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Linear Hybrid Deterministic Dynamic Modeling for Time-to-Event Processes: State and Parameter Estimations

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Abstract

In this work, we initiate an innovative alternative modeling approach for time-to-event dynamic processes. The proposed approach is composed of the following basic components: (1) development of continuous-time state of dynamic process, (2) introduction of discrete-time dynamic intervention process, (3) formulation of continuous and discrete-time interconnected dynamic system, (4) utilizing Euler-type discretized schemes, and (5) introduction of conceptual and computational state and parameter estimation procedures. The presented approach is motivated by state and parameter estimation of time-to-event processes in biological, chemical, engineering, epidemiological, medical, military, multiple-markets and social dynamic processes under the influence of discrete-time intervention processes. The role and scope of our approach is exhibited by presenting several well-known hazard/risk rate and survival function estimates as special cases. Moreover, conceptual algorithms are illustrated by time-series data sets under the influence of intervention processes.

Keywords: Kaplan-Meier estimator, hazard/risk rate function, piecewise exponential estimator, time-to-event closed process, totally discrete-time hybrid system

1. Introduction

In the survival and reliability data analysis, the main interest is focused on a nonnegative random variable, say T which describes a time-to-event process characterizing an occurrence of time until a certain event. Historically well-known time-to-event processes are deaths in population dynamic and component failures in mechanical systems (Kalbfleisch & Prentice, 2011). The human mobility, electronic communications, technological changes, advancements in engineering, medical, and social sciences have diversified the role and scope of time-to-event processes in cultural, epidemiological, financial, military and social sciences (Ladde, 2015; Chandra & Ladde, 2014; Ladde & Ladde, 2012; Wanduku & Ladde, 2011; Anis, 2009).

The study of survival analysis rests on the concept of time-to-event. The mathematical statistics development of time-to-event analysis is based on the probabilistic approach and the concept of hazard rate. Moreover, the time-to-event is described by the closed form expressions of survival function that is determined by the concept of hazard rate (Kalbfleisch & Prentice, 2011; Lawless, 2011; Miller, 2011). We note that in general, hazard rate is unknown. This leads to a problem of determining hazard rate function. This is based on a feasible approach of collecting data set for the time-to-event processes in biological, chemical, engineering, epidemiological, medical, multiple-markets and social sciences. The hazard/risk rate and survival function estimation problems in the survival and reliability analysis are centered around the idea of "right censored data" (Miller, 2011). In fact, the common conventional understanding for resolving ties between censored and uncensored observations is adopted by shifting the censored observations slightly to the left of uncensored observations (Whittemore & Keller, 1983). In short, the items/individuals/objects in a given sample are decomposed into two mutually exclusive groups, namely, (a) deaths/failure /removal/non-operational/inactive, and (b) censored/losses/ withdrawals.

In the survival and reliability data analysis, parametric and nonparametric methods are applied to estimate the hazard/risk rate and survival functions (Kalbfleisch & Prentice, 2011; Lawless, 2011). A parametric approach is based on the assumption that the underlying survival distribution belongs to some specific family of distributions (e.g. normal, Weibull, exponential). On the other hand, a nonparametric approach is centered around the best-fitting member of a class of survival distribution functions (Kaplan & Meier, 1958). Moreover, Kaplan-Meier(KME) (Kaplan & Meier, 1958) and Nelson-Aalen (Aalen, 1978; Nelson, 1969) type nonparametric approach do not assume neither distribution class, nor closed-form distributions. In fact, it just depends on a data. The Kaplan-Meier and Nelson-Aalen type nonparametric estimation approaches are systematically analyzed by our totally discrete-time hybrid dynamic modeling process.

In the existing literature (Kalbfleisch & Prentice, 2011; Lawless, 2011), the closed-form expression for a survival function is based on the usage of probabilistic analysis approach. The closed-form representation of the survival function coupled with mathematical statistics method (parametric approach) is used to estimate both survival and hazard/risk rate functions. In fact, the parametric approach/model has advantages of simplicity, the availability of likelihood based inference procedures and the ease of use for a description, comparison, prediction, or decision (Lawless, 2011). In this work, we initiate an innovative alternative approach for modeling time-to-event dynamic processes. This approach leads to the development for estimating survival and hazard/risk rate functions. The presented approach is motivated by a simple observation regarding the probabilistic definition of the survival function (Kalbfleisch & Prentice, 2002). Moreover, this approach does not require a knowledge of either a closed-form solution distribution or a class of distributions.

Historically, exponential distributions have been widely used in analyzing survival/reliability data (Lawless, 2011; Davis, 1952). This was partly due to the mathematical simplicity and the availability of simple statistical methods. An application of the exponential model with covariates to medical survival data was initiated in Feigl and Zelen (1965). The assumption of a constant hazard/risk rate function is very restrictive. In fact, it is often violated. This is due to the fact that in some real life applications, sudden changes in the hazard rate at unknown times can be encountered due to a major maintenance in a mechanical system or a new treatment procedure in medical sciences (Anis, 2009). For example, usually a machine component functions with a constant hazard/risk rate function λ_1, until it suffers a shock. After this shock, the component may continue to operate but with a different constant hazard/risk rate function λ_2. In the medical field, there is usually a high initial risk after a major operation which settles down to a lower constant long-term risk rate (Anis, 2009). This type of change could occur in multiple times. In view of this, one is often interested in detecting the locations of such changes and estimating the sizes of the detected changes. Recently, several authors (Han, Schell & Kim 2014; He & Su, 2013; Fang & Su, 2011, Goodman, Li, & Tiwari, 2011) have proposed estimators based on change point hazard models. A Bayesian approach for estimating the piecewise exponential distribution (Gamerman, 1994) and estimating the grid of time-points (Demarqui, Loschi, & Colosimo, 2008) for the piecewise exponential model are also available in the literature. In order to incorporate these types of sudden changes (intervention process) in the hazard rate function, we modify the developed continuous state dynamic model to an interconnected hybrid dynamic model that is composed of both continuous time state and discrete time state (intervention process) dynamic processes.

Employing the total time on test (TTT) for undefined censored data beyond the last observation, the idea of Piecewise Exponential Estimator (PEXE) of a survival function was introduced by (Kitchin, Langberg, & Proschan, 1980) and applied for estimating life distribution from incomplete data. The PEXE has been modified to address the issues regarding the presence of ties in the data by Whittemore and Keller (1983).

The comparison of the PEXE with the KME (Kim & Proschan, 1991) exhibits the advantage of the PEXE over the KME. For example, the PEXE is a continuous survival function. Moreover, it exhibits the complete information that is coming from the censored data. Using a total time test and the PEXE based approach, the estimators of the hazard/risk rate and cumulative distribution functions on the left closed pairwise consecutive failure time intervals are determined in Kulasekera and White (1996). The PEXE is further extended by Malla and Mukerjee (2010) with an exponential tail extension in the framework of the Kaplan and Meier (1958) nonparametric estimator approach. Under the presented dynamic framework, we develop the PEXE and new PEXE of Malla and Mukerjee (2010) types in a systematic and unified way. In short, the presented novel approach incorporates all the existing features such as: incomplete data, issues regarding the ties, exponential tail extensions in the framework of Kaplan and Meier (1958), and so on in a coherent manner.

The organization of the presented work is as follows. In Section 2, recognizing the classical probabilistic analysis model of time-to-event as a dynamic process, we initiate a linear hybrid deterministic dynamic model for time-to-event processes. Moreover, a fundamental mathematical result that provides a basis for interconnected continuous-discrete-time and totally discrete-time dynamic processes, is developed. Utilizing the dynamic model and the main result developed in Section 2, basic conceptual analytic algorithms and its special cases for interconnected continuous-discrete-time and totally discrete-time linear hybrid dynamic models for time-to-event processes are presented in Section 3. In Section 4, we outline conceptual computational schemes. In Section 5, we present a very general conceptual and computational algorithm for estimating a hazard/risk rate function for multiple censoring times between consecutive failure times. These general results include the presented results in Section 4 as special cases. In Section 6, conceptual computational and simulation algorithms are developed. The developed computational schemes are applied to estimate hazard/risk rate and survival functions in a systematic and unified way. Moreover, several well-known results are exhibited as special cases. A few conclusions are drawn in Section 7 to exhibit the role and scope of linear hybrid deterministic modeling for time-to-event processes. Moreover, further extensions and generalizations to both deterministic and stochastic nonlinear and non-stationary hybrid modeling for time-to-event processes are currently underway. In addition, currently, a complex time-to-event dynamic analysis is also undertaken by the authors. These results will appear elsewhere. Finally, proofs of theorems and corollaries in Sections 2, 3, 4 and 5 are outlined in supplementary Section 8.

2. Linear Hybrid Dynamic Modeling of Time-to-event Process

In this section, based on the probabilistic definition of the survival function, we develop a model for time-to-event dynamic processes. From the probabilistic definition of the survival function (Kalbfleisch & Prentice, 2011; Lawless, 2011; Miller, 2011) and differential calculus (Apostol, 1967), we recognize that

$$\lambda(t)\Delta t \approx \frac{S(t) - S(t+\Delta t)}{S(t)}, \tag{1}$$

where S and λ are survival and hazard/risk rate functions, respectively. Moreover, from (1) and differential calculus (Apostol, 1976), we have

$$\mathrm{d}S = -\lambda(t)S\,\mathrm{d}t, \quad S(t_0) = S_0, \quad t \in [t_0, \infty), \tag{2}$$

where dS is a differential of a survival function S. In fact, (2) is a differential equation, and it is an initial value problem (IVP) (Ladde & Ladde, 2012). Based on continuous-time dynamic modeling (Ladde & Ladde, 2012), (2) represents a continuous-time linear dynamic model of time-to-event processes. In fact, we consider time-to-event processes to be probabilistic dynamic processes. The state of the process is represented by survival/infective/operational/radical and its complementary state, failure/removal/death/non-operational/normal, and it is measured by a probability distribution function. Employing Newtonian modeling approach, the instantaneous rate of change of survival state is directly proportional to the magnitude of the survival. The negative sign in (2) signifies that the state of survival is decaying/diminishing/decreasing. λ is a positive constant of proportionality. In general, it is a function of time. This is because of the fact that in general, the time-to-event processes are non-stationary. The solution of (2) on the interval $[t_0, \infty)$ is given by

$$S(t) = S_0 \exp\left[-\Lambda(t)\right], \tag{3}$$

where

$$\Lambda(t) = \int_0^t \lambda(u)\mathrm{d}u, \tag{4}$$

and it is the cumulative hazard/risk rate function.

Remark 2.1. If $\lambda(t) = \lambda$ for $t \geq 0$, $t_0 = 0$, $S(0) = 1$, then (3) reduces to the following well-known exponential distribution function:

$$S(t) = \exp[-\lambda t], \quad t \in [0, \infty), \tag{5}$$

and a complementary state of the survival state of time-to-event process is represented by

$$F(t) = 1 - S(t) = 1 - \exp[-\lambda t], \quad t \in [0, \infty),$$

and it is referred as a failure distribution function. Furthermore, we note that survival state dynamic model (2) signifies that the time-to-event process is closed (Rosen, 1970), that is, $S(t) + F(t) = 1$. It is analogous to epidemiological dynamic modeling process without removal (Ladde & Ladde, 2012; Wanduku & Ladde, 2011).

The presented motivational observation coupled with the introduction of the idea of continuous-time state dynamic process (2) operating under the discrete-time intervention processes further leads to a development of a linear hybrid dynamic model (Ladde & Ladde, 2012) for time-to-event processes. It is known (Ladde & Ladde, 2012) that many real world time-to-event dynamic processes are subject to intervention processes (internal or external). Therefore, it is natural that time-to-event dynamic processes undergo state adjustment processes. This causes a modification of the presented state dynamic processes that are described by simple state dynamic model (2). We note that the dynamic state adjustment processes are caused by periodic changes in science, technology, medicine, culture, socio-economic, environmental conditions and general behavior.

In the following, we introduce a type of hazard/risk rate function. Moreover, using dynamic approach, we present a development of PEXE (Kitchin et al., 1980; Kim & Proschan, 1991) in a systematic and unified way.

Definition 2.1. Let $\tau_0 < \tau_1 < \tau_2 < \ldots < \tau_k < \tau_{k+1}$ be a given partition of a time interval $[\tau_0, \mathscr{T}]$, with $\tau_0 = 0$ and $\tau_{k+1} = \infty$. Let $\lambda_1, \lambda_2, \ldots, \lambda_{k+1}$ be model parameters. A hazard/risk rate function for a nonnegative random variable T that characterizes time-to-event processes, is of the following form:

$$\lambda(t) = \sum_{i=1}^{k+1} = \lambda_j I_{[\tau_{j-1},\tau_j)}(t)\,, \quad t \in \mathbf{R}_+ = [0,\infty)\,, \tag{6}$$

where λ_j are positive real numbers for $j \in I(1, k+1)$, $(I(1,l) = \{1,2,\ldots,l\})$; $I_{[\tau_{j-1},\tau_j)}$ is the characteristic function with respect to $[\tau_{j-1}, \tau_j)$. Moreover, T is said to have a piecewise constant hazard function.

Definition 2.2. $\prod\limits_{i|\tau_j\le t}$ denotes the symbol for a product of objects for all positive integers $i \in I(1,\infty)$ that satisfy the conditions $\tau_i \le \tau_j$ and $\tau_j \le t < \tau_{j+1}$ for some $j \in I(1,n)$ and for $\tau_i, \tau_{j-1}, \tau_{j+1}, t \in [\tau_0, \mathscr{T}]$.

From Definition 2.1, we recognize that the sudden changes in the hazard/risk rate function are encountered due to various types of intervention processes (internal or external) (Ladde & Ladde, 2012). This causes to interrupt the current continuous-time state dynamic process (2). Following the linear hybrid dynamic model (Ladde & Ladde, 2012), a modified version of time-to-event dynamic model (2) is represented by:

$$\begin{cases} \mathrm{d}S = -\lambda(t)S\,\mathrm{d}t\,, \quad S(\tau_{j-1}) = S_{j-1}\,, \quad t \in [\tau_{j-1},\tau_j)\,, \\ S_j = S(\tau_j^-, \tau_{j-1}, S_{j-1})\,, \quad S(\tau_0) = S_0\,, \quad j \in I(1,k+1)\,, \end{cases} \tag{7}$$

where $S(\tau_j^-) = S(\tau_j^-|\lambda, \tau_{j-1}, S_{j-1})$ describes a very simpler form of intervention process generated at an intervention time τ_j; τ_j^- stands for $t \in [\tau_{j-1}, \tau_j)$, that is less than τ_j and very close to τ_j. We note that system (7) is interconnected hybrid dynamic system composed of both continuous and discrete time state dynamic systems. Imitating the procedure described in Ladde and Ladde (2012), the solution process of the IVP (7) is as follows:

$$S(t, \tau_{j-1}, S_{j-1}|\lambda) = S_{j-1} \exp\left[-\int_{\tau_{j-1}}^{t} \lambda(u)\mathrm{d}u\right]\,, \text{ for all } t \in [\tau_{j-1}, \tau_j)\,. \tag{8}$$

Furthermore, the solution process of the overall time-to-event dynamic process (7) on $[\tau_0, \mathscr{T})$ is

$$S(t, \tau_{j-1}, S_0|\lambda) = S_0 \prod_{m=1}^{j-1} \exp\left[-\int_{\tau_{m-1}}^{\tau_m} \lambda(u)\mathrm{d}u\right] \exp\left[-\int_{\tau_{j-1}}^{t} \lambda(u)\mathrm{d}u\right],\ t \in [\tau_0, \mathscr{T})\ ,\ j \in I(1,k+1)\,. \tag{9}$$

Remark 2.2. From (7) and (8), we note that the solution process (8) is indeed PEXE (Kitchin et al., 1980; Kim & Proschan, 1991).

In the following, we present a very simple fundamental auxiliary result that would be used, subsequently. Moreover, it exhibits an analytic unified bridge and basis for (7) and its complete discrete-time version.

Theorem 2.1. *Let $\{\tau_j\}_0^n$ be a partition of $[0, \mathscr{T}]$ and let β be a monotonic nondecreasing function defined by*

$$\beta(t) = \begin{cases} 0, & t \in [\tau_{j-1}, \tau_j)\,, \\ 1, & t = \tau_j\,, \end{cases} \tag{10}$$

for each $j \in I(1,n)$. Let x be a state dynamic process in biological, engineering, epidemiological, human, medical, military, physical and social sciences under the influence of time-to-event processes. Let x be described by:

$$\begin{cases} \mathrm{d}x = [-\alpha(t)\,x + \gamma(t)]\,\mathrm{d}\beta(t), \quad t \in [\tau_{j-1}, \tau_j)\,, \\ x_j = (1-\alpha_j)x(\tau_j^-, \tau_{j-1}, x_{j-1}) + \gamma_j\,, \quad x(\tau_0) = x_0\,, \end{cases} \tag{11}$$

where α and γ are real-valued continuous functions defined on $[0,\infty)$; $\alpha_j = \alpha(\tau_j)$ and $\gamma_j = \gamma(\tau_j)$. Then

$$x(t) = \prod_{k|\tau_j\le t} (1-\alpha_k)x_0 + \sum_{i=1}^{j-1} \Phi(t,\tau_i)\gamma_i + \gamma_j\,, \quad \textit{for} \quad t \ge \tau_0\,, \tag{12}$$

where j is the largest integer so that $\tau_j \le t < \tau_{j+1}$, $\tau_k \le \tau_j$ and

$$\Phi(t,\tau_i) = \prod_{\tau_i\le\tau_j\le t} (1-\alpha_i)\,, \quad \Phi(\tau_i,\tau_i) = 1 \quad \text{for} \quad i \in I(0,n)\,.$$

Proof. The proof of Theorem 2.1 is given in the supplementary Section 8.

Remark 2.3. From (10), the hybrid dynamic system (11), is equivalent to the hybrid dynamic system

$$\begin{cases} \mathrm{d}x = 0\,\mathrm{d}t\,, \quad x(\tau_{j-1}) = x_{j-1}\,, \quad t \in [\tau_{j-1}, \tau_i)\,, \\ x_j = (1-\alpha_j)x(\tau_j^-, \tau_{j-1}, x_{j-1}) + \gamma_j\,, \quad x(\tau_0) = x_0\,, \end{cases} \tag{13}$$

for $j \in I(1, n)$. The solution process of (13) is represented in (12).

In the following, we present a couple of special cases of Theorem 2.1. These special cases illustrate a systematic way for exhibiting the existing results in Kaplan and Meier (1958), Nelson (1969), Aalen (1978) and Malla and Mukerjee (2010) in the framework of presented innovative dynamic approach.

Corollary 2.1. *If functions α and γ in Theorem 2.1 are replaced by functions λ and $\gamma = 0$, then (12) reduces to*

$$x(t) = \prod_{j|\tau_j \le t} (1-\lambda_j)x_0\,, \quad t \ge \tau_0\,. \tag{14}$$

Corollary 2.2. *If $\alpha = 0$ and $x_0 = 0$ in Theorem 2.1, then the conclusion of Theorem 2.1 reduces to*

$$x(t) = \sum_{i|\tau_{j-1} \le t} \gamma_i\,, \quad t \ge \tau_0 \quad \textit{and} \quad t \in [\tau_{j-1}, \tau_j)\,. \tag{15}$$

In the following, we present a definition of cumulative jump process (Malla & Mukerjee, 2010) in the framework of hybrid dynamic model.

Example 2.1. Let $T_1, T_2, \ldots, T_n$ be discrete failure times for the discrete-time event process, and $0 = a_0 < a_1 \le a_2 \le \ldots \le a_m$ be jumps of a survival function in magnitude. Then the dynamic for the cumulative jump process is as described in Corollary 2.2, and its solution process is exhibited in (15).
In this example, applying Corollary 2.2 in the context of $\gamma_0 = 0$, $\gamma_i = a_i$, the cumulative jump process is represented by

$$x(t) = \begin{cases} A_{j-1} = \sum\limits_{i=1}^{j-1} a_i\,, \quad \text{for} \quad t \in [\tau_{j-1}, \tau_j)\,, \\ A_j = \sum\limits_{i=1}^{j} a_i\,, \quad t = \tau_j\,. \end{cases} \tag{16}$$

From (16), we recognize that the cumulative jump defined in Malla and Mukerjee (2010) is indeed recast as the discrete time intervention process described by the hybrid dynamic system illustrated in Corollary 2.2 at the discrete time τ_j for $j \in I(1, m)$ with $\gamma_0 = a_0 = 0$ and $\gamma_i = a_i$.

Example 2.2. Under the conditions of Example 2.1, the magnitude of the survival function at the failure times is represented by

$$S(t) = \begin{cases} 1 - A_{j-1}\,, \quad \text{for} \quad t \in [\tau_{j-1}, \tau_j)\,, \\ 1 - A_j\,, \quad t = \tau_j, \quad j \in I(1, m)\,, \end{cases} \tag{17}$$

where $\gamma_0 = 1$ and $x(\tau_j) = A_j$. The $S(t)$ in (17) is the magnitude of the survival function determined by the cumulative jump (Malla & Mukerjee, 2010) process described in Example 2.1.

Remark 2.4. We remark that the continuous-time dynamic model can be exhibited by the cumulative hazard/risk rate function. In fact, from (2), we have

$$\mathrm{d}\ln S = -\lambda(t)\mathrm{d}t\,, \quad \ln S(\tau_0) = S_0\,. \tag{18}$$

Based on the solution processes of (2) and (7), the solution process of (18) can be represented as:

$$-\ln\left[\frac{S(t)}{S(\tau_0)}\right] = \Lambda(t, \tau_0, S_0|\lambda) = \int_{\tau_0}^{t} \lambda(u)\mathrm{d}u\,. \tag{19}$$

and

$$-\ln\left[\frac{S(t)}{S(\tau_0)}\right] = \Lambda(t, \tau_0|\lambda) = \sum_{m=1}^{j-1} \int_{\tau_{m-1}}^{\tau_m} \lambda(u)\mathrm{d}u + \int_{\tau_{j-1}}^{t} \lambda(u)\mathrm{d}u\,,\ t \in [\tau_{j-1}, \tau_j)\,. \tag{20}$$

respectively. Furthermore, we set $x = \ln S$, $S_0 = 1$ and $\gamma(t) = -\lambda(t)$ where S and λ are defined in (18). From Corollary 2.2, we have

$$\ln S(t) = -\Lambda(t), \tag{21}$$

where $\Lambda(t) = \sum_{i|\tau_i \le t} \lambda_i$ is a cumulative hazard function.

Remark 2.5. We remark that if x is replaced by survival function, S in Corollary 2.1, and x and γ are replaced by S and λ in Corollary 2.2, then (14) and (15) are replaced by:

$$S(t) = \prod_{j|\tau_j \le t} (1 - \lambda_j) S_0, \quad t \ge \tau_0 \tag{22}$$

and

$$S(t) = \sum_{i|\tau_i \le t} \lambda_i, \quad t \ge \tau_0, \tag{23}$$

respectively. Moreover, (22) is the solution process of the discrete-time dynamic system described by Corollary 2.1. Furthermore, dynamic system outlined in Corollary 2.1 provides an innovative alternative approach for finding the discrete-time survival function (Kaplan & Meier, 1958) in a systematic manner.

We utilize the above presented concepts and results in subsequent sections in a systematic and unified way.

3. Fundamental Results for Continuous and Discrete-Time to Event Dynamic Processes

In this section, we utilize hybrid dynamic model (7) and fundamental analytic Theorem 2.1 for time-to-event process to develop a general fundamental result. The developed result provides basic analytic and computational tools for estimating survival state and parameters. The presented approach also provides a systematic and unified way of estimating the parameters and survival functions.

Let $x(t)$ be the total number of units/individuals operating/alive (or survivals) at time t, for $t \in [\tau_0, \mathscr{T}]$. It is described by (11). Let λ and S be hazard/risk rate and survival functions of the units/patients/infectives/species/individuals, respectively. Employing a dynamic model for number of units/species/ individuals coupled with survival state dynamic model (2) or (7), we present an interconnected hybrid dynamic model below.

Following the argument used in developing dynamic models (Ladde & Ladde, 2012), we introduce the following interconnected system of differential equations:

$$\begin{cases} \mathrm{d}S = -\lambda(t) S\,\mathrm{d}t, \quad t \in [\tau_{j-1}, \tau_j), \\ S_j = (1 - \beta_j) S(\tau_j^-, \tau_{j-1}, S_{j-1}), \quad S(\tau_0) = 1, \\ dx = (-\alpha(t)x + \gamma(t))\mathrm{d}\beta(t), \quad x(\tau_0) = x_0, \quad t \in [\tau_{j-1}, \tau_j), \\ x_j = (1 - \alpha_j) x(\tau_j^-, \tau_{j-1}, x_{j-1}) + \gamma_j, \end{cases} \tag{24}$$

Remark 3.1. We outline a few important observations that exhibit the role and scope of dynamic approach to illustrate the existing results (Han et al., 2014; Kim & Proschan, 1991; Thaler, 1984; Kitchin et al., 1980; Kaplan & Meier, 1958) as special cases.

(i) Dynamic system (24) in the context of (13) (Remark 2.3) is reduced to

$$\begin{cases} \mathrm{d}S = -\lambda(t) S\,\mathrm{d}t, \quad t \in [\tau_{j-1}, \tau_j), \\ S_j = (1 - \beta_j) S(\tau_j^-, \tau_{j-1}, S_{j-1}), \quad S(\tau_0) = 1, \\ dx = 0\,\mathrm{d}t, \quad x(\tau_0) = x_0, \quad t \in [\tau_{j-1}, \tau_j), \\ x_j = (1 - \alpha_j) x(\tau_j^-, \tau_{j-1}, x_{j-1}) + \gamma_j\ . \end{cases} \tag{25}$$

(ii) From Corollary 2.1 in the context of Remark 2.5, in particular (22), system (24) becomes:

$$\begin{cases} \mathrm{d}S = 0\,\mathrm{d}t, \quad t \in [\tau_{j-1}, \tau_j), \\ S_j = (1 - \lambda_j) S_{j-1}, \\ \mathrm{d}x = 0\,\mathrm{d}t, \quad x(\tau_0) = x_0, \\ x_j = (1 - \alpha_j) x_{j-1} + \gamma_j\ . \end{cases} \tag{26}$$

We note that (26) is a special version of (24). In addition, we refer to system (26) as a totally discrete-time hybrid dynamic system.

Now, we are ready to present a basic result regarding continuous and discrete time interconnected dynamic of survival species or objects or thoughts operating under the time-to-event intervention processes. Prior to the formulation of the fundamental result, we introduce a concept of number of survivals.

Definition 3.1. Let z be a function defied by $z(t) = x(t)S(t)$, where S and x are solution process of (24) for $t \in [\tau_0, \mathscr{T}]$. Moreover, for each $t \in [\tau_0, \mathscr{T}]$, $z(t)$ stands for the number of survivals at t under an influence of time-to-event process.

Theorem 3.1. *Let (x, S) be a solution process of (24). Then the interconnected hybrid dynamic population model for time-to-event process (24) and corresponding intervention iterative process are described by:*

$$\begin{cases} dz = -\lambda(t)z dt, \quad z(\tau_{j-1}) = z_{j-1}, \quad for \quad t \in [\tau_{j-1}, \tau_j), \quad j \in I(1,k), \\ z(\tau_j) = (1-\alpha_j)(1-\beta_j)z(\tau_j^-) + \gamma_j(1-\beta_j), \end{cases} \tag{27}$$

and

$$z(\tau_j) = (1 - \lambda(\tau_j)\Delta\tau_j)(1-\alpha_j)(1-\beta_j)z(\tau_{j-1}) + \gamma_j(1-\beta_j) . \tag{28}$$

respectively, where z is defined in Definition 3.1 and $\Delta\tau_j = \tau_j - \tau_{j-1}$ for $j \in I(1,k)$.

Proof. For the detailed proof of Theorem 3.1, the readers are encouraged to read the supplementary Section 8.

In the following, we present a few special/trivial cases that exhibit existing results in the framework of hybrid dynamic of time-to-event interconnected system.

Corollary 3.1. *Let us consider a very special/trivial case of Theorem 3.1 as follows:*

$$\begin{cases} dS = -\lambda(t)S\, dt, \quad t \geq \tau_0, \\ dx = 0\, dt, \quad t \geq \tau_0, \\ x(\tau_j) = x(\tau_j^-, \tau_{j-1}, x_{j-1}), \quad x(\tau_0) = x_0, \quad j \in I(1,k) . \end{cases} \tag{29}$$

Applying Theorem 3.1 and using (27) and (28), (29) reduces to

$$\begin{cases} dz = -\lambda(t)z dt, \quad z(\tau_{j-1}) = z_{j-1}, \quad t \in [\tau_{j-1}, \tau_j), \\ z(\tau_j) = z(\tau_j^-, \tau_{j-1}, z_{j-1}) = z(\tau_{j-1}), \quad j \in I(1,k), \end{cases} \tag{30}$$

and

$$z(\tau_j) = \left(1 - \lambda(\tau_j)\Delta\tau_j\right) z(\tau_{j-1}) . \tag{31}$$

Corollary 3.2. *Let us consider a special case of (24) as follows:*

$$\begin{cases} dS = -\lambda(t)S\, dt, \quad S(\tau_{j-1}) = S_{j-1}, \quad t \in [\tau_{j-1}, \tau_j), \\ S(\tau_j) = S(\tau_j^-, \tau_{j-1}, S_{j-1}), \end{cases} \tag{32}$$

where a_j is defined in Example 2.1. Then applying Euler-type discretization scheme (Atkinson, 2008) on $[\tau_{j-1}, \tau_j^-]$, yields

$$S(\tau_j^-) - S(\tau_{j-1}) = -\lambda(\tau_{j-1})\Delta\tau_j S(\tau_{j-1}) . \tag{33}$$

Moreover, from (32) and (33), we have

$$S(\tau_j) - S(\tau_{j-1}) = -\lambda(\tau_j)\Delta\tau_j S(\tau_{j-1}) . \tag{34}$$

Corollary 3.3. *Under the assumptions of Theorem 3.1 in the context of Remark 3.1(ii), (26) becomes:*

$$\begin{cases} dz = 0\, dt, \quad z(\tau_{j-1}) = z_{j-1}, \quad t \in [\tau_{j-1}, \tau_j), \\ z(\tau_j) = (1-\lambda_j)(1-\alpha_j)z_{j-1} + \gamma_j, \end{cases} \tag{35}$$

and

$$z(\tau_j) = (1-\lambda_j)(1-\alpha_j)z(\tau_{j-1}) + \gamma_j . \tag{36}$$

This corollary is indeed a totally discrete-time version of hybrid dynamic system operating under discrete-time intervention process.

Using Definition 3.1 and the discrete-time iterative process (28), we introduce a couple of definitions.

Definition 3.2. Let τ_{j-1} and τ_j be a pair of consecutive observation times belonging to $[0, \mathscr{T}]$. $z(\tau_{j-1})$ stands for the number of survivals at the time τ_{j-1} for each $j \in I(1,k)$. Moreover, $z(\tau_{j-1})$ is the number of survivals under observation over the sub-interval of time $[\tau_{j-1}, \tau_j)$. $z(\tau_{j-1})\Delta\tau_j$ is the amount of time spent under observation/testing/evaluation by $z(\tau_{j-1})$ survivals over the length $\Delta\tau_j$ of time interval $[\tau_{j-1}, \tau_j)$.

Definition 3.3. For $j \in I(1,k)$, $z(\tau_{j-1}) - z(\tau_j)$ stands for the change in number of survivals over the interval of time $[\tau_{j-1}, \tau_j]$ of length $\Delta\tau_j$.

Remark 3.2. The discrete-time processes (28), (31), (34) and (36) are referred as our numerical schemes with respect to interconnected hybrid dynamic models for a survival population dynamic processes. Moreover, from (28), we will introduce three more special numerical schemes, namely, time-to-event: (i) failure/death/removal/infective, (ii) censored/withdrawn, and (iii) admission/joining/susceptible/relapsed processes. We further note that the presented numerical schemes allow "ties" with deaths/failure or censored/quiting process. In addition, the population under the presented observation/-supervision process includes the patient/objects population as a special case.

(i) For each $j \in I(1,k)$, let us assume that either τ_{j-1} and τ_j are consecutive failure/death/removal/infective times of individual/machine/species, or τ_{j-1} and τ_j are censored and failure times, respectively. For $\alpha_j = \gamma_j = \beta_j = 0$, the numerical scheme (28) for failure/death/removal/infective/etc process data set is described by

$$z(\tau_j) = (1 - \lambda(\tau_j)\Delta\tau_j)z(\tau_{j-1})\,, \tag{37}$$

and hence

$$z(\tau_j) - z(\tau_{j-1}) = -\lambda(\tau_j)z(\tau_{j-1})\Delta\tau_j\,, \tag{38}$$

where τ_{j-1} is either the failure or censored time.

Moreover, $\alpha_j = \gamma_j = \beta_j = 0$ in (28) coupled with (94) is equivalent to the Kaplan and Meier (1958) assumption, namely,

$$x(\tau_j^-) - x(\tau_j) = \text{the number of deaths at } \tau_j\,.$$

That is

$$z(\tau_{j-1}) - z(\tau_j^-) = 0 \quad \text{and} \quad z(\tau_j) = z(\tau_j^+)\,.$$

This implies that $z(t)$ is left discontinuous and right continuous at τ_j.

(ii) Let us assume that either τ_{j-1} and τ_j are consecutive censored times, or τ_{j-1} and τ_j are failure and censored times, respectively. For $\alpha_j = \beta_j = 0$, and γ_j^c stands for the number of censored objects/infectives/etc at a time τ_j. The numerical scheme (28) for censored/listed/identified process data set is described by

$$z(\tau_j) = \left(1 - \lambda(\tau_j)\Delta\tau_j\right)z(\tau_{j-1}) - \gamma_j^c\,, \tag{39}$$

where τ_{j-1} is either a failure or censored time.

Thus

$$z(\tau_j) - z(\tau_{j-1}) = -\lambda(\tau_j)z(\tau_{j-1})\Delta\tau_j - \gamma_j^c \tag{40}$$

Again, we note that $\alpha_j = \beta_j = 0, \gamma_j^c$, in the context of (94) is equivalent to the Kaplan and Meier (1958) assumption, namely,

$$z(\tau_j) = z(\tau_j^-) \quad \text{and} \quad z(\tau_j) - z(\tau_j^+) = \gamma_j^c\,.$$

This implies that $z(t)$ is left continuous and right discontinuous at τ_j.

(iii) Let us assume that τ_{j-1} is either failure or censored time, and τ_j is a joining/admitting/relapsing time. For $\alpha_j = 0$ and γ_j^a denoting the number of objects/infectives that joined the observation process at time τ_j. The numerical scheme (28) for admission/joining/sustainable/recruiting/relapsing process is

$$z(\tau_j) = \left(1 - \lambda(\tau_j)\Delta\tau_j\right)z(\tau_{j-1}) + \gamma_j^a\,. \tag{41}$$

The scheme determined by $\alpha_j = 0$ in (28) with (94) and the addition γ_j^a in (41) is equivalent to $z(\tau_j) - z(\tau_j^-) = \gamma_j^a$ and $z(\tau_j) = z(\tau_j^+)$.

(iv) Remarks (i), (ii) and (iii) remain valid for the iterative processes (28), (31) and (36).

(I) For $\alpha_j = 0 = \beta_j = \gamma_j$ in (28), (34) reduces to (38); for $\alpha_j = 0 = \beta_j = \gamma_j$, (36) reduces to $z(\tau_j) = (1-\lambda_j)z(\tau_{j-1})$.

(II) For $\alpha_j = 0 = \beta_j$ and $\gamma_j = -\gamma^c_j$ in (28), (28) reduces to (40); for $\alpha_j = 0 = \lambda_j$ and $\gamma_j = -\gamma^c_j$, (36) becomes

$$z(\tau_j) - z(\tau_{j-1}) = (1-\lambda_j)z(\tau_{j-1}) - \gamma^c_j\,. \tag{42}$$

(III) For $\alpha_j = 0 = \beta_j$ and $\gamma_j = \gamma^a_j$ in (28), and $\alpha_j = 0 = \lambda_j$ and $\gamma_j = \gamma^a_j$ in (36), (28) reduces to (41), and (36) reduces to

$$z(\tau_j) - z(\tau_{j-1}) = (1-\lambda_j)z(\tau_{j-1}) + \gamma^a_j. \tag{43}$$

4. Estimations of Risk Rate and Survival Functions

Now, we are ready to find an estimate for the hazard/risk rate and survival functions for interconnected continuous and discrete-time survival state dynamic processes. For the sake of completeness and clarity, we first introduce a couple of definitions.

Definition 4.1. For $j \in I(1,k)$, let τ_{j-1} and τ_j be consecutive change times under continuous-time state survival dynamic process. The parameter estimate at τ_j is defined by the quotient of change of objects over the consecutive time change interval $[\tau_{j-1}, \tau_j)$ and the total time spent by the objects under observation over the time interval of length $\Delta\tau_j$.

Definition 4.2. For $j \in I(1,k)$, let τ_{j-1} and τ_j be consecutive change times for discrete-time state survival dynamic process. The parameter estimate at τ_j is defined by the quotient of the change in the number of survival state over the consecutive time change interval $[\tau_{j-1}, \tau_j)$ and the number of objects at the immediate past time, that is, either the change time or the censored time.

Remark 4.1. We observe that the Definitions 4.1 and 4.2 are consistent with each other. This statement can be justified in the context of discrete-time iterative scheme (95) and the continuous and discrete-time hybrid-type descriptions of survival state dynamic model (25) and totally discrete-time hybrid dynamic system (26).

Now, we are ready to present a main result regarding parameter and survival state estimation problems. This result includes several existing results as special cases. In the following, we simply state a conceptual computational algorithm. The detailed proof is given in the supplementary section.

Theorem 4.1. *Let us assume that the conditions of Theorem 3.1 in the context of Remarks 3.1 and 3.2(i),(ii) are satisfied.*

(a) For $j \in I(1,k)$, if τ_{j-1} and τ_j are consecutive risk/failure/removal/death/non-operational times in $[\tau_0, \mathscr{T}]$ then an estimate for the hazard/risk rate function at τ_j is determined by:

$$\hat{\lambda}(\tau_j) = \frac{z(\tau_{j-1}) - z(\tau_j)}{z(\tau_{j-1})\Delta\tau_j}\,, \tag{44}$$

and an estimate for the hazard/risk rate function is

$$\hat{\lambda}(t) = \hat{\lambda}(\tau_j)\,, \quad \textit{for} \quad t \in [\tau_{j-1}, \tau_j) \quad \textit{and} \quad j \in I(1,k)\,. \tag{45}$$

(b) For $j \in I(1,k)$, if $\tau_{j-1} < \tau^c_j < \tau_j$, and τ^c_j is censored time between a pair of consecutive failure times τ_{j-1} and τ_j in $[\tau_0, \mathscr{T})$, then,

(i) a change in the number of items/subjects/thoughts that are under observation over the subinterval $[\tau_{j-1}, \tau_j)$ of the time interval of study $[\tau_0, \mathscr{T}]$ is

$$z(\tau_{j-1}) - z(\tau_j) - \gamma^c_j\,. \tag{46}$$

(ii) a total amount of time spent under the observation/testing/evaluation of $z(\tau_{j-1}) - z(\tau_j) - \gamma^c_j$ items/patients/infectives/radicals/subjects over the time interval $[\tau_{j-1}, \tau_j)$ is

$$z(\tau_{j-1})\Delta\tau^c_j + z(\tau^c_j)\Delta\tau_{jc}\,, \quad \Delta\tau_{jc} = \tau_j - \tau^c_j. \tag{47}$$

(iii) an estimate for the hazard/risk rate function at τ_j is defined as:

$$\hat{\lambda}(\tau_j) = \frac{z(\tau_{j-1}) - z(\tau_j) - \gamma_j^c}{z(\tau_{j-1})\Delta\tau_j^c + z(\tau_j^c)\Delta\tau_{jc}}\,, \tag{48}$$

and an estimate for the hazard/risk rate function is

$$\hat{\lambda}(t) = \hat{\lambda}(\tau_j)\,, \quad \text{for} \quad t \in [\tau_{j-1}, \tau_j) \quad \text{and} \quad j \in I(1,k)\,. \tag{49}$$

(iv) Moreover, an estimate for the survival function in (24) *is*

$$\hat{S}(t) = S_0 \exp\left[\sum_{m=1}^{j-1} \hat{\lambda}_m(\tau_m - \tau_{m-1}) + \hat{\lambda}_j\left(t - \tau_{j-1}\right)\right],\ t \in [\tau_{j-1}, \tau_j). \tag{50}$$

Remark 4.2. We note that if $\tau_j^c = \tau_j$ in Theorem 4.1(b), then we have "ties" between censored and failure times. In this case, $\Delta\tau_j^c = \Delta\tau_j$ and $\Delta\tau_{jc} = 0$. From this, (47) and (48) reduce to

$$z(\tau_{j-1})\Delta\tau_j\,, \tag{51}$$

and

$$\hat{\lambda}(\tau_j) = \frac{z(\tau_{j-1}) - z(\tau_j) - \gamma_j^c}{z(\tau_{j-1})\Delta\tau_j} \quad \text{for} \quad j \in I(1,k)\,. \tag{52}$$

This observation justifies Remark 3.2 regarding the mixed "ties."

In the following, we exhibit the role and scope of Theorem 4.1. This is achieved by presenting the well-known hazard/risk rate and survival functions as special cases.

Corollary 4.1. *Let us assume that conditions of Corollary 3.3 in the context of Remark 3.2(iv)(I) and (II) are satisfied.*

(a) For $j \in I(1,k)$, if τ_{j-1} and τ_j are consecutive risk/failure times in $[\tau_0, \mathscr{T}]$, then employing Remark 3.2(iv)(I) and Definitions 3.2, 3.3 and 4.2, an estimate for the risk/hazard rate function at τ_j is determined by:

$$\hat{\lambda}(\tau_j) = \frac{z(\tau_{j-1}) - z(\tau_j)}{z(\tau_{j-1})}\,, \tag{53}$$

and

$$\lambda(t) = \hat{\lambda}(\tau_j)\,, \quad t \in [\tau_{j-1}, \tau_j)\,. \tag{54}$$

Substituting (53) into (22), an estimate for the survival function is obtained as:

$$\begin{aligned} S(t) &= \prod_{i|\tau_{j-1} \le t} \left(1 - \hat{\lambda}_i\right) = \prod_{i|\tau_{j-1} \le t} \left(1 - \frac{z(\tau_{i-1}) - z(\tau_i)}{z(\tau_{i-1})}\right) \\ &= \prod_{i|\tau_{j-1} \le t} \left(1 - \frac{d_i}{z(\tau_{i-1})}\right), \quad t \ge \tau_0\,, \end{aligned} \tag{55}$$

where $d_i = z(t_{i-1}) - z(\tau_i)$ is the number of deaths over the consecutive risk/failure time interval $[\tau_{i-1}, \tau_i)$, $\tau_i \le \tau_{j-1} \le t < \tau_j$ for some $j \in I(1,k)$.

(b) For $j \in I(1,k)$, if $\tau_{j-1} < \tau_j^c < \tau_j$, and τ_j^c is censored time between a pair of consecutive risk/failure times τ_{j-1} and τ_j in $[\tau_0, \mathscr{T})$, then, employing Remark 3.2(iv)(II) and Definitions 3.2, 3.3 and 4.2, an estimate for the risk/hazard rate function at τ_j is determined by:

$$\hat{\lambda}(\tau_j) = \frac{z(\tau_{j-1}) - z(\tau_j) - \gamma_j^c}{z(\tau_j^c)}\,, \tag{56}$$

and

$$\lambda(t) = \hat{\lambda}(\tau_j)\,, \quad t \in [\tau_{j-1}, \tau_j)\,. \tag{57}$$

Substituting (56) into (22), an estimate for the survival function when τ_j^c is a censored time between consecutive failure times, τ_{j-1} and τ_j is given by:

$$S(t) = \prod_{i|\tau_{j-1}\le t}\left(1-\hat{\lambda}_i\right) = \prod_{i|\tau_{j-1}\le t}\left(1-\frac{z(\tau_{i-1})-z(\tau_i)-\gamma_i^c}{z(\tau_i^c)}\right)$$
$$= \prod_{i|\tau_{j-1}\le t}\left(1-\frac{d_i}{z(\tau_i^c)}\right), \quad t\ge\tau_0\,, \tag{58}$$

where i runs over the positive integers for which $\tau_i \le \tau_{j-1}$, $\tau_{j-1} \le t < t$ for some $j \in I(1,k)$; τ_{i-1}, τ_i are consecutive failure times for $i \in I(1, j)$, and $d_i = z(t_{i-1}) - z(\tau_i) - \gamma_i^c$ is the number of deaths over the consecutive failure time interval $[\tau_{j-1}, \tau_j)$.

Remark 4.3. (a) We remark that (55) and (58) are indeed the Kaplan and Meier (1958)-type survival estimate functions.

(b) In the literature (Kalbfleisch & Prentice, 2011; Lawless, 2011), the numbers in the denominator of (55) and (58) are referred to as the number of individuals at rist at τ_{j-1} and τ_j^c respectively. Denoting this by n_j, we can write both (55) and (58) as

$$S(t) = \prod_{i|\tau_{j-1}\le t}\left(\frac{n_i-d_i}{n_i}\right). \tag{59}$$

This is the well-known formula cited in the literature (Kalbfleisch & Prentice, 2011; Lawless, 2011).

(c) From Remark 2.4, we obtain

$$\hat{\Lambda}(t) = \sum_{\tau_j\le t}\hat{\lambda}_j = \sum_{\tau_j\le t}\frac{d_j}{n_j}, \quad t\ge\tau_0\ , \tag{60}$$

where

$$n_j = \begin{cases} z(\tau_{j-1}) & \text{if there are no censors in} \quad [\tau_{j-1},\tau_j)\,,\\ z(\tau_j^c) & \text{if} \quad \tau_j^c \quad \text{is a censored time in} \quad [\tau_{j-1},\tau_j)\,. \end{cases} \tag{61}$$

This is the estimator introduced by Nelson (1969) and Aalen (1978). These special cases exhibit the role and scope of the presented innovative alternative dynamic approach.

In the following, we state a corollary that further illustrates the role and scope of our dynamic approach. Further details regarding the proof is outlined in the supplementary section.

Corollary 4.2. *Let us assume that the conditions of Corollary 3.2 and Example 2.1 in the context of Remark 3.2(iii) are satisfied. For $j \in I(1,n)$, if τ_{j-1} and τ_j are consecutive risk/failure times in $[\tau_0, \mathscr{T}]$, then employing Definitions 3.2, 3.3 and 4.2, an estimate for the risk/hazard rate function at τ_j is determined by:*

$$\hat{\lambda}(\tau_j) = \frac{a_j}{(1-A_{j-1})\Delta\tau_j}\,, \tag{62}$$

and

$$\hat{\lambda}(t) = \hat{\lambda}(\tau_j)\,, \quad t\in[\tau_{j-1},\tau_j)\,, \tag{63}$$

where a_j and A_{j-1} are defined in Example 2.1.

Moreover, an estimate for the survival function is represented by

$$\hat{S}(t) = S_{j-1}\exp\left[-\hat{\lambda}_j(t-\tau_{j-1})\right], \quad \textit{for} \quad t\in[\tau_{j-1},\tau_j)\,. \tag{64}$$

Remark 4.4. The PEXE of Kitchin et al. (1980), as well as Kim and Proschan (1991) is undefined beyond the last observed failure time. To rectify that, Malla and Mukerjee (2010) provided the following exponential tail hazard/risk rate estimate:

$$\hat{\lambda}_{\text{tail}} = \frac{\exp(-\hat{\Lambda}_m)}{\sum_{i=1}^{m}(I_j-J_j)} \tag{65}$$

where

$$I_j = \int_{\tau_{j-1}}^{\tau_j} \hat{S}^{KM}(t)\mathrm{d}t = (1 - A_{j-1})(\tau_j - \tau_{j-1})$$

and

$$J_j = \int_{\tau_{j-1}}^{\tau_j} \hat{S}^{MN}(t) = \exp(-\hat{\Lambda}_{j-1})\frac{(1 - A_{j-1})(\tau_j - \tau_{j-1})}{a_j}\left[1 - \exp\left(-\frac{a_j}{1 - A_{j-1}}\right)\right] .$$

Thus, under the following assumptions: (i) no ties among the failure times, (ii) the last observation is uncensored, a new PEXE of Malla and Mukerjee (2010) is given by

$$S(t) = \begin{cases} \exp(-\Lambda_{j-1})\exp\left(\frac{-a_j(t-\tau_{j-1})}{(1-A_{j-1})(\tau_j-\tau_{j-1})}\right), & \tau_{j-1} \le t < \tau_j, \quad j \in I(1,m) \\ \exp(-\hat{\Lambda}_m)\exp(-\hat{\lambda}_{\text{tail}}(t-\tau_m)), & \tau_m \le t < \infty . \end{cases} \tag{66}$$

We further note that the presented dynamic approach does not require the failure function to be invertible.

5. Multiple Censored Times between Consecutive Failure Times

In this section, we further apply the conceptual dynamic results developed in Sections 2 and 3 to multiple censored times between consecutive failure times. We present a result that provides a very general algorithm for estimating a hazard rate function for multiple censoring times between consecutive failure times τ_{j-1} and τ_j with $\tau_{j-1}, \tau_j \in [\tau_0, \mathscr{T})$. We further note that the presented results in this section extend the results of Section 4 in a systematic and unified manner.

Theorem 5.1. *Let the hypotheses of Theorem 3.1 in the context of Remarks 3.1, 3.2(i) and 3.2(ii) be satisfied. For each $j \in I(1,m)$, let τ_{j-1} and τ_j be consecutive failure times. Let $\{\tau_{j-1l}\}_{l=1}^{k_j}$ be a finite sequence of censored time observations over a time interval $[\tau_{j-1}, \tau_j]$. Let γ_j^l be the number of objects censored at time τ_{j-1l}, for $l \in I(1,k_j)$ and $\{\gamma_j^l\}_{l=1}^{k_j}$ be a corresponding sequence of observed number of objects/species/patients/etc. Then*

1. *$z(\tau_{j-1}) - z(\tau_j) - \sum_{l=1}^{k_j} \gamma_j^l$ is a change in the number of items/subjects that is under the observation over the sub-interval $[\tau_{j-1}, \tau_j]$ of the time interval of study $[\tau_0, \mathscr{T}]$*
2. *$\sum_{l=1}^{k_j+1} z(\tau_{j-1l-1})\Delta(\tau_{j-1l})$ is a total amount of time spent under the observation/testing/evaluation/monitoring of $z(\tau_{j-1l-1})$ items/patients/ infectives/subjects on the interval $[\tau_{j-1l-1}, \tau_{j-1l})$ for $l \in I(1,k_j))$ and $j \in I(1,n)$.*
3. *an estimate for the hazard rate function at τ_j is determined by*

$$\hat{\lambda}(\tau_j) = \frac{z(\tau_{j-1}) - z(\tau_j) - \sum_{l=1}^{k_j} \gamma_j^l}{\sum_{l=1}^{k_j+1} z(\tau_{j-1l-1})\Delta(\tau_{j-1l})}, \tag{67}$$

and an estimate for the hazard rate function is

$$\hat{\lambda}(t) = \hat{\lambda}(\tau_j), \quad \text{for} \quad t \in [\tau_{j-1}, \tau_j) \quad \text{and} \quad j \in I(1,n) \tag{68}$$

Proof. The detailed proof of Theorem 5.1 is given in the supplementary section 8.

Corollary 5.1. *Under the conditions of Theorem 5.1 and assumptions of Corollary 3.3 in the context of Remark 3.2(iv), an estimate for the hazard rate function at τ_j is determined by*

$$\hat{\lambda}(\tau_j) = \frac{z(\tau_{j-1}) - z(\tau_j) - \sum_{l=1}^{k_j} \gamma_j^l}{z(\tau_{j-1k_j})}, \tag{69}$$

and an estimate for the hazard rate function is $\hat{\lambda}(t) = \hat{\lambda}(\tau_j)$, for $t \in [\tau_{j-1}, \tau_j)$ and $j \in I(1,n)$. An estimate for the survival function is thus given by

$$\hat{S}(t) = \prod_{i|\tau_{j-1}<t} (1 - \hat{\lambda}(\tau_i)),\ t \ge \tau_0,\ \tau_i \le \tau_{j-1} \le t < \tau_j \text{ for some } j \in I(1,n). \tag{70}$$

Corollary 5.2. *Under the conditions of Theorem 5.1 and estimate for the cumulative hazard/risk rate and survival functions are represented by:*

$$\hat{\Lambda}(t,\tau_0) = \sum_{m=1}^{j-1} \hat{\lambda}_m(\tau_m - \tau_{m-1}) + \hat{\lambda}_j\left(t - \tau_{j-1}\right)\ ,\ t \in [\tau_{j-1}, \tau_j)$$

and

$$\hat{S}(t,\tau_0) = S_0 \exp\left[\sum_{m=1}^{j-1} \hat{\lambda}_m(\tau_m - \tau_{m-1}) + \hat{\lambda}_j\left(t - \tau_{j-1}\right)\right]\ ,\ t \in [\tau_{j-1}, \tau_j)$$

for $t \geq \tau_0,\ \tau_{j-1} \leq t < \tau_j$ for some $j \in I(1,n)$.

Remark 5.1. (a) We remark that the innovative dynamic approach for the development of computational parameter estimation algorithm (67) is an alternative approach for the algorithm proposed by Kim and Proschan (1991).

(b) The estimates (67) in the context of (20) yields the estimate obtained by Kulasekera and White (1996) as special cases.

(c) For continuous-time interconnected hybrid state survival dynamic process, if $k_j = 0$, for some $j \in I(1,n)$, then $l = 0$ and $\gamma_j^0 = 0$ and (67) reduces to (44). On the other hand, if $k_j = 1$ for some $j \in I(1,n)$, then $l = 0$ and $\gamma_j^1 = \gamma_j^c$ and (67) implies (48).

(d) For discrete-time interconnected hybrid state survival dynamic process, if $k_j = 0$, for some $j \in I(1,n)$, then $l = 0$ and $\gamma_j^0 = 0$ and (69) reduces to (53). On the other hand, if $k_j = 1$, for some $j \in I(1,n)$, then $l = 0$ and $\gamma_j^1 = \gamma_j^c$ and (69) implies (56).

The presented innovative approach of parameter and state estimation includes the Thaler (1984)-type hazard rate estimation problem as a particular case. To justify this statement, we first introduce a concept of hazard/risk rate function for responder and non-responder states. In addition, we state a corollary of Theorem 5.1 without its proof. The proof is outlined in the supplementary section.

Definition 5.1. For $i \in I(0,1)$, Let $\lambda_0(t)$ and $\lambda_1(t)$ represent the hazard/risk rate functions in the non-responder and responder states, respectively, at time t (Thaler, 1984).

Corollary 5.3. *Let us assume that the conditions of Corollary 3.1 in the context of Remark 3.2(i) are satisfied. For $j \in I(1,n_0)$, let τ_{j-1} and τ_j be consecutive risk/failure times in state 0. For $j' \in (1,n_1)$, let $\tau_{j'-1}$ and $\tau_{j'}$ be consecutive failure times in state 1. Let $z_0(\tau_j)$ be the number of survivals at τ_j in state 0. Let $z_1(\tau_{j'})$ be the number of survivals at $\tau_{j'}$ in state 1. Then an estimate for the hazard/risk rate function at τ_j is determined by:*

$$\hat{\lambda}_0(\tau_j) = \frac{\sum\limits_{m=1}^{j} [z_0(\tau_{m-1}) - z_0(\tau_m)]}{\sum\limits_{m=1}^{j} z_0(\tau_{m-1})\Delta\tau_m} = \frac{\sum\limits_{m=1}^{j} d_{0j}}{\sum\limits_{m=1}^{j} z_0(\tau_{m-1})\Delta\tau_m}\ , \tag{71}$$

where d_{0j} is the number of deaths/failures at the jth distinct failure time in state i, and an estimate for the hazard rate function is

$$\hat{\lambda}_0(t) = \hat{\lambda}_0(\tau_j)\ , \quad for \quad t \in [\tau_{j-1}, \tau_j) \quad and \quad j \in I(1,n_0)\ . \tag{72}$$

An estimate for the hazard/risk rate function at $\tau_{j'}$ is determined by:

$$\hat{\lambda}_1(\tau_{j'}) = \frac{\sum\limits_{m=1}^{j'} [z_1(\tau_{m-1}) - z_1(\tau_m)]}{\sum\limits_{m=1}^{j'} z_1(\tau_{m-1})\Delta\tau_m} = \frac{\sum\limits_{m=1}^{j} d_{1j'}}{\sum\limits_{m=1}^{j} z_1(\tau_{m-1})\Delta\tau_m}\ , \tag{73}$$

where $d_{1j'}$ is the number of deaths/failures at the j'th distinct failure time in state 1, and an estimate for the hazard rate function is

$$\hat{\lambda}_1(t) = \hat{\lambda}_1(\tau_{j'})\ , \quad for \quad t \in [\tau_{j'-1}, \tau_{j'}) \quad and \quad j' \in I(1,n_1)\ . \tag{74}$$

The hazard/risk ratio rate function estimate is given by:

The corresponding estimate of the log *hazard/risk rate ratio function for patients currently in a response compared to a nonresponse state is given by:*

$$\hat{\rho}(t) = \ln\left[\frac{\hat{\lambda}_0(\tau_j)}{\hat{\lambda}_1(\tau_{j'})}\right] \text{ for}, \tau_{j-1} < t \leq \tau_j \text{ and } \tau_{j'-1} \leq t < \tau_{j'} \,. \tag{75}$$

Remark 5.2. We remark that (71), (73) and (75) are identical to the result obtained in Thaler (1984). Moreover, the estimates in (71), (73) and (75) were obtained in the framework of an innovative dynamic approach.

In the following, we state a general theorem that provides a theoretical estimate for the hazard/risk rate function between two successive change point times, τ_{j-1} and τ_j.

Theorem 5.2. *Let the hypothesis of Theorem 5.1 be satisfied. Let $\{T_i^j\}_{i=1}^n$ be a sequence of times(failure/ censor/arrival) that fall between the change point times τ_{j-1} and τ_j for $j = I(1,k)$. Then an estimate for the hazard rate function at τ_j is determined by*

$$\hat{\lambda}(\tau_j) = \hat{\lambda}(\tau_j) = \frac{z(\tau_{j-1}) - z(\tau_j) - \sum\limits_{m=1}^{l} \eta_m^j}{\sum\limits_{m=1}^{l+1} z(T_m^j)\Delta(T_m^j)} \,, \quad j \in I(1,k+1)\,. \tag{76}$$

where

$$\eta_m^j = \begin{cases} 0 & \textit{if} \quad T_m^j \textit{ is failure time} \\ \gamma_m^{jc} & \textit{if} \quad T_m^j \textit{ is censored time} \\ -\gamma_m^{ja} & \textit{if} \quad T_m^j \textit{ is arrival time} \end{cases}; \tag{77}$$

γ_m^{jc} is the number of objects/items/individuals censored at time T_m^j; γ_m^{ja} is the number of objects/items/individuals joining/arriving at time T_m^j, and an estimate for the hazard rate function is $\lambda(t) = \hat{\tau}_j$ for $t \in [\tau_{j-1}, \tau_j)$.

Proof. The proof of Theorem 5.2 is outlined in the supplementary section.

6. Computational Algorithms

In this section, we outline very general conceptual computational, data organizational and simulation schemes. The computational and simulation algorithms are based on fundamental theoretical result (Theorem 5.1) developed in Section 5.

6.1 Conceptual Computational Parameter and State Estimation Scheme

The theoretical computational algorithm for interconnected continuous-time hybrid dynamic process (24), is as follows:

$$z(\tau_{j-1}) - z(\tau_j) - \sum_{l=1}^{k_j} \gamma_j^l = \hat{\lambda}(\tau_j) \sum_{l=1}^{k_j+1} z(\tau_{j-1l-1})\Delta(\tau_{j-1l}), \tag{78}$$

and the conceptual computational algorithm for totally discrete-time hybrid dynamic process (26) is

$$z(\tau_{j-1}) - z(\tau_j) - \sum_{l=1}^{k_j} \gamma_j^l = \hat{\lambda}(\tau_j) z(\tau_{j-1k_j})\,. \tag{79}$$

Here $\mathscr{P}_0^{\mathscr{T}} : \tau_0 < \tau_1 < \ldots < \tau_{j-1} < \tau_j < \ldots < \tau_n$ is a partition of failure times over the time interval $[0, \mathscr{T})$. Let $\mathscr{P}_j$ be a partition corresponding to a given finite sequence of censored times over the failure time interval $[\tau_{j-1}, \tau_j)$, and let it be represented by

$$\mathscr{P}_j : \tau_{j-1} = \tau_{j-10} < \tau_{j-11} < \ldots < \tau_{j-1l-1} < \tau_{j-1l} < \ldots < \tau_{j-1k_{j-1}} < \tau_{j-1k_j}\,. \tag{80}$$

For $j \in I(1,n)$, λ is the hazard rate function; $z(t)$ stands for the number of survivals at time t; γ_j^l denotes the number of objects censored at the time τ_{j-1l}, $j \in I(1,m)$ and $l \in I(0,k_j)$, $k_j \in I(0,\infty)$.

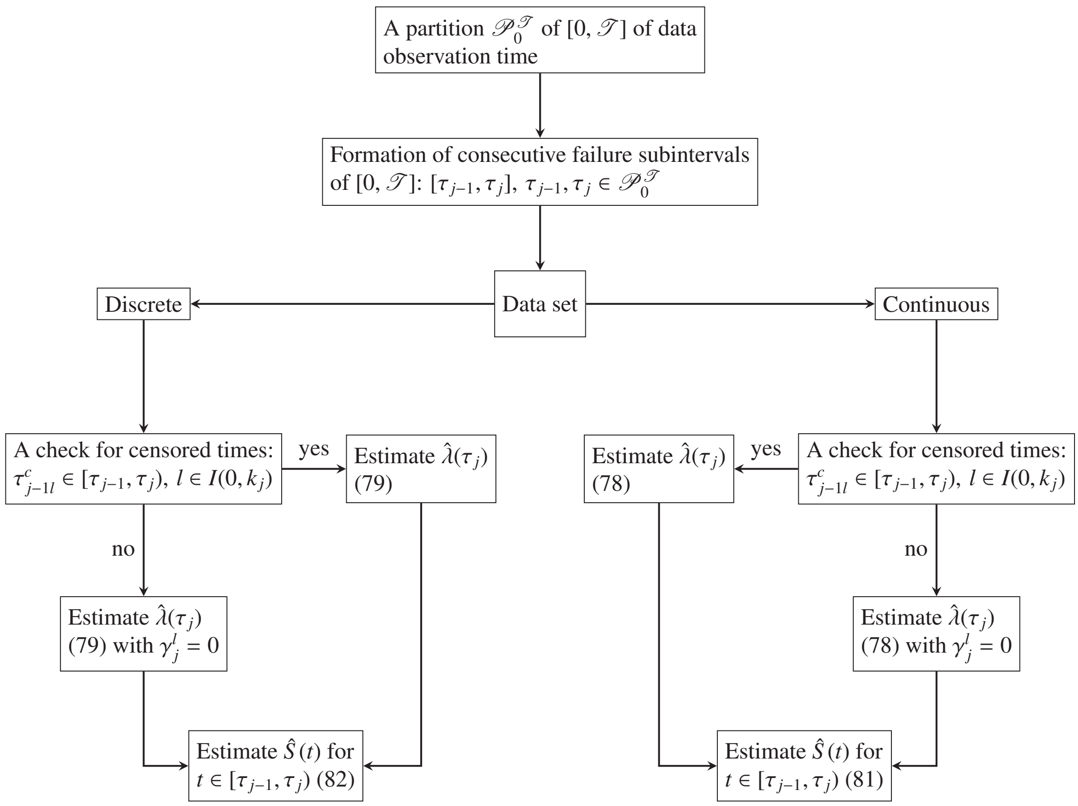
For the continuous-time hybrid dynamic process (24), an estimate of the survival function is represented by

$$\hat{S}(t,\tau_0) = S_0 \exp\left[\sum_{m=1}^{j-1} \hat{\lambda}_m(\tau_m - \tau_{m-1}) + \hat{\lambda}_j\left(t - \tau_{j-1}\right)\right], \; t \in [\tau_{j-1}, \tau_j) \text{ for } t \geq \tau_0\,. \tag{81}$$

For the totally discrete-time hybrid dynamic process (26), an estimate of the survival function is represented by

$$\hat{S}(t) = \prod_{i|\tau_{j-1}<t} (1 - \hat{\lambda}(\tau_i)),\ t \geq \tau_0. \tag{82}$$

First, we construct a detailed flowchart for the general conceptual computational algorithm developed in Section 5.



Flowchart 1. Conceptual Computational Algorithm

We observe that the conceptual computational algorithm (Flowchart 1) is composed of two sub-conceptual computational algorithms, namely, continuous-time and discrete-time hybrid dynamic processes.

6.2 Conceptual and Computational Simulation Algorithms

A pseudocode for a simulation scheme for both interconnected continuous-time and totally discrete-time hybrid dynamic processes are outlined below:

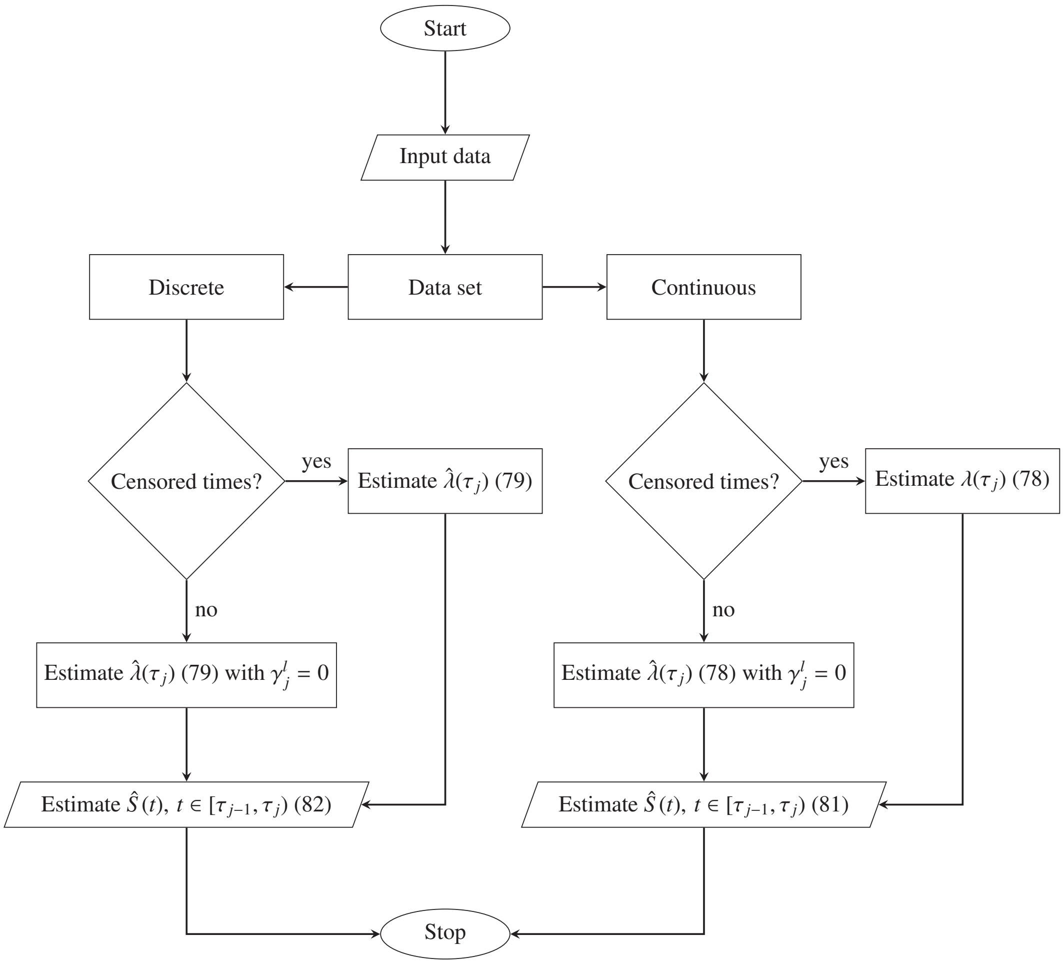
for $j = 1$ to N **do**
Compute k_j, $z(\tau_{j-1}), z(\tau_j)$
  if $k_j = 0$ **then**
Compute $z(\tau_{j-1})\Delta\tau_j$
  else
Compute $\sum_{l=1}^{k_j} \gamma^l_j$, $\sum_{l=1}^{k_j+1} z(\tau_{j-1l-1})\Delta(\tau_{j-1l})$
  end if
Compute $\hat{\lambda}(\tau_j)$, $\hat{S}(t)$
end for

Simulation Scheme 1a. Pseudocode for interconnected continuous-time hybrid dynamic process

for $j = 1$ to N **do**
Compute k_j, $z(\tau_{j-1}), z(\tau_j)$
  if $k_j = 0$ **then**
Compute $z(\tau_{j-1})$
  else
Compute $\sum_{l=1}^{k_j} \gamma^l_j$, $z(\tau_{j-1k_j})$
  end if
Compute $\hat{\lambda}(\tau_j)$, $\hat{S}(t)$
end for

Simulation Scheme 1b. Pseudocode for totally discrete-time hybrid dynamic process

Moreover, a flowchart for the simulation algorithm for parameter and state estimation problems for interconnected continuous-time (24) and discrete-time (26) hybrid dynamic processes are provided in Flowchart 2.



Flowchart 2. Simulation Algorithm for interconnected hybrid dynamic processes

We note that flowchart for simulation algorithm (Flowchart 2) is composed of two sub-simulation algorithms, namely, continuous-time and totally discrete-time hybrid dynamic processes.

In the following, using the conceptual computational algorithm, we exemplify our theoretical procedure by estimating hazard rate and survival functions of two data sets in a systematic and unified way. The first data set can be found in Kaplan and Meier (1958).

Illustration 6.1. Suppose that out of a sample of 8 items the following are observed:

Table 1. Dataset used by Kaplan and Meier (1958)

Order of Observation	Time of Cessation of Observation	Cause of Cessation	Time Notation
1	0.8	Failure	τ_1
2	1.0	Censored	τ_{11}
3	2.7	Censored	τ_{12}
4	3.1	Failure	τ_2
5	5.4	Failure	τ_3
6	7.0	Censored	τ_{31}
7	9.2	Failure	τ_4
8	12.1	Censored	

We note that the data set in Table 1 is for the totally discrete-time hybrid time-to-event dynamic process (26). In view of this, we apply the totally discrete-time parameter and state estimation schemes (79) and (82). In short, we utilize the discrete-time conceptual computational sub-algorithm (Simulation Scheme 1b) "pseudocode" and simulation sub-algorithm (Flowchart 2).

For $t \in [\tau_0, \tau_1)$, there are no censored times between $[\tau_0, \tau_1)$. Therefore, $k_j = 0$, and from Remark 5.1(d) and hence using (79) we have

$$\hat{\lambda}(\tau_1) = \hat{\lambda}_1 = \frac{z(\tau_0) - z(\tau_1)}{z(\tau_0)} = \frac{1}{8}\,.$$

Utilizing (82), the corresponding survival function is given by

$$\hat{S}(t) = \begin{cases} 1\,, & \text{for} \quad t \in [\tau_0, \tau_1)\,, \\ 1 - \lambda_1 = \frac{7}{8}\,, & \text{for} \quad t = \tau_1\,. \end{cases}$$

For $t \in [\tau_1, \tau_2)$, we note that there are two censored times between τ_1 and τ_2. So, $k_j = k_2 = 2$. Hence

$$\sum_{l=1}^{2} \gamma_2^l = \gamma_2^1 + \gamma_2^2 = 1 + 1 = 2\,.$$

Also, $z(\tau_{j-1k_j}) = z(\tau_{12}) = 5$. Thus, from Remark 5.1(d) and hence applying (79), we have

$$\hat{\lambda}(\tau_2) = \hat{\lambda}_2 = \frac{z(\tau_1) - z(\tau_2) - \sum\limits_{l=1}^{2} \gamma_2^l}{z(\tau_{12})} = \frac{1}{5}\,.$$

Utilizing (82), the corresponding survival function is thus given by

$$\hat{S}(t) = \begin{cases} \frac{7}{8}\,, & \text{for} \quad t \in [\tau_1, \tau_2)\,, \\ \prod\limits_{k|\tau_j \le t} (1 - \hat{\lambda}_j) = \prod\limits_{j=1}^{2} (1 - \hat{\lambda}_j) = \frac{7}{10}\,, & \text{for} \quad t = \tau_2\,. \end{cases}$$

There is no censoring time between the interval $[\tau_2, \tau_3) = [3.1, 5.4)$. Therefore, $k_j = 0$, and from Remark 5.1(d) and hence using (79) we obtain

$$\hat{\lambda}(\tau_3) = \frac{z(\tau_2) - z(\tau_3)}{z(\tau_2)} = \frac{1}{4}\,.$$

Once again, utilizing (82), the corresponding survival function is thus given by

$$\hat{S}(t) = \begin{cases} \frac{7}{10}\,, & \text{for} \quad t \in [\tau_2, \tau_3)\,, \\ \prod\limits_{j=1}^{3} (1 - \hat{\lambda}_j) = \frac{21}{40}\,, & \text{for} \quad t = \tau_3\,. \end{cases}$$

Continuing in this manner, we record the estimates for hazard rate and survival functions in the following table with the last column exhibiting the survival function estimate as obtained by Kaplan and Meier (1958).

Table 2. Kaplan and Meier Survival estimates for data set given in Kaplan and Meier (1958).

Failure Times τ_j	Survivals $z(\tau_j)$	Hazard Rate Function $\hat{\lambda}(\tau_j)$	Survival Function $\hat{S}(\tau_j)$
0.8	7	1/8	7/8
3.1	4	1/5	7/10
5.4	3	1/4	21/40
9.2	1	1/2	21/80
(12.1)	0	1/2	21/80

Using the dataset in Kim and Proschan (1991) and theoretical computational algorithm, Theorem 5.1, we illustrate the estimation of hazard rate and survival functions, systematically.

Illustration 6.2. Suppose that seven items (new) are put on test at time 0. Each item is observed until it fails or until it is withdrawn, whichever occurs first. The resulting set of observation (Kim & Proschan, 1991) is shown in Table 3 in order of occurrence.

Table 3. Data from Kim and Proschan (1991)

Order of Observation	Time of Cessation of Observation	Cause of Cessation	Time Notation	Finite sequence of censored Time	Size of sequence	Number of Censored
0	0					
1	2.0	Failure	$\tau_1 = \tau_{01} = \tau_{10}$			
2	3.5	Censored	τ_{11}	$\{\tau_{j-1l}\}_{l=1}^{2}$	$k_2 = 2$	$\{\gamma_2^l\}_{l=1}^{2}$
3	4.5	Censored	τ_{12}			
4	6.2	Failure	$\tau_2 = \tau_{13} = \tau_{20}$			
5	8.0	Censored	τ_{21}	$\{\tau_{j-1l}\}_{l=1}^{1}$	$k_3 = 1$	$\{\gamma_3^l\}_{l=1}^{2}$
6	8.8	Failure	$\tau_3 = \tau_{22}$			
7	11.3	Failure	τ_4			

The data set in Table 3 is for the interconnected continuous-time hybrid dynamic time-to-event dynamic process (24). In view of this, we apply the continuous-time parameter and state estimation schemes (78) and (81). In short, we utilize the continuous-time conceptual computational sub-algorithm (Simulation Scheme 1a) "pseudocode" and simulation sub-algorithm (Flowchart 2).

For $[0, \tau_1)$, since there are no censored times in between $[0, \tau_1)$, $k_j = k_1 = 0$. Thus from Remark 5.1(c) and using (78) we have

$$\hat{\lambda}(\tau_1) = \frac{z(\tau_0) - z(\tau_1)}{z(\tau_0)(\tau_{01} - \tau_0)} = \frac{1}{14}.$$

Thus $\hat{\lambda}(t) = \frac{1}{14} \approx 0.0714$ for $t \in [\tau_0, \tau_1) = [0, 2.0)$.

For the estimate on $[\tau_1, \tau_2) = [2.0, 6.2)$, we note that there are two censoring times between $[\tau_1, \tau_2)$, hence $k_j = k_2 = 2$ and

$$\sum_{l=1}^{2} \gamma_2^l = \gamma_2^1 + \gamma_2^2 = 1 + 1 = 2.$$

Thus from Remark 5.1(c) and thus applying (78), we have

$$\hat{\lambda}(\tau_2) = \frac{z(\tau_1) - z(\tau_2) - \sum\limits_{l=1}^{k_2} \gamma_2^l}{\sum\limits_{l=1}^{k_2+1} z(\tau_{1l-1})\Delta\tau_{1l}} = \frac{z(\tau_1) - z(\tau_2) - \sum\limits_{l=1}^{2} \gamma_2^l}{\sum\limits_{l=1}^{3} z(\tau_{1l-1})\Delta\tau_{1l}} = \frac{1}{20.8}.$$

Thus, $\hat{\lambda}(t) = \frac{1}{20.8}$, for $t \in [2.0, 6.2)$.

On the interval $[\tau_2, \tau_3) = [6.2, 8.8)$, we have only one censoring time in between the two failure times. So, $k_j = k_3 = 1$. Thus from Remark 5.1(c) and hence, using (67), we obtain

$$\hat{\lambda}(\tau_3) = \frac{z(\tau_2) - z(\tau_3) - \sum\limits_{l=1}^{1} \gamma_3^l}{\sum\limits_{l=1}^{2} z(\tau_{2l-1})\Delta\tau_{2l}} = \frac{3 - 1 - 1}{z(\tau_{20})\Delta\tau_{21} + z(\tau_{21})\Delta\tau_{22}} = \frac{1}{7}.$$

Hence, $\hat{\lambda}(t) = \frac{1}{7}$, for $t \in [6.2, 8.0)$.

There is no censoring in the interval $[\tau_3, \tau_4)$. Thus,

$$\hat{\lambda}(\tau_4) = \frac{z(\tau_3) - z(\tau_4)}{z(\tau_3)\Delta\tau_4} = \frac{1}{2.5},$$

which implies that $\hat{\lambda}(t) = \frac{1}{2.5} = 0.4$, for $t \in [8.0, 11.3)$.

Following this estimation procedure we have

$$\hat{\lambda}(t) = \begin{cases} 0.0714 & 0 \leq t < \tau_1 = 2 \\ 0.0481 & \tau_1 \leq t < \tau_2 = 6.8 \\ 0.1429 & \tau_2 \leq t < \tau_3 = 8.8 \\ 0.4 & \tau_3 \leq t < \tau_4 = 11.3\,. \end{cases} \tag{83}$$

To obtain the estimate of survival function, we use (81) or we apply the solution process described in Section 2 regarding (7) and obtain exponential pieces on successive intervals between failure times that are joined to form a continuous function. Thus,

$$\hat{S}(t) = \begin{cases} \exp(-0.0714t)\,, & 0 \leq t < 2 \\ \exp\left[-0.1429 - 0.0481(t-2)\right]\,, & 2 \leq t < 6.2 \\ \exp\left[0.3448 - 0.1429(t-6.2)\right]\,, & 6.2 \leq t < 8.8 \\ \exp\left[0.4591 - 0.4(t-8.8)\right]\,, & 8.8 \leq t < 11.3 \\ \text{no estimator}, & t \geq 11.3 \end{cases} \tag{84}$$

Remark 6.1. These are the same results obtained by using the method proposed Kim and Proschan (1991).

7. Conclusions

Most of the research work in the area of survival and reliability analysis is centered around the probabilistic analysis approach. In general, a closed-form solution is not feasible. In addition, a hazard rate function is nonlinear in covariate state processes and non-stationary. The presented linear hybrid deterministic dynamic modeling is more suitable for a complex time-to-event processes. This innovative approach does not require a closed-form solution distribution. The influence of both continuous and discrete-time states can be easily incorporated as an interconnected hybrid dynamic model for time-to-event processes. In fact, it allows to have a time-varying covariate state influence on the dynamic of a complex survival/reliability of systems. The influence of human mobility, electronic communications, rapid technological changes, advancements in biological, engineering, medical, military, physical and social sciences is motivated to initiate, formulate and to develop an innovative interconnected alternative modeling approach for time-to-event processes in biological, chemical, engineering, epidemiological, medical, multiple-markets and social dynamic processes through discrete-time intervention processes. The presented innovative modeling approach further enhanced our motivation to develop state and parameter estimation procedures. Moreover, the parameter and state estimation approach is dynamic. The dynamic nature rather than the existing algebraic approach plays a very significant role in state and parameter estimation problems in systematic and unifying way. The discrete-time dynamic is exhibited by the two flowcharts and Simulation algorithms 1(a) and 1(b). Furthermore, the significance of the conceptual computational algorithms are also exhibited by illustrations. At the initial level of our objective, we began with a very simple observation of the probabilistic definition of the survival function. This has led to the development of this approach. The role and scope of the presented dynamic approach is exhibited through several existing results (Han et al., 2014; Malla & Mukerjee, 2010; Kim & Proschan, 1991; Thaler, 1984; Aalen, 1978; Nelson, 1969; Kaplan & Meier, 1958) as corollaries, illustrations and remarks. In fact, the full force of the role and scope of hybrid deterministic modeling for time-to-event processes is currently being explored (Appiah E. A. *Time-To-Event Dynamic Processes: Modeling, Methods and Estimations*-Ph.D Dissertation, 2017) for both deterministic and stochastic nonlinear and non-stationary hybrid modeling for time-to-event processes. Furthermore, a complex time-to-event dynamic study is also currently undertaken by Ladde and his team. These developed results will be reported elsewhere.

8. Supplements: Proofs of Theorems

In this supplementary section, proofs of a few theorems and corollaries stated in sections 2, 3, 4 and 5 are presented.

***Proof of Theorem 2.1*:** The theorem is proved by the principle of mathematical induction (PMI) (Ladde & Ladde, 2012). From (11), for $j = 1$, we have

$$\mathrm{d}x = [-\alpha(t)\,x + \gamma(t)]\mathrm{d}\beta(t),\ \ x(t_0) = x_0,\ \ t \in [\tau_0, \tau_1)\,.$$

From (10) and the definition of Riemann-Stieltjes integral (Apostol, 1974), we have

$$x(t) - x(\tau_0) = \int_{\tau_0}^{t} [-\alpha(s)\,x(s) + \gamma(s)]\mathrm{d}\beta(s) = 0,\ \text{for}\ t \in [\tau_0, \tau_1)\,. \tag{85}$$

We define

$$x(t) = x(t, \tau_0, x_0) = x_0(t, \tau_0, x_0)\,, \quad x_0(\tau_0) = x_0, \text{ for } t \in [\tau_0, \tau_1)\,. \tag{86}$$

From (10), (11), (85), and $x_0(t, \tau_0, x_0) = x_0(\tau_1^-, \tau_0, x_0)$ for $t \in [\tau_0, \tau^-]$, we have

$$x_0(\tau_1) - x_0(\tau_0) = 0 + \int_{\tau_1^-}^{t} [-\alpha(s)\, x(s) + \gamma(s)]\, \mathrm{d}\beta(s), \text{ for } t \in [\tau_0, \tau_1]\,.$$

From this, the continuity of α and γ, the definitions of Riemann-Stieltjes integral (Apostol, 1974) and the initial value problem (Ladde & Ladde, 2012), we have

$$\begin{aligned} x_0(\tau_1, \tau_0, x_0) &= x_0(\tau_0) + \beta(\tau_1)[-\alpha(t_1^*)x(t_1^*) + \gamma(t_1^*)] - \beta(t_1^*)[-\alpha(t_1^*)x(t_1^*) + \gamma(t_1^*)] \\ &= x_0(\tau_0) - \alpha_1 x_0(\tau_1^-, \tau_0, x_0) + \gamma_1\,, \end{aligned} \tag{87}$$

for $t_1^* \in [\tau_1^-, \tau_1]$. From (87) and setting $x_0(\tau_1, \tau_0, x_0) = x(\tau_1) = x_1$ and again $x(\tau_1^-, \tau_0, x_0) = x_0$, we obtain

$$\begin{aligned} x_1 &= x(\tau_1^-, \tau_0, x_0) - \alpha_1 x(\tau_1^-, \tau_0, x_0) + \gamma_1 \\ &= (1 - \alpha_1)x_0 + \gamma_1\,. \end{aligned} \tag{88}$$

Continuing the above argument, we can establish the induction hypothesis (Ladde & Ladde, 2012) as:

$$x_j = \Phi(\tau_j, \tau_0)x_0 + \sum_{i=1}^{j} \Phi(\tau_j, \tau_i)\gamma_i\,, \quad \text{for} \quad x(\tau_j) = x_j\,,$$

where

$$\Phi(\tau_j, \tau_i) = \prod_{k=i}^{j}(1 - \alpha_k)\,, \Phi(\tau_i, \tau_i) = 1 \quad \text{for} \quad i \in I(0, n)\,.$$

Now, we consider

$$\mathrm{d}x = [-\alpha(t)\, x + \gamma(t)]\, \mathrm{d}\beta(t), \quad x(\tau_j) = x_j,\ t \in [\tau_j, \tau_{j+1})\,.$$

From the definitions of x_j and Φ, and using the above argument, one can establish the following:

$$x_j(t) = x(t, \tau_j, x_j) = \prod_{k=1}^{j}(1 - \alpha_k)x_0 + \sum_{i=1}^{j-1} \Phi(\tau_j, \tau_i)\gamma_i + \gamma_j \quad \text{for } t \in [\tau_j, \tau_{j+1})\,. \tag{89}$$

Hence

$$\begin{cases} x(\tau_{j+1}^-, \tau_j, x_j) = \prod_{k=1}^{j}(1 - \alpha_k)x_0 + \sum_{i=1}^{j} \Phi(\tau_j, \tau_i)\gamma_i\,, \\ x_{j+1}(\tau_{j+1}, \tau_j, x_j) = (1 - \alpha_{j+1})x_j + \gamma_{j+1}\,. \end{cases} \tag{90}$$

Therefore, from (89) and (90), we have

$$\begin{aligned} x_{j+1} &= (1 - \alpha_{j+1})x_j + \gamma_{j+1} \\ &= \prod_{k=1}^{j+1}(1 - \alpha_k)x_0 + \sum_{i=1}^{j+1} \Phi(\tau_{j+1}, \tau_i)\gamma_i\,. \end{aligned}$$

By the application of PMI and the definition of the IVP regarding hybrid dynamic system (Ladde & Ladde, 2012), we have

$$x(t) = \prod_{k|\tau_j \le t}(1 - \alpha_k)x_0 + \sum_{i=1}^{j-1} \Phi(t, \tau_i)\gamma_i + \gamma_j\,,$$

for $t \ge \tau_0$ and $t \in [\tau_{j-1}, \tau_{j+1})$. This establishes the proof of the theorem.

***Proof of Theorem 3.1*:** For $t \in [\tau_{j-1}, \tau_j)$, $j \ge 1$, from Definition 3.1, Remark 3.1 and the nature of S, we have

$$\mathrm{d}z(t) = -\lambda(t)z(t)\mathrm{d}t\,. \tag{91}$$

This establishes the continuous-time dynamic equation in (27). The proof of the discrete-time dynamic part in (27) and iterative process in (28) are outlined below.

Multiplying the discrete-time iterative process in (24) by $S(\tau_j^-)$ and noting the fact that $S(\tau_j) = S(\tau_j^-)$, we obtain

$$x(\tau_j)S(\tau_j) = (1-\alpha_j)(1-\beta_j)x(\tau_j^-)S(\tau_j^-) + \gamma_j(1-\beta_j)S(\tau_j^-)\ . \tag{92}$$

Moreover, using the definition of z, (92) reduces to

$$z(\tau_j) = (1-\alpha_j)(1-\beta_j)z(\tau_j^-) + \gamma_j(1-\beta_j)\ . \tag{93}$$

This establishes (27).

Applying the Euler-type numerical scheme (Atkinson, 2008) to (91) over an interval $[\tau_{j-1}, \tau_j^-]$, we obtain

$$z(\tau_j^-) - z(\tau_{j-1}) = -\lambda(\tau_{j-1})z(\tau_{j-1})\Delta\tau_j\ . \tag{94}$$

From (93) and (94) , we have

$$z(\tau_j) = (1-\lambda(\tau_j)\Delta\tau_j)(1-\alpha_j)(1-\beta_j)z(\tau_{j-1}) + \gamma_j(1-\beta_j)\ . \tag{95}$$

(95) exhibits the discrete time dynamic for survival process corresponding to the continuous-time dynamic process described in (27) and the discrete-time intervention process. Moreover, (95) exhibits the validity of (28). This establishes proof of Theorem 3.1.

***Proof of Theorem 4.1*:**

(a) Using the discrete-time iterative scheme (28), Remark 3.2(i)(38) and Definitions 3.2, 3.3 and 4.1, we have

$$\lambda(t) = \hat{\lambda}(\tau_j) = \frac{z(\tau_{j-1}) - z(\tau_j)}{z(\tau_{j-1})\Delta\tau_j}$$

for $t \in [\tau_{j-1}, \tau_j)$ and $j \in I(1,k)$. This establishes (a).

(b) Let τ_j^c be a censoring time between two consecutive risk/failure times, τ_{j-1} and τ_j. We consider a partition of $[\tau_{j-1}, \tau_j]$: $\tau_{j-1} < \tau_j^c < \tau_j$.

Employing iterative processes in (40) and (38) on respective subintervals $[\tau_{j-1}, \tau_j^c]$ and $[\tau_j^c, \tau_j]$, we have

$$\begin{aligned} z(\tau_j) - z(\tau_{j-1}) &= z(\tau_j^c) - z(\tau_{j-1}) + z(\tau_j) - z(\tau_j^c) \\ &= -\lambda(\tau_{j-1})\Delta\tau_j^c - \gamma_j^c - \lambda(\tau_j)z(\tau_j^c)\Delta\tau_{jc} \\ &= -\lambda(\tau_j)\left[z(\tau_{j-1})\Delta\tau_j^c + z(\tau_j^c)\Delta\tau_{jc}\right] - \gamma_j^c\ . \end{aligned} \tag{96}$$

From (96), we obtain:

$$z(\tau_{j-1}) - z(\tau_j) - \gamma_j^c = \lambda(\tau_j)\left[z(\tau_{j-1})\Delta\tau_j^c + z(\tau_j^c)\Delta\tau_{jc}\right]\ . \tag{97}$$

From (97) and knowing that $\lambda(\tau_j)$ is the hazard/risk rate of change per unit time per unit object/subject, we conclude that $z(\tau_{j-1}) - z(\tau_j) - \gamma_j^c$ is the number of failure/non-operating objects and $z(\tau_{j-1})\Delta\tau_j^c + z(\tau_j^c)\Delta\tau_{jc}$ denotes the total amount of time spent by $z(\tau_{j-1}) - z(\tau_j) - \gamma_j^c$ over the the interval $[\tau_{j-1}, \tau_j)$. This establishes (i) and (ii).

To complete the proofs of (iii) and (iv), we utilize Definition 4.1 and (97), and obtain

$$\hat{\lambda}(\tau_j) = \frac{z(\tau_{j-1}) - z(\tau_j) - \gamma_j^c}{z(\tau_{j-1})\Delta\tau_j^c + z(\tau_j^c)\Delta\tau_{jc}} \quad \text{for} \quad j \in I(1,k)\ .$$

and hence

$$\lambda(t) = \hat{\lambda}(\tau_j), \quad t \in [\tau_{j-1}, \tau_j)\ , \quad j \in I(1,k)\ .$$

This establishes proof of the theorem.

Proof of Corollary 4.2: Under the conditions of Example 2.1 and using the relationship between S, the cumulative jumps in Example 2.2, Corollary 3.2(in particular (34)), an estimate for the risk/hazard rate function at τ_j is obtained as:

$$\hat{\lambda}(\tau_j) = \frac{a_j}{(1 - A_{j-1})\Delta\tau_j}\,, \tag{98}$$

and an estimate for the risk/hazard rate function is

$$\hat{\lambda}(t) = \hat{\lambda}(\tau_j)\,, \quad \text{for} \quad t \in [\tau_{j-1}, \tau_j) \quad \text{and} \quad j \in I(1, m) \tag{99}$$

From (32), using (8) and (99), an estimate for the survival function is given by:

$$\hat{S}(t) = \exp(-\Lambda_{j-1}) \exp\left(\frac{-a_j(t - \tau_{j-1})}{(1 - A_{j-1})(\tau_j - \tau_{j-1})}\right), \quad \tau_{j-1} \le t < \tau_j\,, \tag{100}$$

where

$$\Lambda_j = \sum_{i=1}^{j} \frac{a_i}{1 - A_{i-1}},\ 1 \le j \le m,\ \Lambda_0 := 0,$$

and Λ_j is the cumulative hazard function. This establishes the proof of the corollary.

Proof of Theorem 5.1: For each $j \in I(1, n)$ and $\tau_{j-1}, \tau_j \in \mathscr{P}_0^{\mathscr{T}}$, objects/subjects are censored k_j times over a partition of $[\tau_{j-1}, \tau_j]$ of consecutive failure times. Let $\mathscr{P}_j$ be a partition corresponding to a given finite sequence of censored times over the failure time interval $[\tau_{j-1}, \tau_j)$, and let it be represented by

$$\mathscr{P}_j : \tau_{j-1} = \tau_{j-10} < \tau_{j-11} < \ldots < \tau_{j-1l-1} < \tau_{j-1l} < \ldots < \tau_{j-1k_{j-1}} < \tau_{j-1k_j}\,. \tag{101}$$

where $\mathscr{P}_j$ is a partition of $[\tau_{j-1}, \tau_j]$.

For each $j \in I(1, n)$, using the iterative schemes (38) and (40) we have

$$\begin{aligned} z(\tau_j) - z(\tau_{j-1}) &= \sum_{l=1}^{k_j} \left[z(\tau_{j-1l}) - z(\tau_{j-1l-1})\right] + [z(\tau_j) - z(\tau_{j-1k_j})] \\ &= -\lambda(\tau_j)\left[\sum_{l=1}^{k_j+1} z(\tau_{j-1l-1})\Delta\tau_{j-1l}\right] - \sum_{l=1}^{k_j} \gamma_j^l\,, \end{aligned} \tag{102}$$

and hence

$$z(\tau_{j-1}) - z(\tau_j) - \sum_{l=1}^{k_j} \gamma_j^l = \lambda(\tau_j) \sum_{l=1}^{k_j+1} z(\tau_{j-1l-1})\Delta(\tau_{j-1l})\,. \tag{103}$$

Thus, $z(\tau_{j-1}) - z(\tau_j) - \sum_{l=1}^{k_j} \gamma_j^l$ is a change in the number of items/subjects that are under observation over the subinterval $[\tau_{j-1}, \tau_j]$, and $\sum_{l=1}^{k_j+1} z(\tau_{j-1l-1})\Delta(\tau_{j-1l})$ is a total amount of time spent under the observation/testing/evaluation/monitoring of $z(\tau_{j-1l})$ items/patients/infectives/subjects on the interval $[\tau_{j-1l-1}, \tau_{j-1l})$ for $l \in I(1, k_j))$ and $j \in I(1, n)$. These statements establish conclusions 1 and 2 of Theorem 5.1.

Finally, from Definition 4.1, we obtain an estimate for a hazard rate function at $\tau_j \in [\tau_0, \mathscr{T})$ as:

$$\hat{\lambda}(\tau_j) = \frac{z(\tau_{j-1}) - z(\tau_j) - \sum_{l=1}^{k_j} \gamma_j^l}{\sum_{l=1}^{k_{j+1}} z(\tau_{j-1l-1})\Delta(\tau_{j-1l})}\,.$$

This establishes (67).

Moreover,

$$\hat{\lambda}(t) = \hat{\lambda}(\tau_j)\,, \quad \text{for} \quad t \in [\tau_{j-1}, \tau_j) \quad \text{and} \quad j \in I(1, n)\,. \tag{104}$$

This completes the proof of the theorem.

***Proof of Theorem 5.2*:**

Let $0 = \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_{j-1} < \tau_j < \ldots < \tau_k$ be the partition of $[\tau_0, \mathscr{T})$ corresponding to change point times.
For $j = 1, 2, \ldots, k$, we consider a partition of $[\tau_{j-1}, \tau_j]$ as follows:

$$\mathscr{P}_j^\tau : \tau_{j-1} = T_0^j < T_1^j < T_2^j < T_3^j < \ldots < T_{l-1}^j < T_l^j < \ldots < T_{n-1}^j < T_n^j < T_{n+1}^j = \tau_j \,. \tag{105}$$

Imitating the proof of Theorem 5.1, we have

$$\begin{aligned} z(\tau_j) - z(\tau_{j-1}) &= \sum_{m=1}^{l} \left[z(T_m^j) - z(T_{m-1}^j) \right] + [z(\tau_j) - z(T_l^j)] \\ &= \sum_{m=1}^{l} \left[-\lambda(T_{m-1}^j) z(T_{m-1}^j) \Delta T_m^j - \eta_m^j \right] + [-\lambda(T_l^j) z(T_l^j) \Delta\tau_j] \\ &- \lambda(\tau_j) \left[\sum_{m=1}^{l} z(T_{m-1}^j) \Delta T_m^j \right] - \sum_{m=1}^{l} \eta_m^j - \lambda(\tau_j) z(t_l^j) \Delta\tau_j \\ &= -\lambda(\tau_j) \left[\sum_{m=1}^{l+1} z(T_{m-1}^j) \Delta T_m^j \right] - \sum_{m=1}^{l} \eta_m^j \,, \end{aligned} \tag{106}$$

and hence

$$z(\tau_{j-1}) - z(\tau_j) - \sum_{m=1}^{l} \eta_m^j = \lambda(\tau_j) \sum_{m=1}^{l+1} z(T_{m-1}^j) \Delta T_m^j \tag{107}$$

Thus, $z(\tau_{j-1}) - z(\tau_j) - \sum_{m=1}^{l} \eta_m^j$ is a change in the number of items/subjects that is under the observation over the subinterval $[\tau_{j-1}, \tau_j]$ of the time interval of study $[\tau_0, \mathscr{T}]$ and $\sum_{m=1}^{l+1} z(T_m^j) \Delta T_m^j$ is a total amount of time spent under the observation/testing/evaluation of $z(T_m^j)$ items/patients/infectives/subjects on the interval $[T_{m-1}^j, T_m^j)$ for $m \in I(1, l))$ and $j \in I(1, k)$. These statements establish conclusions 1 and 2 of Theorem 5.1.
Finally, from Definition 4.1, we obtain an estimate for a hazard rate function at $\tau_j \in [\tau_0, \mathscr{T})$ as:

$$\hat{\lambda}(\tau_j) = \frac{z(\tau_{j-1}) - z(\tau_j) - \sum_{m=1}^{l} \eta_m^j}{\sum_{m=1}^{l+1} z(T_{m-1}^j) \Delta T_m^j} \,,$$

Moreover,

$$\hat{\lambda}(t) = \hat{\lambda}(\tau_j) \,, \quad \text{for} \quad t \in [\tau_{j-1}, \tau_j) \quad \text{and} \quad j \in I(1, k) \,. \tag{108}$$

This establishes proof of the theorem.

Proof of Corollary 5.3: Let $\tau_0 < \tau_1 < \ldots < \tau_{m-1} < \tau_m < \ldots < \tau_{j-1} < \tau_j < \ldots < \tau_n = \mathscr{T}$ be a partition of $[\tau_0, \mathscr{T}]$. Using (31), for fixed $i = 0$ and $j \in I(1, n_0)$, we have

$$z_0(\tau_m) - z_0(\tau_{m-1}) = -\lambda_0(\tau_m) z_0(\tau_{m-1}) \Delta\tau_m \,. \tag{109}$$

Summing (109) from $m = 1$ to j, we obtain

$$\begin{aligned} \sum_{m=1}^{j} [z_0(\tau_m) - z_0(\tau_{m-1})] &= \sum_{m=1}^{j} -\lambda_0(\tau_m) z_0(\tau_{m-1}) \Delta_m \\ &= -\lambda_0(\tau_j) \sum_{m=1}^{j} z_0(\tau_{m-1}) \Delta\tau_m \,. \end{aligned} \tag{110}$$

Rearranging (110) establishes (71). The proof of (73) is similar to the proof of (71). (75) is obtained by taking the natural log of the ratio of (71) and (73) . This establishes the proof of the corollary.

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