mathcal{X}$ (because Z is measurable) and has, due to (19), finite measure as $r > 0$. Then it follows from (13) – (15) by Lemma 2.5 that for any $r > 0$

$$\int \zeta_t I\{Z > r\}\mathrm{d}\mu \xrightarrow{\mathsf{P}} 0.$$

Consequently, for any positive r and ε

$$\varlimsup_{t\to\infty} \mathsf{P}\left\{\int \zeta_t \mathrm{d}\mu > 2\varepsilon\right\} \le \varlimsup_{t\to\infty} \mathsf{P}\left\{\int \zeta_t I\{Z \le r\}\mathrm{d}\mu > \varepsilon\right\}. \tag{20}$$

Condition (15) implies that Z is nonnegative, whence with account of (14) we get by the DCT

$$\lim_{r\to 0} \int ZI\{Z \le r\}\mathrm{d}\mu = 0,$$

which together with (15) yields

$$\lim_{r\to 0} \varlimsup_{t\to\infty} \mathsf{P}\left\{\int \zeta_t I\{Z \le r\}\mathrm{d}\mu > \varepsilon\right\} = 0$$

for any $\varepsilon > 0$. Now, (16) follows from (20). □

Let n and d be natural numbers, P and S be $n \times d$-matrices, and B be a $d \times d$ matrix. The identity

$$PBP^\top - SBS^\top = (P-S)BP^\top + SB(P-S)^\top$$

yields the estimate

$$\left\|PBP^\top - SBS^\top\right\| \le \|P-S\|\,\|B\|\,\left\|P^\top\right\| + \|S\|\,\|B\|\,\left\|(P-S)^\top\right\|.$$

In particular, for any $B \in \mathfrak{S}_+$ (so that $\|B\| \le \operatorname{tr} B$) and $p, q \in \mathbb{R}^{d*}$

$$\left\|pBp^\top - qBq^\top\right\| \le (|p| + |q|)|p-q| \operatorname{tr} B.$$

Hence the following conclusion is immediate.

Lemma 2.7. *Let V be an increasing $\mathfrak{S}_+$-valued function on $[a, b]$. Then for any continuous $\mathbb{R}^{d*}$-valued functions φ and ψ on $[a, b]$*

$$\left|\int_a^b \varphi(s)\mathrm{d}V(s)\varphi(s)^\top - \int_a^b \psi(s)\mathrm{d}V(s)\psi(s)^\top\right| \le \int_a^b (|\varphi(s)| + |\psi(s)|)|\varphi(s) - \psi(s)|\,\mathrm{d}\operatorname{tr} V(s).$$

Lemma 2.8. *For arbitrary $z \in \mathbb{R}^{d*},\ x, y \in \mathbb{R}^d$ $\quad |e^{izx} - e^{izy}| \le 2(|z| \vee 2)\frac{|x-y|}{1+|x-y|}$.*

Proof. For $y = 0$ this is Lemma 7.1 in (Yurachkivsky, 2013). It remains to note that $|e^{izx} - e^{izy}| = |e^{iz(x-y)} - 1|$. □

Lemma 2.9. *Let K be the solution of equation (3) on $\mathbb{R}_+^2$, where ι is a continuous numeral function of locally bounded variation, and A is a matrix-valued Borel function such that for any $t > 0$ (2) holds. Then for every Borel function $L(\cdot,\cdot)$ that has locally bounded variation in the first argument the solution of the equation*

$$\phi(t, s) = L(t, s) + \int_s^t A(u)\phi(u, s)\mathrm{d}\iota(u) \tag{21}$$

is given by the formula

$$\phi(t, s) = K(t, s)L(s, s) + \int_s^t K(t, \tau)\mathrm{d}_\tau L(\tau, s). \tag{22}$$

Proof. Without loss of generality $s = 0$. Then this statement is a particular case of Corollary 3.19 in (Yurachkivsky, 2013). □

Throughout below, the prime does not signify differentiation.

Corollary 2.10. *Let ι, A, K be as in Lemma 2.9, and K' be the solution of the equation*

$$K'(t, s) = \mathbf{I} + \int_s^t A'(\tau)K'(\tau, s)\mathrm{d}\iota(\tau), \tag{23}$$

on $\mathbb{R}_+^2$, where A' is a matrix-valued measurable function such that for any $t > 0$

$$\int_0^t \|A'(u)\|\,|\mathrm{d}\iota(u)| < \infty. \tag{24}$$

Then for all $t \ge s \ge 0$

$$K(t, s) - K'(t, s) = \int_s^t K(t, \tau)(A(\tau) - A'(\tau))K'(\tau, s)\mathrm{d}\iota(\tau).$$

Proof. To deduce this statement from Lemma 2.9 it suffices to write, on the basis of (3) and (23),

$$K(t, s) - K'(t, s) = \int_s^t (A(\tau) - A'(\tau))K'(\tau, s)\mathrm{d}\iota(\tau) + \int_s^t (A(\tau)(K(\tau, s) - K'(\tau, s))\mathrm{d}\iota(\tau)$$

□

Corollary 2.11. *Let ι, A, A', K and K' be as in Corollary 2.10. Suppose also that*

$$\|K(t, s)\| \vee \|K'(t, s)\| \le e^{q(t-s)}$$

for some $q \in \mathbb{R}$ and all $t, s \in \mathbb{R}_+$. Then for all $t \ge s \ge 0$

$$\|K(t, s) - K'(t, s)\| \le e^{q(t-s)} \int_s^t \|A(\tau) - A'(\tau)\|\,|\mathrm{d}\iota(\tau)|. \tag{25}$$

3. Special Preliminaries

From now on, $\int_\Theta$ will be written shortly as $\int$.

We impose two more assumptions:

12. There exists an $\mathbb{M}_+(\mathfrak{C})$-valued random process π such that the equality

$$\int_0^t \int \chi(s,\theta)\widetilde{\nu}(\mathrm{d}s\mathrm{d}\theta) = \int_0^t \mathrm{d}s \int \chi(s,\theta)\pi(s,\mathrm{d}\theta) \tag{26}$$

holds for every nonnegative $\mathcal{F}(0)\otimes\mathcal{B}_+\otimes\mathfrak{C}$-measurable random function χ on $\mathbb{R}_+\times\Theta$ and all $t>0$.

13. There exists an $\mathbb{M}_+(\mathfrak{C})$-valued random processes ϖ such that the equality

$$\int_0^t \int \chi(s,\theta)\widetilde{\upsilon}(\mathrm{d}s\mathrm{d}\theta) = \int_0^t \mathrm{d}s \int \chi(s,\theta)\varpi(s,\mathrm{d}\theta) \tag{27}$$

holds for every nonnegative $\mathcal{F}(0)\otimes\mathcal{B}_+\otimes\mathfrak{C}$-measurable random function χ on $\mathbb{R}_+\times\Theta$ and all $t>0$.

For an $\mathbb{M}_+(\mathfrak{C})$-valued random process κ we denote by $\mathcal{E}^\kappa$ the class of all $\mathbb{C}$-valued $\mathcal{F}(0)\otimes\mathcal{B}_+\otimes\mathfrak{C}$-measurable random functions χ on $\mathbb{R}_+\times\Theta$ such that

$$\int_0^t \mathrm{d}s \int |\chi(s,\theta)|\kappa(s,\mathrm{d}\theta) < \infty$$

for all $t>0$.

The next statement is obvious.

Lemma 3.1. *If assumption **12** is satisfied, then equality* (26) *holds for all* $\chi\in\mathcal{E}^\pi$ *and* $t>0$*; under assumption **13**, equality* (27) *holds for all* $\chi\in\mathcal{E}^\varpi$ *and* $t>0$.

Lemma 3.2. *Let assumptions **2**, **3**, **8** – **10**, **12** and **13** be satisfied. Then*

$$F_t(z) = \int_0^t \mathrm{d}s \int \left(e^{izK(t,s)f(s,\theta)} - 1\right)\pi(s,\mathrm{d}\theta), \tag{28}$$

$$G_t(z) = \int_0^t \mathrm{d}s \int \left(e^{izK(t,s)g(s,\theta)} - 1 - izK(t,s)g(s,\theta)\right)\varpi(s,\mathrm{d}\theta). \tag{29}$$

Proof. Let us fix $v>0$, $z\in\mathbb{R}^{d*}$ and denote $\varphi(s,\theta) = e^{izK(v,s)f(s,\theta)} - 1$. By construction and assumptions **2** and **3** the random process $zK(v,\cdot)$ is $\mathcal{F}(0)\otimes\mathcal{B}_+$-measurable, so by assumption **8** φ is $\mathcal{F}(0)\otimes\mathcal{B}_+\otimes\mathfrak{C}$-measurable. Herein

$$|\varphi(s,\theta)| \le (|zK(v,s)f(s,\theta)|\,|f(s,\theta)|)\wedge 2 \le (|z|\,\|K(v,s)\|\vee 2)(|f(s,\theta)|\wedge 1),$$

whence

$$\int_0^t \mathrm{d}s \int |\varphi(s,\theta)|\pi(s,\mathrm{d}\theta) \le \left(|z|\max_{s\le t}\|K(v,s)\| + 2\right)\int_0^t \mathrm{d}s \int (|f(s,\theta)|\wedge 1)\pi(s,\mathrm{d}\theta). \tag{30}$$

The random function $|f|\wedge 1$ is by assumption **8** $\mathcal{F}(0)\otimes\mathcal{B}_+\otimes\mathfrak{C}$-measurable, so by assumption **12**

$$\int_0^t \mathrm{d}s \int (|f(s,\theta)|\wedge 1)\pi(s,\mathrm{d}\theta) = \int_0^t \int (|f(s,\theta)|\wedge 1)\widetilde{\nu}(\mathrm{d}s,\mathrm{d}\theta),$$

whence in view of **10** and (30) we get $\int_0^t \mathrm{d}s\int_\Theta |\varphi(s,\theta)|\pi(s,\mathrm{d}\theta)<\infty$. Now, Lemma 3.1 asserts equality (26). Setting $v=t$, we turn it into

$$\int_0^t \int \left(e^{izK(t,s)f(s,\theta)} - 1\right)\widetilde{\nu}(\mathrm{d}s\mathrm{d}\theta) = \int_0^t \mathrm{d}s \int \left(e^{izK(t,s)f(s,\theta)} - 1\right)\pi(s,\mathrm{d}\theta).$$

And this is, in view of (7), none other than (28).

Likewise from **2**, **3**, **8**, **9**, (27) and the evident inequality

$$|e^{ip} - 1 - ip| \le p^2\wedge 2|p|, \qquad p\in\mathbb{R}, \tag{31}$$

we get by Lemma 3.1

$$\int_0^t \int \left(e^{izK(t,s)g(s,\theta)} - 1 - izK(t,s)g(s,\theta)\right)\widetilde{\upsilon}(\mathrm{d}s\mathrm{d}\theta) = \int_0^t \int \left(e^{izK(t,s)g(s,\theta)} - 1 - izK(t,s)g(s,\theta)\right)\varpi(s,\mathrm{d}\theta),$$

which together with (8) proves (9). □

We continue the list of assumptions:

14. There exist a nonrandom σ-finite measure Π on $\mathfrak{C}$ and a positive random variable Ξ such that for all $t \in \mathbb{R}_+$ and $B \in \mathfrak{C}$ $\quad \pi(t, B) \le \Xi\Pi(B)$.

15. There exist a nonrandom σ-finite measure Σ on $\mathfrak{C}$ and a positive random variable Ξ such that for all $t \in \mathbb{R}_+$ and $B \in \mathfrak{C}$ $\quad \varpi(t, B) \le \Xi\Sigma(B)$.

16. There exists a nonrandom measurable function $\mathfrak{f}$ on Θ such that (i) $|f(s,\theta)| \le \mathfrak{f}(\theta)$ and (ii) $\int \ln(1+\mathfrak{f})\mathrm{d}\Pi < \infty$.

17. There exists a nonrandom measurable function $\mathfrak{g}$ on Θ such that (i) $|g(s,\theta)| \le \mathfrak{g}(\theta)$ and (ii) $\int \left(\mathfrak{g}^2 \wedge \mathfrak{g}\right)\mathrm{d}\Sigma < \infty$.

In the next two assumptions and four statements, K_0 and K are continuous matrix-valued random function on $\mathbb{R}_+^2$; the assumption that K satisfies equation (3) is not used. We assume the following:

18. There exists a positive random variable $\varkappa$ such that for all $t > s$

$$\|K_0(t,s)\| \vee \|K(t,s)\| \le e^{\varkappa(s-t)}.$$

19. There exists an increasing random process Λ such that for all $t > s$

$$\|K(t,s) - K_0(t,s)\| \le e^{\varkappa(s-t)}(\Lambda(t) - \Lambda(s)),$$

where $\varkappa$ is the same as in **18**.

If assumption **19** is imposed and **18** is not, then $\varkappa$ will signify simply an $\mathcal{F}(0)$-measurable positive random variable.

Throughout below, we tacitly use the fact that the function $x \mapsto \frac{x}{1+x}$ increases on $\mathbb{R}_+$.

Lemma 3.3. *Let assumptions **14**, **16** and **19** be satisfied, and let*

$$\lim_{t\to\infty} \Lambda(t) < \infty. \tag{32}$$

Then for any $z \in \mathbb{R}^{d*}$

$$\int_0^t \mathrm{d}s \int \left(e^{izK(t,s)f(s,\theta)} - e^{izK_0(t,s)f(s,\theta)}\right)\pi(s,\mathrm{d}\theta) \to 0.$$

Proof. Denote $H(t,s) = \|K(t,s) - K_0(t,s)\|$, $\epsilon = \lim_{t\to\infty}(\Lambda(t) - \Lambda(0))$ ($< \infty$ by condition (32)). It suffices, in view of Lemma 2.8, the definition of H, the Fubini – Tonelli theorem and assumptions **14** and **16**(i), to show that

$$\int \Pi(\mathrm{d}\theta) \int_0^t \frac{\mathfrak{f}(\theta)H(t,s)}{1+\mathfrak{f}(\theta)H(t,s)}\mathrm{d}s \to 0. \tag{33}$$

Assumption **19** and the definition of ϵ yield

$$\frac{\mathfrak{f}(\theta)H(t,s)}{1+\mathfrak{f}(\theta)H(t,s)} \le \frac{\epsilon\mathfrak{f}(\theta)e^{\varkappa(s-t)}}{1+\epsilon\mathfrak{f}(\theta)e^{\varkappa(s-t)}},$$

whence for any $c \in [0,\ 1]$

$$\int_0^{ct} \frac{\mathfrak{f}(\theta)H(t,s)}{1+\mathfrak{f}(\theta)H(t,s)}\mathrm{d}s \le \ln\left(1+\epsilon e^{-(1-c)\varkappa t}\mathfrak{f}(\theta)\right) - \ln\left(1+\epsilon e^{-\varkappa t}\mathfrak{f}(\theta)\right).$$

And this together with **16**(ii) implies by the DCT that for any $c \in [0,\ 1[$

$$\int \Pi(\mathrm{d}\theta) \int_0^{ct} \frac{\mathfrak{f}(\theta)H(t,s)}{1+\mathfrak{f}(\theta)H(t,s)}\mathrm{d}s \to 0. \tag{34}$$

Denote $\rho(t,c) = \Lambda(t) - \Lambda(ct)$. It follows from (32) that for any $c > 0$

$$\lim_{t\to\infty} \rho(t,c) = 0. \tag{35}$$

By assumption **19** $H(t,s) \le e^{\varkappa(s-t)}\rho(t,c)$ as $s \ge ct$, so

$$\int_{ct}^{t} \frac{\mathfrak{f}(\theta)H(t,s)}{1+\mathfrak{f}(\theta)H(t,s)}\mathrm{d}s \le \ln\left(1+\rho(t,c)\mathfrak{f}(\theta)\right) - \ln\left(1+\rho(t,c)e^{-(1-c)\varkappa t}\mathfrak{f}(\theta)\right).$$

And this together with **16**(ii) and (35) implies by the DCT that for any $c \in]0,\ 1]$

$$\int \Pi(\mathrm{d}\theta)\int_{ct}^{t} \frac{\mathfrak{f}(\theta)H(t,s)}{1+\mathfrak{f}(\theta)H(t,s)}\mathrm{d}s \to 0,$$

whence with account of (34) relation (33) follows. □

Denote

$$\gamma(s,\theta,z) = e^{izg(s,\theta)} - izg(s,\theta) - 1.$$

Lemma 3.4. *Let assumptions **15**, **17** – **19** be satisfied and condition* (32) *be fulfilled. Then for any* $z \in \mathbb{R}^{d*}$

$$\int_0^t \mathrm{d}s \int (\gamma(s,\theta,zK(t,s)) - \gamma(s,\theta,zK_0(t,s)))\varpi(s,\mathrm{d}\theta) \to 0.$$

Proof. It suffices, in view of assumptions **15** and the Fubini – Tonelli theorem, to show that

$$\int_0^t \mathrm{d}s \int |\gamma(s,\theta,zK(t,s)) - \gamma(s,\theta,zK_0(t,s))|\,\Sigma(\mathrm{d}\theta) \to 0. \tag{36}$$

Denote $B_\varepsilon = \{\theta : \mathfrak{g}(\theta) \le \varepsilon\}$, $B^\varepsilon = \{\theta : \mathfrak{g}(\theta) > \varepsilon\}$. Assumption **17**(ii) implies, obviously, that for any $\varepsilon > 0$

$$\int_{B_\varepsilon} \mathfrak{g}^2 \mathrm{d}\Sigma < \infty \tag{37}$$

and

$$\int_{B^\varepsilon} \mathfrak{g}\mathrm{d}\Sigma < \infty. \tag{38}$$

Inequality (31) implies that

$$|\gamma(s,\theta,zK(t,s)) - \gamma(s,\theta,zK_0(t,s))| \le 2|z|^2|g(s,\theta)|^2(\|K_0(t,s)\|^2 + \|K(t,s)\|^2),$$

whence in view of **17**(i) and **18**

$$\int_0^t \mathrm{d}s \int_{B_\varepsilon} |\gamma(s,\theta,zK(t,s)) - \gamma(s,\theta,zK_0(t,s))|\,\Sigma(\mathrm{d}\theta) \le 4|z|^2 \int_0^t e^{2\varkappa(s-t)}\mathrm{d}s \int_{B_\varepsilon} \mathfrak{g}^2 \mathrm{d}\Sigma.$$

Hence, writing

$$\int_0^t e^{2\varkappa(s-t)}\mathrm{d}s = \frac{1-e^{-2\varkappa t}}{2\varkappa}$$

and noting that

$$\lim_{\varepsilon\to 0}\int_{B_\varepsilon} \mathfrak{g}^2\,\mathrm{d}\Sigma = 0$$

because of (37), we get

$$\lim_{\varepsilon\to 0}\varlimsup_{t\to\infty}\int_0^t \mathrm{d}s \int_{B_\varepsilon} |\gamma(s,\theta,zK(t,s)) - \gamma(s,\theta,zK_0(t,s))|\,\Sigma(\mathrm{d}\theta) = 0. \tag{39}$$

On the other hand, $|e^{ia} - ia - (e^{ib} - ib)| \le 2|a-b|$ for any $a, b \in \mathbb{R}$, so

$$|\gamma(s,\theta,zK(t,s)) - \gamma(s,\theta,zK_0(t,s))| \le 2|z|\,|g(s,\theta)|\,\|K_0(t,s) - K(t,s)\|,$$

which together with **17**(i) and **19** yields

$$\int_0^t \mathrm{d}s \int_{B^\varepsilon} |\gamma(s,\theta,zK(t,s)) - \gamma(s,\theta,zK_0(t,s))|\,\Sigma(\mathrm{d}\theta) \le 2|z| \int_0^t e^{\varkappa(s-t)}(\Lambda(t) - \Lambda(s))\mathrm{d}s \int_{B^\varepsilon} \mathfrak{g}\mathrm{d}\Sigma. \tag{40}$$

Writing, for an arbitrary $c \in]0,\ 1[$,

$$\int_0^t e^{\varkappa(s-t)}(\Lambda(t) - \Lambda(s))\mathrm{d}s = \int_0^{ct} e^{\varkappa(s-t)}(\Lambda(t) - \Lambda(s))\mathrm{d}s + \int_{ct}^t e^{\varkappa(s-t)}(\Lambda(t) - \Lambda(s))\mathrm{d}s$$

and arguing as in the previous proof, we get from (32) and the assumption that Λ increases

$$\lim_{t\to\infty} \int_0^t e^{\varkappa(s-t)}(\Lambda(t) - \Lambda(s))\mathrm{d}s = 0. \tag{41}$$

And this jointly with (40) and (38) implies that for all $z \in \mathbb{R}^{d*}$ and $\varepsilon > 0$

$$\lim_{t\to\infty} \int_0^t \mathrm{d}s \int_{B^\varepsilon} |\gamma(s, \theta, zK(t, s)) - \gamma(s, \theta, zK_0(t, s))|\, \Sigma(\mathrm{d}\theta) = 0.$$

Hence and from (39) relation (36) follows. □

Lemma 3.5. *Let assumptions **7**, **18** and **19** be satisfied and condition* (32) *be fulfilled. Suppose also that there exists a positive random variable β such that for all $t > s \geq 0$*

$$\mathrm{tr}\langle W\rangle(t) - \mathrm{tr}\langle W\rangle(s) \leq \beta(t - s). \tag{42}$$

Then for any $t \in \mathbb{R}_+$

$$\int_0^t K(t, s)\mathrm{d}\langle W\rangle(s)K(t, s)^\top - \int_0^t K_0(t, s)\mathrm{d}\langle W\rangle(s)K_0(t, s)^\top \to 0.$$

Proof. By Lemma 2.7

$$\left|\int_0^t zK(t, s)\mathrm{d}\langle W\rangle(s)(zK(t, s))^\top - \int_0^t zK_0(t, s)\mathrm{d}\langle W\rangle(s)(zK_0(t, s))^\top\right| \leq |z|^2 \int_0^t (\|K_0(t, s)\| + \|K(t, s)\|)H(t, s)\, \mathrm{d}\, \mathrm{tr}\langle W\rangle(s). \tag{43}$$

Assumptions **18**, **19**, condition (42) and the inequality $e^{2p} \leq e^p$ for $p \leq 0$ imply that

$$\int_0^t (\|K_0(t, s)\| + \|K(t, s)\|)H(t, s)\, \mathrm{d}\, \mathrm{tr}\langle W\rangle(s) \leq 2\beta \int_0^t e^{\varkappa(s-t)}(\Lambda(t) - \Lambda(s))\, \mathrm{d}s,$$

which together with (41) (emerging from (32)) and (43) proves the lemma. □

Lemma 3.6. *Let assumptions **4** and **19** be satisfied and condition* (32) *be fulfilled. Suppose also that there exists a positive random variable β such that for all $t > s \geq 0$*

$$\operatorname*{var}_{[s,\, t]} U \leq \beta(t - s). \tag{44}$$

Then

$$\int_0^t K(t, s)\mathrm{d}U(s) - \int_0^t K_0(t, s)\mathrm{d}U(s) \to 0.$$

Proof. Writing on the basis of (44) and **19**

$$\left|\int_0^t K(t, s)\mathrm{d}U(s) - \int_0^t K_0(t, s)\mathrm{d}U(s)\right| \leq \beta \int_0^t e^{\varkappa(s-t)}(\Lambda(t) - \Lambda(s))\mathrm{d}s,$$

we deduce the desired conclusion from (41) (emerging from (32)). □

4. The Main Result

Theorem 4.1. *Let assumptions **1** – **8** and **11** – **17** be satisfied. Suppose also that there exist:*
- *an $\mathbb{R}^d$-valued random variable ϑ,*
- *random matrices A_0 and Υ,*
- *random σ-finite measures π_0 and ϖ_0 on $\mathfrak{C}$,*
- *positive random variables $\varkappa$, α and β,*
- *$\mathbb{R}^d$-valued measurable random functions f_0 and g_0 on Θ,*
- *a positive number a,*

• *an increasing random process* Λ *with property* (32)
— *such that:*

$$e^{tA_0}\int_0^t e^{-sA_0}\mathrm{d}U(s)\xrightarrow{\mathsf{P}}\vartheta, \tag{45}$$

$$\int_0^t e^{(t-s)A_0}\mathrm{d}\langle W\rangle(s)e^{(t-s)A_0^\top}\xrightarrow{\mathsf{P}}\Upsilon; \tag{46}$$

for any $B\in\mathfrak{C}$

$$\pi_0(B)\le\Xi\Pi(B), \tag{47}$$

$$\varpi_0(B)\le\Xi\Pi(B), \tag{48}$$

where Π, Σ *and* Ξ *are from assumptions* ***14*** *and* ***15****; for all* $u\in\mathbb{R}_+$

$$\left\|e^{uA_0}\right\|\le e^{-au}; \tag{49}$$

for all $t>s\ge 0$

$$\|K(t,s)\|\le e^{\varkappa(s-t)}, \tag{50}$$

$$\left\|K(t,s)-e^{(t-s)A_0}\right\|\le e^{\alpha(s-t)}(\Lambda(t)-\Lambda(s)) \tag{51}$$

and inequalities (42) *and* (44) *hold; for any* $\theta\in\Theta$

$$f(t,\theta)\xrightarrow{\mathsf{P}}f_0(\theta), \tag{52}$$

$$g(t,\theta)\xrightarrow{\mathsf{P}}g_0(\theta); \tag{53}$$

for any $z\in\mathbb{R}^{d*}$

$$\int(e^{izf_0(\theta)}-1)\pi(t,\mathrm{d}\theta)\xrightarrow{\mathsf{P}}\int(e^{izf_0(\theta)}-1)\pi_0(\mathrm{d}\theta), \tag{54}$$

$$\int(e^{izg_0(\theta)}-1-izg(\theta))\varpi(t,\mathrm{d}\theta)\xrightarrow{\mathsf{P}}\int(e^{izg_0(\theta)}-1-izg_0(\theta))\varpi_0(\mathrm{d}\theta). \tag{55}$$

Then the distribution of $\xi(t)$ *weakly converges, as* $t\to\infty$*, to the distribution with characteristic function*

$$\mathsf{E}e^{iz\vartheta-z\Upsilon z^\top/2+\Phi(z)+\Gamma(z)},$$

where

$$\Phi(z)=\int_0^\infty\mathrm{d}u\int\left(e^{ize^{uA_0}f_0(\theta)}-1\right)\pi_0(\mathrm{d}\theta), \tag{56}$$

$$\Gamma(z)=\int_0^\infty\mathrm{d}u\int\left(e^{ize^{uA_0}g_0(\theta)}-1-ize^{uA_0}g_0(\theta)\right)\varpi_0(\mathrm{d}\theta). \tag{57}$$

Proof. 1°. Denote $D(u)=e^{uA_0}$, $K_0(t,s)=D(t-s)$,

$$\eta_t(u,z)=\int\left(e^{izD(u)f(t-u,\theta)}-1\right)\pi(t-u,\mathrm{d}\theta), \tag{58}$$

$$\psi_t(u,z)=\int\left(e^{izD(u)g(t-u,\theta)}-1-izD(u)g(t-u,\theta)\right)\varpi(t-u,\mathrm{d}\theta). \tag{59}$$

Rewriting condition (49) in the form

$$\|D(u)\|\le e^{-au}, \tag{60}$$

and taking to account (50), (51) and the assumed properties of Λ, we see that K and K_0 satisfy assumptions **18** and **19** (with $\varkappa\wedge\alpha$ as the former $\varkappa$). Hence and from assumptions **14** – **17** we get by Lemmas 3.3 and 3.4

$$F_t(z)-\int_0^t\eta_t(u,z)\mathrm{d}u\to 0, \tag{61}$$

$$G_t(z)-\int_0^t\psi_t(u,z)\mathrm{d}u\to 0, \tag{62}$$

where $F_t(z)$ and $G_t(z)$ are defined by equalities (28) and (29), respectively.

2°. Let us show that

$$\int_0^t \eta_t(u,z)\mathrm{d}u - \int_0^t \mathrm{d}u \int \left(e^{izD(u)f_0(\theta)} - 1\right)\pi(t-u,\mathrm{d}\theta) \xrightarrow{\mathsf{P}} 0. \tag{63}$$

Denote $\delta_t(u,\theta) = |f(t-u,\theta) - f_0(\theta)|$. From (58) we have by Lemma 2.8, assumption **14** and the Fubini – Tonelli theorem

$$\left|\int_0^t \eta_t(u,z)\mathrm{d}u - \int_0^t \mathrm{d}u \int \left(e^{izD(u)f_0(\theta)} - 1\right)\pi(t-u,\mathrm{d}\theta)\right| \le (|z| \vee 2)\Xi \int \Pi(\mathrm{d}\theta) \int_0^t \frac{\|D(u)\|\delta_t(u,\theta)}{1+\|D(u)\|\delta_t(u,\theta)}\mathrm{d}u. \tag{64}$$

On the strength of (60)

$$\frac{\|D(u)\|\delta_t(u,\theta)}{1+\|D(u)\|\delta_t(u,\theta)} \le \kappa_t(u,\theta), \tag{65}$$

where

$$\kappa_t(u,\theta) = \frac{e^{-au}\delta_t(u,\theta)}{1+e^{-au}\delta_t(u,\theta)}. \tag{66}$$

For convenience of the subsequent derivations, we set $\kappa_t(u,\theta) = 0$ as $u > t$.

By construction $0 \le \kappa_t(u,\theta) < 1$. By condition (52)

$$\kappa_t(u,\theta) \xrightarrow{\mathsf{P}} 0,$$

whence by Lemma 2.5 for any $T > 0$

$$\int_0^T \kappa_t(u,\theta)\mathrm{d}u \xrightarrow{\mathsf{P}} 0. \tag{67}$$

By assumption **16**(i) $|f(t-u,\theta)| \le \mathfrak{f}(\theta)$, which together with (52) yields

$$|f_0(\theta)| \le \mathfrak{f}(\theta). \tag{68}$$

Hence and (again) from **16**(i) we get $\delta_t(u,\theta) \le 2\mathfrak{f}(\theta)$, whence in view of (66)

$$\kappa_t(u,\theta) \le \frac{2\mathfrak{f}(\theta)e^{-au}}{1+2\mathfrak{f}(\theta)e^{-au}},$$

and therefore for any $[a,\, b] \subset \mathbb{R}_+$

$$\int_a^b \kappa_t(u,\theta)\mathrm{d}u \le \ln(1+2e^{-\alpha a}\mathfrak{f}(\theta)).$$

Herein, obviously, $\ln(1+2x) \le 2\ln(1+x)$. Thus

$$\int_0^T \kappa_t(u,\theta)\mathrm{d}u \le 2\ln(1+\mathfrak{f}(\theta)),$$

which together with (66) and finiteness (by assumption **16**(ii)) of $\int \ln(1+\mathfrak{f}(\theta))\Pi(\mathrm{d}\theta)$ implies by Lemma 2.6 (applied to $X = \Theta,\ \mathcal{X} = \mathfrak{C},\ \mu = \Pi,\ \zeta_t(\theta) = \int_0^T \kappa_t(u,\theta)\mathrm{d}u$) that

$$\operatorname*{l.i.p.}_{t\to\infty} \int \Pi(\mathrm{d}\theta) \int_0^T \kappa_t(u,\theta)\mathrm{d}u = 0. \tag{69}$$

Further, for any $t \ge T$

$$\int \Pi(\mathrm{d}\theta) \int_T^t \kappa_t(u,\theta)\mathrm{d}u \le \int \ln\left(1+2e^{-\alpha T}\mathfrak{f}(\theta)\right)\Pi(\mathrm{d}\theta),$$

which together with **16**(ii) and the DCT yields

$$\lim_{T\to\infty} \sup_{t\ge T} \int \Pi(\mathrm{d}\theta) \int_T^t \kappa_t(u,\theta)\mathrm{d}u = 0.$$

And this jointly with (69) entails (63).

3°. Let us denote

$$\gamma_t(u,\theta,z) = e^{izD(u)g(t-u,\theta)} - 1 - izD(u)g(t-u,\theta),$$
$$\gamma_0(u,\theta,z) = e^{izD(u)g_0(\theta)} - 1 - izD(u)g_0(\theta)$$

and show that

$$\int_0^t \psi_t(u,z)\mathrm{d}u - \int_0^t \mathrm{d}u \int \gamma_0(u,\theta,z)\varpi(t-u,\mathrm{d}\theta) \xrightarrow{\mathsf{P}} 0. \tag{70}$$

Recalling (59), we see that $\psi_t(u,z) = \int \gamma_t(u,\theta,z)\varpi(t-u,\mathrm{d}\theta)$, so it suffices, in view of **15**, to prove the relation

$$\int_0^t \mathrm{d}u \int |\gamma_t(u,\theta,z) - \gamma_0(u,\theta,z)|\,\Sigma(\mathrm{d}\theta) \to 0. \tag{71}$$

Inequalities (31) and (60) imply that

$$|\gamma_t(u,\theta,z) - \gamma_0(u,\theta,z)| \le 2|z|^2 e^{-2au}(|g(t-u,\theta)|^2 + |g_0(\theta)|^2).$$

By assumption **17**(i) $|g(t-u,\theta)| \le \mathfrak{g}(\theta)$, which together with (53) yields

$$|g_0(\theta)| \le \mathfrak{g}(\theta). \tag{72}$$

Thus

$$|\gamma_t(u,\theta,z) - \gamma_0(u,\theta,z)| \le 4|z|^2 e^{-2au}\mathfrak{g}(\theta)^2.$$

Hence and from (37) we get

$$\lim_{\varepsilon\to 0}\overline{\lim_{t\to\infty}} \int_0^t \mathrm{d}u \int_{B_\varepsilon} |\gamma_t(u,\theta,z) - \gamma_0(u,\theta,z)|\,\Sigma(\mathrm{d}\theta) = 0. \tag{73}$$

On the other hand, $|e^{ib} - ib - (e^{ic} - ic)| \le 2|b-c|$ for any $b, c \in \mathbb{R}$, so

$$|\gamma_t(u,\theta,z) - \gamma_0(u,\theta,z)| \le 2e^{-au}|z|h_t(u,\theta),$$

where

$$h_t(u,\theta) = |g(t-u,\theta) - g_0(\theta)|.$$

Thus

$$\int_0^t \mathrm{d}s \int_{B^\varepsilon} |\gamma_t(u,\theta,z) - \gamma_0(u,\theta,z)|\,\Sigma(\mathrm{d}\theta) \le 2|z| \int_0^t e^{-au} q_t(u,\varepsilon)\mathrm{d}u, \tag{74}$$

where

$$q_t(u,\varepsilon) = \int_{B^\varepsilon} h_t(u,\theta)\Sigma(\mathrm{d}\theta). \tag{75}$$

Condition (53) means that $h_t(u,\theta) \xrightarrow{\mathsf{P}} 0$ for all u and θ; inequality (38) and the definition of B^ε show that $\Pi(B^\varepsilon) < \infty$; assumption **17**(i) and inequality (72) imply that

$$h_t(u,\theta) \le 2\mathfrak{g}(\theta). \tag{76}$$

Hence and from (38) we get by Lemma 2.5

$$q_t(u,\varepsilon) \xrightarrow{\mathsf{P}} 0 \tag{77}$$

for all positive u and ε. Herein

$$q_t(u,\theta) \le C \equiv 2\int_{B^\varepsilon} \mathfrak{g}\mathrm{d}\Pi \tag{78}$$

in view of (75) and (76). Then again by Lemma 2.5

$$\int_0^T e^{-au} q_t(u,\varepsilon)\mathrm{d}u \xrightarrow{\mathsf{P}} 0 \tag{79}$$

for all positive T and ε. Besides, inequality (78) implies that for any $t > T$

$$\int_T^t e^{-au} q_t(u,\varepsilon)\mathrm{d}u \le C\alpha^{-1}e^{-aT}$$

and therefore

$$\lim_{T\to\infty}\sup_{t>T}\int_T^t e^{-au}q_t(u,\varepsilon)\mathrm{d}u = 0,$$

which together with (79) implies

$$\int_0^t e^{-au}q_t(u,\varepsilon)\mathrm{d}u \xrightarrow{\mathsf{P}} 0.$$

This relation jointly with (74) and (73) proves (70).

4°. Lemma 2.8 and inequality (60) yield

$$\left|e^{izD(u)f_0(\theta)} - 1\right| \le 2(|z|\vee 2)\frac{e^{-au}|f_0(\theta)|}{1+e^{-au}|f_0(\theta)|},$$

whence, using the Fubini – Tonelli theorem, we get

$$\int_0^\infty \mathrm{d}u \int \left|e^{izD(u)f_0(\theta)} - 1\right|\pi_0(\mathrm{d}\theta) \le 2(|z|\vee 2)\int \pi_0(\mathrm{d}\theta)\int_0^\infty \frac{e^{-au}|f_0(\theta)|}{1+e^{-au}|f_0(\theta)|}\mathrm{d}u,$$

which together with (47), (68) and **16**(ii) results in

$$\int_0^\infty \mathrm{d}u \int \left|e^{izD(u)f_0(\theta)} - 1\right|\pi_0(\mathrm{d}\theta) < \infty.$$

Thus the definition $\Phi(z)$ by formula (56) is correct and there holds the estimate

$$|\Phi(z)| \le 2\Xi(|z|\vee 2)\int \ln(1+\mathfrak{f})\mathrm{d}\Pi$$

showing that Φ satisfies condition (10).

Let us show that

$$\int_0^t \mathrm{d}u \int \left(e^{izD(u)f_0(\theta)} - 1\right)\pi(t-u,\mathrm{d}\theta) \xrightarrow{\mathsf{P}} \Phi(z). \tag{80}$$

This will be done if we establish the relation

$$\int_0^\infty \zeta_t(u,z)\mathrm{d}u \xrightarrow{\mathsf{P}} 0, \tag{81}$$

where

$$\zeta_t(u,z) = \frac{1}{\Xi}\left|\int \left(e^{izD(u)f_0(\theta)} - 1\right)\pi(t-u,\mathrm{d}\theta) - \int \left(e^{izD(u)f_0(\theta)} - 1\right)\pi_0(\mathrm{d}\theta)\right|,$$

and π is defined for negative values of the temporal argument by zero. (The divider Ξ will enable us to construct a nonrandom majorant for ζ_t and thereon to apply Lemma 2.6.)

By construction, Lemma 2.8, assumption **14**, condition (47) and formula (68)

$$0 \le \zeta_t(u,z) \le Z(u,z), \tag{82}$$

where

$$Z(u,z) = 4(|z|\vee 2)\int \frac{e^{-au}\mathfrak{f}(\theta)}{1+e^{-au}\mathfrak{f}(\theta)}\Pi(\mathrm{d}\theta). \tag{83}$$

By condition (54)

$$\zeta_t(u,z) \xrightarrow{\mathsf{P}} 0 \tag{84}$$

for all $u \in \mathbb{R}_+$ and $z \in \mathbb{R}^{d*}$. By construction $Z(u,z)$ is nonrandom. Formula (83) and assumption **16**(ii) imply that

$$\int_0^\infty Z(u,z)\mathrm{d}u < \infty \tag{85}$$

and (by the DCT) $Z(u,z) \to 0$ as $u\to\infty$, so that for any $r>0$ the Lebesgue measure of the set $\{u : Z(u,z) > r\}$ is finite. Now, (81) follows from (82) and (84) by Lemma 2.6.

5°. Inequalities (31) and (60) yield

$$\left|e^{izD(u)g_0(\theta)} - 1 - izD(u)g_0(\theta)\right| \le 2(|z|^2\vee|z|)e^{-au}(|g_0(\theta)|^2\wedge|g_0(\theta)|),$$

whence, using the Fubini – Tonelli theorem, we get

$$\int_0^\infty \mathrm{d}u \int \left|e^{izD(u)g_0(\theta)} - 1 - izD(u)g_0(\theta)\right| \varpi_0(\mathrm{d}\theta) \le 2(|z|^2 \vee |z|)a^{-1} \int (|g_0|^2 \wedge |g_0|)\mathrm{d}\varpi_0,$$

which together with (48), (72) and **17**(ii) results in

$$\int_0^\infty \mathrm{d}u \int \left|e^{izD(u)g_0(\theta)} - 1 - izD(u)g_0(\theta)\right| \varpi_0(\mathrm{d}\theta) < \infty. \tag{86}$$

Thus the definition $\Gamma(z)$ by formula (57) is correct and there holds the estimate

$$|\Gamma(z)| \le 2(|z|^2 \vee |z|)a^{-1} \int (\mathfrak{g}_0^2 \wedge \mathfrak{g}_0)\mathrm{d}\Sigma$$

showing that Γ satisfies condition (11).

Let us show that

$$\int_0^t \mathrm{d}u \int \left(e^{izD(u)g_0(\theta)} - 1 - izD(u)g_0(\theta)\right) \varpi(t-u, \mathrm{d}\theta) \xrightarrow{\mathsf{P}} \Gamma(z). \tag{87}$$

This will be done if we establish relation (81), where this time (unlike item 4°)

$$\zeta_t(u,z) = \frac{1}{\Xi}\left|\int \left(e^{izD(u)g_0(\theta)} - 1 - izD(u)g_0(\theta)\right) \varpi(t-u, \mathrm{d}\theta) - \int \left(e^{izD(u)g_0(\theta)} - 1 - izD(u)g_0(\theta)\right) \varpi_0(\mathrm{d}\theta)\right|,$$

and ϖ is defined for negative values of the temporal argument by zero.

The above expression, assumption **15**, condition (48) and formulas (72), (31) imply (82), where this time (unlike item 4°)

$$Z(u,z) = 4(|z|^2 \vee |z|)e^{-au} \int (\mathfrak{g}^2 \wedge \mathfrak{g})\mathrm{d}\Sigma. \tag{88}$$

By condition (55) relation (84) holds for all $u \in \mathbb{R}_+$ and $z \in \mathbb{R}^{d*}$. Obviously, $Z(u,z) \to 0$ as $u \to \infty$, so that for any $r > 0$ the Lebesgue measure of the set $\{u : Z(u,z) > r\}$ is finite. Now, (81) follows from (82) and (84) by Lemma 2.6.

6°. Obviously, condition (50) entails (9).

Conditions (51) amounts to assumption **19** for the K_0 defined in item 1°. So Lemma 3.6 whose other conditions are retained in this theorem asserts that

$$\int_0^t K(t,s)\mathrm{d}U(s) - e^{tA_0}\int_0^t e^{-sA_0}\mathrm{d}U(s) \xrightarrow{\mathsf{P}} 0,$$

hereon (45) implies that

$$\int_0^t K(t,s)\mathrm{d}U(s) \xrightarrow{\mathsf{P}} \vartheta. \tag{89}$$

Likewise from (51) and (44) we deduce by Lemma 3.5 that

$$\int_0^t K(t,s)\mathrm{d}\langle W\rangle(s)K(t,s)^\top - \int_0^t e^{(t-s)A_0}\mathrm{d}\langle W\rangle(s)e^{(t-s)A_0^\top} \xrightarrow{\mathsf{P}} 0,$$

hereon (46) implies that

$$\int_0^t K(t,s)\mathrm{d}\langle W\rangle(s)K(t,s)^\top \xrightarrow{\mathsf{P}} \Upsilon. \tag{90}$$

From (61), (63) and (80) we have

$$F_t(z) \xrightarrow{\mathsf{P}} \Phi(z), \tag{91}$$

where $\Phi(z)$ is defined by (56). It follows from (62), (70) and (87) that

$$G_t(z) \xrightarrow{\mathsf{P}} \Gamma(z), \tag{92}$$

where $\Gamma(z)$ is defined by (57). Properties (10) and (11) of these Φ and Γ were verified in items 4° and 5°.

Relations (89) – (92) entail (12) (even in a stronger form — with $\xrightarrow{\mathsf{P}}$ instead of $\xrightarrow{\mathrm{d}}$). Now, noting that assumptions **9** and **10** are satisfied because of **14** – **16** and the evident inequality $p \wedge 1 \le 2\ln(1+p) \quad (p > 0)$, we deduce the conclusion of the theorem from Corollary 2.4. □

Conditions (50) and (51) (the latter accompanied by (32)) of the theorem, unlike the others, involve the function K that is not pre-specified. Its explicit expression is available only in very special cases, so to finalize the main result we must give more elementary conditions guaranteing the fulfilment of (51), (32) and (50). In the next statement, (50) is still an assumption.

Proposition 4.2. *Let condition* (50) *be fulfilled. Suppose also that there exists random variables ϱ, ς and an increasing random process H such that for all $t > s \geq 0$*

$$\|e^{(\iota(t)-\iota(s))A_0}\| \leq e^{\varrho(s-t)}, \tag{93}$$

$$\|e^{(\iota(t)-\iota(s))A_0} - e^{(t-s)A_0}\| \leq e^{\varsigma(s-t)}(H(t) - H(s)). \tag{94}$$

Then: (i) inequality (51) *holds with $\alpha = \varkappa \wedge \varrho \wedge \varsigma$, $\Lambda = H + V$,*

$$V(t) = \int_0^t \|A(u) - A_0\|\,|\mathrm{d}\iota(u)|; \tag{95}$$

(ii) if, moreover, $\lim_{t\to\infty} H(t) < \infty$ and

$$\int_0^\infty \|A(u) - A_0\|\,|\mathrm{d}\iota(u)| < \infty,$$

then $\lim_{t\to\infty} \Lambda(t) < \infty$.

Proof. Denote $K'(t, s) = e^{(\iota(t)-\iota(s))A_0}$. From (50), and (93) and (95) we get by Corollary 2.11

$$\|K(t, s) - K'(t, s)\| \leq e^{(\varkappa\wedge\varrho)(s-t)}(V(t) - V(s)),$$

which together with (94) completes the proof. □

It remains to find elementary sufficient conditions for (50).

Lemma 4.3. *Let K be the solution of equation* (3) *on $\mathbb{R}_+^2$, where ι is a continuous increasing function, and A is a matrix-valued Borel function such that for any $t > 0$*

$$\int_0^t \|A(u)\|\mathrm{d}\iota(u) < \infty.$$

Suppose that there exists a nonnegative Borel function $\mathfrak{a}$ such that

$$x^\top A(t)x \leq -\mathfrak{a}(t)|x|^2 \tag{96}$$

for all $t > 0$ and $x \in \mathbb{R}^d$. Then for all $t > s \geq 0$

$$\|K(t, s)\| \leq \exp\left\{-\int_s^t \mathfrak{a}(u)\mathrm{d}\iota(u)\right\}. \tag{97}$$

Proof. Fix x and denote $X(t, s) = K(t, s)x$. By construction X satisfies the equation

$$X(t, s) = x + \int_s^t A(s)X(u, s)\mathrm{d}\iota(u).$$

Then it follows from (96) by Theorem 3.1 in (Yurachkivsky, 2014) (actually, by its very special particular case) that

$$|X(t, s)| \leq |x| \exp\left\{-\int_s^t \mathfrak{a}(u)\mathrm{d}\iota(u)\right\},$$

which together with the definition of the operator norm implies (97). □

References

Anisimov, V. V., Zakusilo, O. K., & Donchenko, V. S. (1987). *Elements of Queueing Theory and Asymptotic Analysis of Systems* (in Russian). Kiev: Vyshcha Shkola.

Gikhman, I. I. & Skorokhod, A. V. (1982). *Stochastic Differential Equations and Their Applications* (in Russian). Kiev: Naukova Dumka.

Gikhman, I. I. & Skorokhod, A. V. (2009). *Theory of Stochastic Processes* (Vol.3, 2nd ed.). Berlin: Springer.

Ivanenko, D. O. (2007). Existence of a limit distribution of a solution of a linear inhomogeneous stochastic differential equation. *Theory of Probability and Mathematical Statistics, 78*, 49-60.

Jacod, J. & Shiryayev, A. N. (1987). *Limit Theorems for Stochastic Processes*. Berlin: Springer.

Liptser, R. Sh. & Shiryayev, A. N. (1989). *Theory of Martingales*. Dordrecht: Kluwer.

Sato, K. & Yamazato, M. (1984) Operator-self-decomposable distributions as limit distributions of processes of Ornstein–Uhlenbeck type. *Stochastic Processes and Their Applications, 17*, 73-100.

Yurachkivsky, A. (2013) An ergodic-type theorem for Ornstein–Uhlenbeck processes. *Random Operators and Stochastic Equations, 21*, 217-269. http://dx.doi.org/10.1515/rose-2013-0011

Yurachkivsky, A. (2014) A conditional mean square estimate for the solution of a SDE. *International Journal of Statistics and Probability, 3*, 1-10. http://dx.doi.org/10.5539/ijsp.v3n4p1

Copyrights