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Computation of the Survival Probability of Brownian Motion with Drift When the Absorbing Boundary is a Piecewise Affine or Piecewise Exponential Function of Time

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Abstract

A closed form formula is provided for the probability, in a closed time interval, that an arithmetic Brownian motion remains under or above a sequence of three affine, one-sided boundaries (equivalently, for the probability that a geometric Brownian motion remains under or above a sequence of three exponential, one-sided boundaries). The numerical evaluation of this formula can be done instantly and with the accuracy required for all practical purposes. The method followed can be extended to sequences of absorbing boundaries of higher dimension. It is also applied to sequences of two-sided boundaries.

Keywords: boundary crossing probability; survival probability; probability of absorption; first passage time; hitting time; Brownian motion; affine boundary; exponential boundary

1. Introduction

The question of the crossing of a non-constant boundary by a diffusion process is of central importance in many mathematical sciences. As mentioned in Wang and Pötzelberger (2007), it arises in biology, economics, engineering reliability, epidemiology, finance, genetics, seismology and sequential statistical analysis. The probability that a diffusion process will remain under or above some critical threshold over a given time interval can be referred to as a survival probability or probability of non-absorption. The vast majority of the research articles published on this topic either focus on numerical algorithms for general classes of processes or boundaries, usually involving recursive multidimensional quadrature, or they seek to obtain approximate solutions, typically substituting the initial boundary with another one for which computations are easier and then deriving a bound for the error entailed by using the approximating boundary. Much attention has also been paid to asymptotic estimates. However, known closed form results are scarce. By closed form results, we mean fully explicit formulae involving functions whose numerical evaluation can be carried out with the accuracy and the efficiency required for all practical purposes, in contrast to approximate analytical solutions that are quickly computed but inaccurate, and to numerical algorithms that can only produce the required standard of precision through heavy computational burden. The most classical of these closed form results is the so-called Bachelier-Levy formula (Levy, 1948), which provides the first-passage time density of Brownian motion to a linear boundary. This result is extended to a two-sided linear boundary by Doob (1949), but only in infinite time. The generalisation to a closed time interval is given by Anderson (1960), who is also able to integrate the density. The first passage time density of Brownian motion to a quadratic boundary is obtained independently by Salminen (1988) and Groeneboom (1989), while Novikov et al. (1999) manage to derive the hitting time density of Brownian motion to a square root boundary, but the numerical evaluation is quite involved in both cases, requiring infinite series of roots of combinations of Airy functions or confluent hypergeometric functions. By integrating these first passage time densities, the corresponding survival probabilities can be derived, though the integration is not actually performed by the mentioned authors and is far from trivial. Scheike (1992) provides a closed form solution for the survival probability of Brownian motion in infinite time when the boundary consists of two successive linear functions of time but cannot explicitly compute the corresponding integral in finite time. There are also a few closed form results for a Brownian motion (Daniels, 1996; Wang and Pötzelberger, 2007), an Ornstein-Uhlenbeck process (Choi and Nam, 2003; Wang and Pötzelberger, 2007) and a growth process (Wang and Pötzelberger, 2007), that involve very specific forms of the boundary and thus have limited use in practice, although they are quite valuable to test numerical algorithms.

This paper provides new results for the survival probability of Brownian motion. The problem raised by Scheike (1992) is reformulated, extended and analytically solved. The extension with regard to the existing literature can be summarized as follows :

- cumulative distribution functions are provided, i.e. the integration of the first passage time densities is performed
- results are provided for generalised Brownian motion (whether arithmetic or geometric Brownian motion), i.e. the underlying stochastic dynamics include drift and volatility coefficients
- sequences of up to three general affine boundaries (in the case of arithmetic Brownian motion) or exponential boundaries (in the case of geometric Brownian motion) are handled
- sequences of two-sided piecewise affine or exponential boundaries are also tackled, under the assumption that the growth rate of the boundary is identical on the downside and on the upside, i.e. the upper and the lower sides of the boundary are parallel curves

Only distributions in finite time are considered, as they are the ones used in practice in the various mathematical sciences. The choice of affine and exponential boundaries is because they allow to model a reasonably large variety of time-dependent conditions for real life problems, while preserving analytical tractability. There are potentially many applications, for example in the valuation and risk management of various path dependent financial options or insurance contracts as well as in structural models of credit risk (see, e.g., Jeanblanc et al., 2009).

Section 2 of this article states a closed form formula for the survival probability of an arithmetic or a geometric Brownian under or above a sequence of three different one-sided affine or exponential boundaries over a finite time interval and provides a few numerical results, then outlines a proof omitting cumbersome computations, and finally discusses generalization to higher-dimensional boundaries. Section 3 of this article states a closed form formula for the survival probability of an arithmetic or a geometric Brownian motion under and above a sequence of two different two-sided, parallel, affine or exponential boundaries over a finite time interval, provides a few numerical results and outlines the proof.

2. Survival Probability of an Arithmetic or a Geometric Brownian Motion under or Above a Sequence of One-sided Affine or Exponential Boundaries over a Finite Time Interval

2.1 Definitions

Let μ be a real constant, σ be a positive real constant, and $\{B(t), t \geq 0\}$ be a standard Brownian motion defined on a probability space with measure $\mathbb{P}$. Let $\{X_1(t), t \geq 0\}$ be an arithmetic Brownian motion driven, under $\mathbb{P}$, by :

$$dX_1(t) = \mu\, dt + \sigma\, dB(t) \tag{2.1}$$

Let $\{X_2(t), t \geq 0\}$ be a geometric Brownian motion driven, under $\mathbb{P}$, by :

$$dX_2(t) = \mu X_2(t) dt + \sigma X_2(t) dB(t) \tag{2.2}$$

A finite time interval $[0, T]$ is considered and divided into a partition Π of n subintervals $[t_0 = 0, t_1], [t_1, t_2]$, ...$[t_{n-1}, t_n = T]$, which are not necessarily of equal length, with $t_n \geq t_{n-1} \geq ... \geq t_1 \geq t_0$. Let $\mathbb{I}$ denote the indicator function. For a given $n \in \mathbb{N}$, two piecewise affine absorbing boundaries $g_1(t)$ and $g_2(t)$ are defined as follows :

$$g_1(t) = \sum_{i=1}^{n} \left(a_i + b_i (t - t_{i-1})\right) \mathbb{I}_{[t_{i-1}, t_i]}(t), a_i \in \mathbb{R}, b_i \in \mathbb{R},\ i \in \{1, 2, ..., n\} \tag{2.3}$$

$$g_2(t) = \sum_{i=1}^{n} (a_i + b_i t)\, \mathbb{I}_{[t_{i-1},t_i]}(t) \quad , \quad a_i \in \mathbb{R} \quad , \quad b_i \in \mathbb{R}\,, \quad i \in \{1,2,...,n\} \tag{2.4}$$

The difference between $g_1(t)$ and $g_2(t)$ is that $g_1(t)$ is time-homogeneous.

Similarly, we have the two following piecewise exponential boundaries :

$$h_1(t) = \sum_{i=1}^{n} X(0)\exp\left(a_i + b_i(t - t_{i-1})\right) \mathbb{I}_{[t_{i-1},t_i]}(t), \quad a_i \in \mathbb{R} \quad , \quad b_i \in \mathbb{R}\,, \quad i \in \{1,2,...,n\} \tag{2.5}$$

$$h_2(t) = \sum_{i=1}^{n} X(0)\exp\left(a_i + b_i t\right) \mathbb{I}_{[t_{i-1},t_i]}(t), \quad a_i \in \mathbb{R} \quad , \quad b_i \in \mathbb{R}\,, \quad i \in \{1,2,...,n\} \tag{2.6}$$

Consider the cumulative distribution function of a sequence of n maxima or n minima and n endpoints in Π in the two following cases :

- the absorbing boundary is defined either by $g_1(t)$ or $g_2(t)$ and the process under consideration is X_1

- the absorbing boundary is defined either by $h_1(t)$ or $h_2(t)$ and the process under consideration is X_2

Such a function is often referred to as a survival probability. As shown by Wang and Pötzelberger (1997), its value can be approximated by a Monte Carlo simulation scheme drawing on the Markovian nature of X_1 and X_2 in the following manner : the endpoint values of X_1 and X_2 in each time subinterval $[t_{i-1}, t_i]$ are randomly drawn at each performed simulation; if the relevant conditions at each t_i are met, then a cumulative variable records the product of the conditional probabilities that the boundary has not been crossed in each (t_{i-1}, t_i), which admit simple analytical formulae (Siegmund, 1986). This is obviously much more efficient and accurate than discretizing the whole path of the process at each run. For $n > 1$, the survival probability under consideration does not admit any known closed form formula. Although it does not seem possible to come up with an explicit and compact formula for any $n \in \mathbb{N}$, one can actually solve the problem analytically in "moderate" dimension. In this paper, the case $n = 3$ is tackled. More specifically, let $\mathrm{P}_{[...]}(\mu,\sigma,a_1,a_2,a_3,b_1,b_2,b_3,k_1,k_2,k_3,t_1,t_2,t_3)$ be defined as one of the following eight cumulative distribution functions :

$$\mathrm{P}_{[\mathrm{AU1}]}(\mu,\sigma,a_1,a_2,a_3,b_1,b_2,b_3,k_1,k_2,k_3,t_1,t_2,t_3),\ a_1 \in \mathbb{R}_+,\ (a_2,a_3,b_1,b_2,b_3,k_1,k_2,k_3) \in \mathbb{R}^8 \tag{2.7}$$

$$= \mathbb{P}\begin{pmatrix} \left(X_1(t) < a_1 + b_1 t, \forall 0 \le t \le t_1\right) \cap X_1(t_1) < k_1 \cap \left(X_1(t) < a_2 + b_2(t - t_1), \forall t_1 \le t \le t_2\right) \\ \cap X_1(t_2) < k_2 \cap \left(X_1(t) < a_3 + b_3(t - t_2), \forall t_2 \le t \le t_3\right) \cap X_1(t_3) < k_3 \end{pmatrix}$$

$$\mathrm{P}_{[\mathrm{AU2}]}(\mu,\sigma,a_1,a_2,a_3,b_1,b_2,b_3,k_1,k_2,k_3,t_1,t_2,t_3),\ a_1 \in \mathbb{R}_+,(a_2,a_3,b_1,b_2,b_3,k_1,k_2,k_3) \in \mathbb{R}^8 \tag{2.8}$$

$$= \mathbb{P}\begin{pmatrix} \left(X_1(t) < a_1 + b_1 t, \forall 0 \le t \le t_1\right) \cap X_1(t_1) < k_1 \cap \left(X_1(t) < a_2 + b_2 t, \forall t_1 \le t \le t_2\right) \\ \cap X_1(t_2) < k_2 \cap \left(X_1(t) < a_3 + b_3 t, \forall t_2 \le t \le t_3\right) \cap X_1(t_3) < k_3 \end{pmatrix}$$

$$\mathrm{P}_{[\mathrm{GU1}]}\left(\mu,\sigma,a_1,a_2,a_3,b_1,b_2,b_3,k_1,k_2,k_3,t_1,t_2,t_3\right),\left(a_1,k_1,k_2,k_3\right)\in\mathbb{R}_+^4,\left(a_2,a_3,b_1,b_2,b_3\right)\in\mathbb{R}^5 \tag{2.9}$$

$$=\mathbb{P}\left(\begin{array}{l}\left(X_2(t)<X_2(0)\exp\left(a_1+b_1t\right),\forall 0\le t\le t_1\right)\cap X_2\left(t_1\right)<k_1\\ \cap\left(X_2(t)<X_2(0)\exp\left(a_2+b_2\left(t-t_1\right)\right),\forall t_1\le t\le t_2\right)\\ \cap X_2\left(t_2\right)<k_2\cap\left(X_2(t)<X_2(0)\exp\left(a_3+b_3\left(t-t_2\right)\right),\forall t_2\le t\le t_3\right)\cap X_2\left(t_3\right)<k_3\end{array}\right)$$

$$\mathrm{P}_{[\mathrm{GU2}]}\left(\mu,\sigma,a_1,a_2,a_3,b_1,b_2,b_3,k_1,k_2,k_3,t_1,t_2,t_3\right),\left(a_1,k_1,k_2,k_3\right)\in\mathbb{R}_+^4,\left(a_2,a_3,b_1,b_2,b_3\right)\in\mathbb{R}^5 \tag{2.10}$$

$$=\mathbb{P}\left(\begin{array}{l}\left(X_2(t)<X_2(0)\exp\left(a_1+b_1t\right),\forall 0\le t\le t_1\right)\cap X_2\left(t_1\right)<k_1\\ \cap\left(X_2(t)<X_2(0)\exp\left(a_2+b_2t\right),\forall t_1\le t\le t_2\right)\\ \cap X_2\left(t_2\right)<k_2\cap\left(X_2(t)<X_2(0)\exp\left(a_3+b_3t\right),\forall t_2\le t\le t_3\right)\cap X_2\left(t_3\right)<k_3\end{array}\right)$$

$$\mathrm{P}_{[\mathrm{AL1}]}\left(\mu,\sigma,a_1,a_2,a_3,b_1,b_2,b_3,k_1,k_2,k_3,t_1,t_2,t_3\right),\ a_1\in\mathbb{R}_-,\left(a_2,a_3,b_1,b_2,b_3,k_1,k_2,k_3\right)\in\mathbb{R}^8 \tag{2.11}$$

$$=\mathbb{P}\left(\begin{array}{l}\left(X_1(t)>a_1+b_1t,\forall 0\le t\le t_1\right)\cap X_1\left(t_1\right)>k_1\cap\left(X_1(t)>a_2+b_2\left(t-t_1\right),\forall t_1\le t\le t_2\right)\\ \cap X_1\left(t_2\right)>k_2\cap\left(X_1(t)>a_3+b_3\left(t-t_2\right),\forall t_2\le t\le t_3\right)\cap X_1\left(t_3\right)>k_3\end{array}\right)$$

$$\mathrm{P}_{[\mathrm{AL2}]}\left(\mu,\sigma,a_1,a_2,a_3,b_1,b_2,b_3,k_1,k_2,k_3,t_1,t_2,t_3\right),\ a_1\in\mathbb{R}_-,\left(a_2,a_3,b_1,b_2,b_3,k_1,k_2,k_3\right)\in\mathbb{R}^8 \tag{2.12}$$

$$=\mathbb{P}\left(\begin{array}{l}\left(X_1(t)>a_1+b_1t,\forall 0\le t\le t_1\right)\cap X_1\left(t_1\right)>k_1\cap\left(X_1(t)>a_2+b_2t,\forall t_1\le t\le t_2\right)\\ \cap X_1\left(t_2\right)>k_2\cap\left(X_1(t)>a_3+b_3t,\forall t_2\le t\le t_3\right)\cap X_1\left(t_3\right)>k_3\end{array}\right)$$

$$\mathrm{P}_{[\mathrm{GL1}]}\left(\mu,\sigma,a_1,a_2,a_3,b_1,b_2,b_3,k_1,k_2,k_3,t_1,t_2,t_3\right),\ a_1\in\mathbb{R}_-,\left(k_1,k_2,k_3\right)\in\mathbb{R}_+^3,\left(a_2,a_3,b_1,b_2,b_3\right)\in\mathbb{R}^5 \tag{2.13}$$

$$=\mathbb{P}\left(\begin{array}{l}\left(X_2(t)>X_2(0)\exp\left(a_1+b_1t\right),\forall 0\le t\le t_1\right)\cap X_2\left(t_1\right)>k_1\\ \cap\left(X_2(t)>X_2(0)\exp\left(a_2+b_2\left(t-t_1\right)\right),\forall t_1\le t\le t_2\right)\\ \cap X_2\left(t_2\right)>k_2\cap\left(X_2(t)>X_2(0)\exp\left(a_3+b_3\left(t-t_2\right)\right),\forall t_2\le t\le t_3\right)\cap X_2\left(t_3\right)>k_3\end{array}\right)$$

$$\mathrm{P}_{[\mathrm{GL2}]}\left(\mu,\sigma,a_1,a_2,a_3,b_1,b_2,b_3,k_1,k_2,k_3,t_1,t_2,t_3\right),\ a_1\in\mathbb{R}_-,\left(k_1,k_2,k_3\right)\in\mathbb{R}_+^3,\left(a_2,a_3,b_1,b_2,b_3\right)\in\mathbb{R}^5 \tag{2.14}$$

$$=\mathbb{P}\left(\begin{array}{l}\left(X_2(t)>X_2(0)\exp\left(a_1+b_1t\right),\forall 0\le t\le t_1\right)\cap X_2\left(t_1\right)>k_1\\ \cap\left(X_2(t)>X_2(0)\exp\left(a_2+b_2t\right),\forall t_1\le t\le t_2\right)\\ \cap X_2\left(t_2\right)>k_2\cap\left(X_2(t)>X_2(0)\exp\left(a_3+b_3t\right),\forall t_2\le t\le t_3\right)\cap X_2\left(t_3\right)>k_3\end{array}\right)$$

In other words, taking $n=3$,

- $\mathrm{P}_{[\mathrm{AU1}]}$ is the probability that an arithmetic Brownian motion will remain under the piecewise affine time-homogeneous boundary $g_1(t)$ defined by (2.3) and under the successive endpoints k_1,k_2,k_3

- $\mathrm{P}_{[\mathrm{AU2}]}$ is the probability that an arithmetic Brownian motion will remain under the piecewise affine time-inhomogeneous boundary $g_2(t)$ defined by (2.4) and under the successive endpoints k_1,k_2,k_3

- $\mathrm{P}_{[\mathrm{GU1}]}$ is the probability that a geometric Brownian motion will remain under the piecewise exponential time-homogeneous boundary $h_1(t)$ defined by (2.5) and under the successive endpoints k_1, k_2, k_3

- $\mathrm{P}_{[\mathrm{GU2}]}$ is the probability that a geometric Brownian motion will remain under the piecewise exponential time-inhomogeneous boundary $h_2(t)$ defined by (2.6) and under the successive endpoints k_1, k_2, k_3

- $\mathrm{P}_{[\mathrm{AL1}]}$ is the probability that an arithmetic Brownian motion will remain above the piecewise affine boundary $g_1(t)$ defined by (2.3) and above the successive endpoints k_1, k_2, k_3

- $\mathrm{P}_{[\mathrm{AL2}]}$ is the probability that an arithmetic Brownian motion will remain above the piecewise affine boundary $g_2(t)$ defined by (2.4) and above the successive endpoints k_1, k_2, k_3

- $\mathrm{P}_{[\mathrm{GL1}]}$ is the probability that a geometric Brownian motion will remain above the piecewise exponential boundary $h_1(t)$ defined by (2.5) and above the successive endpoints k_1, k_2, k_3

- $\mathrm{P}_{[\mathrm{GL2}]}$ is the probability that a geometric Brownian motion will remain above the piecewise exponential boundary $h_2(t)$ defined by (2.6) and above the successive endpoints k_1, k_2, k_3

2.2 Statement of Formula 1

Formula 1 *Let* $\mathrm{P}_{[\dots]}(\mu, \sigma, a_1, a_2, a_3, b_1, b_2, b_3, k_1, k_2, k_3, t_1, t_2, t_3)$ *be defined as in Subsection 2.1. Then,*

$$\mathrm{P}_{[\dots]}(\mu, \sigma, a_1, a_2, a_3, b_1, b_2, b_3, k_1, k_2, k_3, t_1, t_2, t_3) \tag{2.15}$$

$$= \Phi_3\left[\theta\frac{z_1 - b_1t_1 - \mu_1t_1}{\sigma\sqrt{t_1}}, \theta\frac{z_2 - b_2t_2 - \mu_1t_1 - \mu_2(t_2 - t_1)}{\sigma\sqrt{t_2}}, \theta\frac{z_3 - b_3t_3 - \mu_1t_1 - \mu_2(t_2 - t_1) - \mu_3(t_3 - t_2)}{\sigma\sqrt{t_3}}; \sqrt{t_1 / t_2}, \sqrt{t_2 / t_3}\right]$$

$$-\exp\left(\frac{\lambda_1}{\sigma^2}\right)\Phi_3\left[\theta\frac{z_1 - b_1t_1 - 2a_1 - \mu_1t_1}{\sigma\sqrt{t_1}}, \theta\frac{z_2 - b_2t_2 - 2a_1 - \mu_1t_1 - \mu_2(t_2 - t_1)}{\sigma\sqrt{t_2}}, \theta\frac{z_3 - b_3t_3 - 2a_1 - \mu_1t_1 - \mu_2(t_2 - t_1) - \mu_3(t_3 - t_2)}{\sigma\sqrt{t_3}}; \sqrt{t_1 / t_2}, \sqrt{t_2 / t_3}\right]$$

$$-\exp\left(\frac{\lambda_2}{\sigma^2}\right)\Phi_3\left[\theta\frac{z_1 - b_1t_1 - \mu_1t_1 + 2\mu_2t_1}{\sigma\sqrt{t_1}}, \theta\frac{z_2 - b_2t_2 - 2\alpha_2 + \mu_1t_1 - \mu_2(t_1 + t_2)}{\sigma\sqrt{t_2}}, \theta\frac{z_3 - b_3t_3 - 2\alpha_2 + \mu_1t_1 - \mu_2(t_1 + t_2) - \mu_3(t_3 - t_2)}{\sigma\sqrt{t_3}}; -\sqrt{t_1 / t_2}, \sqrt{t_2 / t_3}\right]$$

$$+\exp\left(\frac{\lambda_3}{\sigma^2}\right)\Phi_3\left[\begin{array}{l}\theta\frac{z_1-b_1t_1-2a_1-\mu_1t_1+2\mu_2t_1}{\sigma\sqrt{t_1}},\theta\frac{z_2-b_2t_2-2\alpha_2+2a_1+\mu_1t_1-\mu_2\left(t_1+t_2\right)}{\sigma\sqrt{t_2}}\\ \theta\frac{z_3-b_3t_3-2\alpha_2+2a_1+\mu_1t_1-\mu_2\left(t_1+t_2\right)-\mu_3\left(t_3-t_2\right)}{\sigma\sqrt{t_3}};-\sqrt{t_1/t_2},\sqrt{t_2/t_3}\end{array}\right]$$

$$-\exp\left(\frac{\lambda_4}{\sigma^2}\right)\Phi_3\left[\begin{array}{l}\theta\frac{z_1-b_1t_1-\mu_1t_1+2\mu_3t_1}{\sigma\sqrt{t_1}},\theta\frac{z_2-b_2t_2-\mu_1t_1-\mu_2\left(t_2-t_1\right)+2\mu_3t_2}{\sigma\sqrt{t_2}},\\ \theta\frac{z_3-b_3t_3-2\alpha_3+\mu_1t_1+\mu_2\left(t_2-t_1\right)-\mu_3\left(t_2+t_3\right)}{\sigma\sqrt{t_3}};\sqrt{t_1/t_2},-\sqrt{t_2/t_3}\end{array}\right]$$

$$+\exp\left(\frac{\lambda_5}{\sigma^2}\right)\Phi_3\left[\begin{array}{l}\theta\frac{z_1-b_1t_1-2a_1+2\mu_3t_1-\mu_1t_1}{\sigma\sqrt{t_1}},\theta\frac{z_2-b_2t_2-2a_1-\mu_1t_1-\mu_2\left(t_2-t_1\right)+2\mu_3t_2}{\sigma\sqrt{t_2}},\\ \theta\frac{z_3-b_3t_3-2\alpha_3+2a_1+\mu_1t_1+\mu_2\left(t_2-t_1\right)-\mu_3\left(t_2+t_3\right)}{\sigma\sqrt{t_3}};\sqrt{t_1/t_2},-\sqrt{t_2/t_3}\end{array}\right]$$

$$+\exp\left(\frac{\lambda_6}{\sigma^2}\right)\Phi_3\left[\begin{array}{l}\theta\frac{z_1-b_1t_1-2\mu_3t_1+2\mu_2t_1-\mu_1t_1}{\sigma\sqrt{t_1}},\theta\frac{z_2-b_2t_2-2\alpha_2+\mu_1t_1-\mu_2\left(t_1+t_2\right)+2\mu_3t_2}{\sigma\sqrt{t_2}},\\ \theta\frac{z_3-b_3t_3-2\alpha_3+2\alpha_2-\mu_1t_1+\mu_2\left(t_1+t_2\right)-\mu_3\left(t_2+t_3\right)}{\sigma\sqrt{t_3}};-\sqrt{t_1/t_2},-\sqrt{t_2/t_3}\end{array}\right]$$

$$-\exp\left(\frac{\lambda_7}{\sigma^2}\right)\Phi_3\left[\begin{array}{l}\theta\frac{z_1-b_1t_1-2a_1-2\mu_3t_1+2\mu_2t_1-\mu_1t_1}{\sigma\sqrt{t_1}},\\ \theta\frac{z_2-b_2t_2-2\alpha_2+2a_1+\mu_1t_1-\mu_2\left(t_1+t_2\right)+2\mu_3t_2}{\sigma\sqrt{t_2}},\\ \theta\frac{z_3-b_3t_3-2\alpha_3+2\alpha_2-2a_1-\mu_1t_1+\mu_2\left(t_1+t_2\right)-\mu_3\left(t_2+t_3\right)}{\sigma\sqrt{t_3}};\\ -\sqrt{t_1/t_2},-\sqrt{t_2/t_3}\end{array}\right]$$

where the function Φ_n is a convolution of gaussian densities defined, for any $n\in\mathbb{N}$, by :

$$\Phi_n\left[x_1,\dots,x_n;\rho_1,\dots,\rho_{n-1}\right]$$

$$=\int_{D^n}\frac{\exp\left(-\frac{y_1^2}{2}-\sum_{i=1}^{n-1}\frac{\left(y_{i+1}-\rho_iy_i\right)^2}{2\left(1-\rho_i^2\right)}\right)}{\left(2\pi\right)^{n/2}\prod_{i=1}^{n-1}\sqrt{1-\rho_i^2}}dy_n\dots dy_1 \tag{2.16}$$

$$D^n=\left]-\infty,x_1\right]\times\left]-\infty,x_2\right]\dots\times\left]-\infty,x_n\right],\ x_i\in\mathbb{R},\ \rho_i\in\left]-1,1\right[\ ,\ i\in\left\{1,\dots,n\right\}$$

The α_i terms, $i\in\left\{2,3\right\}$, in (2.15) are given by :

$$\alpha_2=a_2\left(\mathbb{I}_{\left\{\mathrm{P}_{[\dots]}=\mathrm{P}_{[\mathrm{AU2}]}\right\}}+\mathbb{I}_{\left\{\mathrm{P}_{[\dots]}=\mathrm{P}_{[\mathrm{GU2}]}\right\}}+\mathbb{I}_{\left\{\mathrm{P}_{[\dots]}=\mathrm{P}_{[\mathrm{AL2}]}\right\}}+\mathbb{I}_{\left\{\mathrm{P}_{[\dots]}=\mathrm{P}_{[\mathrm{GL2}]}\right\}}\right)$$

$$+\left(a_2 - b_2 t_1\right)\left(\mathbb{I}_{\left\{P_{[\ldots]}=P_{[AU1]}\right\}} + \mathbb{I}_{\left\{P_{[\ldots]}=P_{[GU1]}\right\}} + \mathbb{I}_{\left\{P_{[\ldots]}=P_{[AL1]}\right\}} + \mathbb{I}_{\left\{P_{[\ldots]}=P_{[GL1]}\right\}}\right)$$

$$\alpha_3 = a_3\left(\mathbb{I}_{\left\{P_{[\ldots]}=P_{[AU2]}\right\}} + \mathbb{I}_{\left\{P_{[\ldots]}=P_{[GU2]}\right\}} + \mathbb{I}_{\left\{P_{[\ldots]}=P_{[AL2]}\right\}} + \mathbb{I}_{\left\{P_{[\ldots]}=P_{[GL2]}\right\}}\right)$$

$$+\left(a_3 - b_3 t_2\right)\left(\mathbb{I}_{\left\{P_{[\ldots]}=P_{[AU1]}\right\}} + \mathbb{I}_{\left\{P_{[\ldots]}=P_{[GU1]}\right\}} + \mathbb{I}_{\left\{P_{[\ldots]}=P_{[AL1]}\right\}} + \mathbb{I}_{\left\{P_{[\ldots]}=P_{[GL1]}\right\}}\right)$$

The λ_i terms, $i \in \left\{1,2,3,4,5,6,7\right\}$, in (2.15) are given by :

$$\lambda_1 = 2\mu_1 a_1$$

$$\lambda_2 = 2\mu_2\alpha_2 - 2\mu_1\mu_2 t_1 + 2\mu_2^2 t_1$$

$$\lambda_3 = 2\mu_1 a_1 + 2\mu_2\alpha_2 - 4\mu_2 a_1 - 2\mu_1\mu_2 t_1 + 2\mu_2^2 t_1$$

$$\lambda_4 = 2\mu_3\alpha_3 + 2\mu_3^2 t_2 - 2\mu_1\mu_3 t_1 - 2\mu_2\mu_3\left(t_2 - t_1\right)$$

$$\lambda_5 = 2\mu_3\alpha_3 + 2\mu_1 a_1 - 4\mu_3 a_1 + 2\mu_3^2 t_2 - 2\mu_1\mu_3 t_1 - 2\mu_2\mu_3\left(t_2 - t_1\right)$$

$$\lambda_6 = 2\mu_3\alpha_3 + 2\mu_2\alpha_2 - 4\mu_3\alpha_2 + 2\left(\mu_3 - \mu_2\right)^2 t_1 + 2\mu_1\left(\mu_3 - \mu_2\right)t_1 + 2\mu_3^2\left(t_2 - t_1\right) - 2\mu_2\mu_3\left(t_2 - t_1\right)$$

$$\lambda_7 = 2\mu_1 a_1 + 2\mu_2\alpha_2 - 4\mu_3\alpha_2 + 2\mu_3\alpha_3 + 2\left(\mu_3 - \mu_2\right)^2 t_1 + 2\mu_3^2\left(t_2 - t_1\right)$$

$$-2\mu_2\mu_3\left(t_2 - t_1\right) + 2\left(\mu_3 - \mu_2\right)\left(2a_1 + \mu_1 t_1\right)$$

The z_i terms, $i \in \left\{1,2,3\right\}$, in (2.15) are given by :

$$z_1 = \min\left(a_1 + b_1 t_1, k_1, a_2\right)\mathbb{I}_{\left\{P_{[\ldots]}=P_{[AU1]}\right\}} + \min\left(a_1 + b_1 t_1, k_1, a_2 + b_2 t_1\right)\mathbb{I}_{\left\{P_{[\ldots]}=P_{[AU2]}\right\}}$$

$$+\min\left(a_1 + b_1 t_1, \ln\left(k_1 / X_2\left(0\right)\right), a_2\right)\mathbb{I}_{\left\{P_{[\ldots]}=P_{[GU1]}\right\}} + \min\left(a_1 + b_1 t_1, \ln\left(k_1 / X_2\left(0\right)\right), a_2 + b_2 t_1\right)\mathbb{I}_{\left\{P_{[\ldots]}=P_{[GU2]}\right\}}$$

$$+\max\left(a_1 + b_1 t_1, k_1, a_2\right)\mathbb{I}_{\left\{P_{[\ldots]}=P_{[AL1]}\right\}} + \max\left(a_1 + b_1 t_1, k_1, a_2 + b_2 t_1\right)\mathbb{I}_{\left\{P_{[\ldots]}=P_{[AL2]}\right\}}$$

$$+\max\left(a_1 + b_1 t_1, \ln\left(k_1 / X_2\left(0\right)\right), a_2\right)\mathbb{I}_{\left\{P_{[\ldots]}=P_{[GL1]}\right\}} + \max\left(a_1 + b_1 t_1, \ln\left(k_1 / X_2\left(0\right)\right), a_2 + b_2 t_1\right)\mathbb{I}_{\left\{P_{[\ldots]}=P_{[GL2]}\right\}}$$

$$z_2 = \min\left(a_2 + b_2\left(t_2 - t_1\right), k_2, a_3\right)\mathbb{I}_{\left\{P_{[\ldots]}=P_{[AU1]}\right\}} + \min\left(a_2 + b_2\left(t_2 - t_1\right), \ln\left(k_2 / X_2\left(0\right)\right), a_3\right)\mathbb{I}_{\left\{P_{[\ldots]}=P_{[GU1]}\right\}}$$

$$+\min\left(a_2 + b_2 t_2, k_2, a_3 + b_3 t_2\right)\mathbb{I}_{\left\{P_{[\ldots]}=P_{[AU2]}\right\}} + \min\left(a_2 + b_2 t_2, \ln\left(k_2 / X_2\left(0\right)\right), a_3 + b_3 t_2\right)\mathbb{I}_{\left\{P_{[\ldots]}=P_{[GU2]}\right\}}$$

$$+\max\left(a_2 + b_2\left(t_2 - t_1\right), k_2, a_3\right)\mathbb{I}_{\left\{P_{[\ldots]}=P_{[AL1]}\right\}} + \max\left(a_2 + b_2\left(t_2 - t_1\right), \ln\left(k_2 / X_2\left(0\right)\right), a_3\right)\mathbb{I}_{\left\{P_{[\ldots]}=P_{[GL1]}\right\}}$$

$$+\max\left(a_2 + b_2 t_2, k_2, a_3 + b_3 t_2\right)\mathbb{I}_{\left\{P_{[\ldots]}=P_{[AL2]}\right\}} + \max\left(a_2 + b_2 t_2, \ln\left(k_2 / X_2\left(0\right)\right), a_3 + b_3 t_2\right)\mathbb{I}_{\left\{P_{[\ldots]}=P_{[GL2]}\right\}}$$

$$z_3 = \min\left(a_3 + b_3\left(t_3 - t_2\right), k_3\right)\mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{AU1}]}\right\}} + \min\left(a_3 + b_3\left(t_3 - t_2\right), \ln\left(k_3 / X_2\left(0\right)\right)\right)\mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{GU1}]}\right\}}$$

$$+\min\left(a_3 + b_3 t_3, k_3\right)\mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{AU2}]}\right\}} + \min\left(a_3 + b_3 t_3, \ln\left(k_3 / X_2\left(0\right)\right)\right)\mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{GU2}]}\right\}}$$

$$+\max\left(a_3 + b_3\left(t_3 - t_2\right), k_3\right)\mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{AL1}]}\right\}} + \max\left(a_3 + b_3\left(t_3 - t_2\right), \ln\left(k_3 / X_2\left(0\right)\right)\right)\mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{GL1}]}\right\}}$$

$$+\max\left(a_3 + b_3 t_3, k_3\right)\mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{AL2}]}\right\}} + \max\left(a_3 + b_3 t_3, \ln\left(k_3 / X_2\left(0\right)\right)\right)\mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{GL2}]}\right\}}$$

The μ_i *terms,* $i \in \left\{1,2,3\right\}$*, in (2.15) are given by :*

$$\mu_1 = \left(\mu - b_1\right)\left(\mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{AU1}]}\right\}} + \mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{AL1}]}\right\}} + \mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{AU2}]}\right\}} + \mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{AL2}]}\right\}}\right)$$

$$+\left(\mu - \frac{\sigma^2}{2} - b_1\right)\left(\mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{GU1}]}\right\}} + \mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{GL1}]}\right\}} + \mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{GU2}]}\right\}} + \mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{GL2}]}\right\}}\right)$$

$$\mu_2 = \left(\mu - b_2\right)\left(\mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{AU1}]}\right\}} + \mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{AL1}]}\right\}} + \mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{AU2}]}\right\}} + \mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{AL2}]}\right\}}\right)$$

$$+\left(\mu - \frac{\sigma^2}{2} - b_2\right)\left(\mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{GU1}]}\right\}} + \mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{GL1}]}\right\}} + \mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{GU2}]}\right\}} + \mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{GL2}]}\right\}}\right)$$

$$\mu_3 = \left(\mu - b_3\right)\left(\mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{AU1}]}\right\}} + \mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{AL1}]}\right\}} + \mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{AU2}]}\right\}} + \mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{AL2}]}\right\}}\right)$$

$$+\left(\mu - \frac{\sigma^2}{2} - b_3\right)\left(\mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{GU1}]}\right\}} + \mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{GL1}]}\right\}} + \mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{GU2}]}\right\}} + \mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{GL2}]}\right\}}\right)$$

θ *is given by :*

$$\theta = 1\left(\mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{AU1}]}\right\}} + \mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{AU2}]}\right\}} + \mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{GU1}]}\right\}} + \mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{GU2}]}\right\}}\right)$$

$$-1\left(\mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{AL1}]}\right\}} + \mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{AL2}]}\right\}} + \mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{GL1}]}\right\}} + \mathbb{I}_{\left\{\mathrm{P}_{[\ldots]}=\mathrm{P}_{[\mathrm{GL2}]}\right\}}\right)$$

End of Formula 1.

From a numerical perspective, Formula 1 raises the question of the evaluation of the function Φ_3. A straightforward calculation yields the following integration rule :

$$\Phi_3\left[x_1, x_2, x_3; \rho_1, \rho_2\right] = \int_{y_2=-\infty}^{x_2} \frac{\exp\left(-y_2^2 / 2\right)}{\sqrt{2\pi}} N\left[\frac{x_1 - \rho_1 y_2}{\sqrt{1-\rho_1^2}}\right] N\left[\frac{x_3 - \rho_2 y_2}{\sqrt{1-\rho_2^2}}\right] dy_2 \tag{2.17}$$

where the function $N\left[.\right]$ is the univariate standard normal cumulative distribution function.

Using (2.17), the numerical evaluation of the function Φ_3 is easy by means of a classical adaptive Gauss-Legendre quadrature. Alternatively, the following identities can be verified :

$$\Phi_3\left[x_1, x_2, x_3; \sqrt{t_1 / t_2}, \sqrt{t_2 / t_3}\right] = N_3\left[x_1, x_2, x_3; \sqrt{t_1 / t_2}, \sqrt{t_1 / t_3}, \sqrt{t_2 / t_3}\right] \tag{2.18}$$

$$\Phi_3\left[x_1,x_2,x_3;-\sqrt{t_1/t_2},-\sqrt{t_2/t_3}\right]=N_3\left[x_1,x_2,x_3;-\sqrt{t_1/t_2},\sqrt{t_1/t_3},-\sqrt{t_2/t_3}\right] \tag{2.19}$$

$$\Phi_3\left[x_1,x_2,x_3;-\sqrt{t_1/t_2},\sqrt{t_2/t_3}\right]=N_3\left[x_1,x_2,x_3;-\sqrt{t_1/t_2},-\sqrt{t_1/t_3},\sqrt{t_2/t_3}\right] \tag{2.20}$$

$$\Phi_3\left[x_1,x_2,x_3;\sqrt{t_1/t_2},-\sqrt{t_2/t_3}\right]=N_3\left[x_1,x_2,x_3;\sqrt{t_1/t_2},-\sqrt{t_1/t_3},-\sqrt{t_2/t_3}\right] \tag{2.21}$$

where the function $N_3\left[.,.,.;.,.,.\right]$ is the trivariate standard normal cumulative distribution function, the numerical evaluation of which can be performed with double precision and computational time of approximately 0.01 second using the algorithm by Genz (2004).

A few numerical results are reported in Table 1, in which the $\mathrm{P}_{[\mathrm{AU1}]}$ survival probability is computed for increasing levels of the volatility coefficient σ and other parameters fixed as follows : $\mu = 0.01$, $t_1 = 0.25$, $t_2 = 0.5$, $t_3 = 1$, $k_1 = 0$, $k_2 = 0.02$, $k_3 = 0.03$, $a_1 = 0.22,$ $b_1 = -0.12$, $a_2 = a_1 + b_1 t_1$, $b_2 = 0.16$, $a_3 = a_2 + b_2\left(t_2 - t_1\right)$, $b_3 = -0.24$. Notice that the absorbing boundary here is continuous at times t_1 and t_2, but non-continuous boundaries can be handled just as easily. Formula 1 is implemented using the algorithm by Genz (2004) for the computation of the trivariate standard normal cumulative distribution function. The results are compared with those obtained using the semi-analytical Monte Carlo algorithm devised by Wang and Pötzelberger (1997), denoted by WP simulation algorithm, that enables to draw only the endpoints of the time subintervals at each run, which is dramatically more efficient and accurate than a basic Monte Carlo simulation. Random numbers are drawn by the Mersenne Twister generator.

For all computed values, a 5-digit convergence can be observed between Formula 1 and the WP algorithm, on condition that a total of 100,000,000 stochastic simulations are performed. The latter method requires a computational time of 411 seconds on an i-7 4GHz personal computer. This is cut to 42 seconds when only 10,000,000 simulations are performed, which achieves 5-digit convergence in 2 cases out of 3 and 4-digit convergence in one case. The numerical computation of Formula 1 takes approximately 0.2 second. The efficiency of the implementation of the WP algorithm could probably be improved, for instance by resorting to low discrepancy sequences instead of a pseudo random number generator, but this is not the subject of this article.

Table 1. Numerical evaluation of the survival probability of an arithmetic Brownian under a one-sided piecewise affine, time-homogeneous, absorbing boundary, as a function of volatility

	Formula 1	WP simulation algorithm 10,000,000 runs	WP simulation algorithm 100,000,000 runs
Volatility = 20%	0.275332974	0.275318288	0.275387164
Volatility = 50%	0.257810712	0.257763484	0.257885116
Volatility = 80%	0.191728749	0.191718319	0.191716445

2.3 Proof of Formula 1

Only sequences of upper boundaries are tackled, since the results for sequences of lower boundaries ensue by symmetry of Brownian paths.

Let us deal with process X_1 first. Let us denote by p the sought probability when the boundary is defined by $g_1\left(t\right)$

in (2.3). The random variables $X_1(t_1)$, $X_1(t_2)$ and $X_1(t_3)$ are absolutely continuous random variables that admit known Gaussian density functions. At time t_1, X_1 must be located below $a_1+b_1t_1, k_1$ and a_2, in order not to be absorbed; at time t_2, it must stand underneath the points $a_2+b_2(t_2-t_1), k_2$ and a_3; at time t_3, it must end below $a_3+b_3(t_3-t_2)$ and k_3. Hence, by conditioning with respect to $X_1(t_1)$, $X_1(t_2)$ and $X_1(t_3)$, and by using the weak Markov property of $\{X_1(t), t\geq 0\}$, one can come up with the following integral formulation of the problem :

$$p=\int_{x_1=-\infty}^{z_1}\int_{x_2=-\infty}^{z_2}\int_{x_3=-\infty}^{z_3}\mathbb{P}\left(\left(X_1(t_1)\in dx_1\right)\cap\left(X_1(t)<a_1+b_1t, \forall 0\leq t\leq t_1\right)\right) \tag{2.22}$$

$$\mathbb{P}\left(\left(X_1(t_2)\in dx_2\right)\cap\left(X_1(t)<a_2+b_2(t-t_1), \forall t_1\leq t\leq t_2\right)\middle|X_1(t_1)\in dx_1\right)$$

$$\mathbb{P}\left(\left(X_1(t_3)\in dx_3\right)\cap\left(X_1(t)<a_3+b_3(t-t_2), \forall t_2\leq t\leq t_3\right)\middle|X_1(t_2)\in dx_2\right)dx_3dx_2dx_1$$

$$=\int_{x_1=-\infty}^{z_1-b_1t_1}\int_{x_2=-\infty}^{z_2-b_2t_2}\int_{x_3=-\infty}^{z_3-b_3t_3}f_1(x_1)f_2(x_1,x_2)f_3(x_2,x_3)dx_3dx_2dx_1 \tag{2.23}$$

where the functions $f_1(x_1)$, $f_2(x_1,x_2)$ and $f_3(x_2,x_3)$ are defined by :

$$f_1(x_1)=\mathbb{P}\left(Y(t_1)\in dx_1, \sup_{0\leq t\leq t_1}Y(t)<a_1\right) \tag{2.24}$$

$$f_2(x_1,x_2)=\mathbb{P}\left(Y(t_2)\in dx_2, \sup_{t_1\leq t\leq t_2}Y(t)<a_2-b_2t_1\middle|Y(t_1)\in dx_1\right) \tag{2.25}$$

$$f_3(x_2,x_3)=\mathbb{P}\left(Y(t_3)\in dx_3, \sup_{t_2\leq t\leq t_3}Y(t)<a_3-b_3t_2\middle|Y(t_2)\in dx_2\right) \tag{2.26}$$

and the process $\{Y(t), t\geq 0\}$ is defined by :

$$dY(t)=\begin{cases}\mu_1dt+\sigma dB(t), \forall 0\leq t<t_1\\ \mu_2dt+\sigma dB(t), \forall t_1\leq t\leq t_2\\ \mu_3dt+\sigma dB(t), \forall t_2\leq t\leq t_3\end{cases} \tag{2.27}$$

$$\mu_i=\mu-b_i, i\in\{1,2,3\}$$

The function $f_1(x_1)$ is obtained by differentiating the classical formula for the joint distribution of the maximum of Brownian motion with drift and its endpoint over the closed time interval $[0,t_1]$ (see, e.g., Karatzas and Shreve, 1991).

To obtain the functions $f_2(x_1, x_2)$ and $f_3(x_2, x_3)$, the following lemma is introduced.

Lemma 1 *Let* $\{Y(t), t \geq 0\}$ *be an arithmetic Brownian motion with constant drift* $\mu \in \mathbb{R}$ *and volatility* $\sigma \in \mathbb{R}_+$ *under a given probability measure* $\mathbb{P}$ *. Let* t_i *and* t_j *be two non-random times such that* $t_j > t_i > t_0 = 0$.

Then, if x_i, x_j *and* h *are real constants with* $x_i < h$ *and* $x_j < h$ *, we have, at time* t_0 *:*

$$\mathbb{P}\left(Y(t_i) \leq x_i, Y(t_j) \leq x_j, \sup_{t_i \leq t \leq t_j} Y(t) \leq h\right) \tag{2.28}$$

$$= N_2\left[\frac{x_i - \mu t_i}{\sigma\sqrt{t_i}}, \frac{x_j - \mu t_j}{\sigma\sqrt{t_j}}; \sqrt{\frac{t_i}{t_j}}\right] - \exp\left(\frac{2\mu h}{\sigma^2}\right) N_2\left[\frac{x_i + \mu t_i}{\sigma\sqrt{t_i}}, \frac{x_j - 2h - \mu t_j}{\sigma\sqrt{t_j}}; -\sqrt{\frac{t_i}{t_j}}\right]$$

where the function $N_2[x_1, x_2; \rho]$ *is the bivariate standard normal cumulative distribution function with upper bounds* x_1 *and* x_2 *and correlation coefficient* ρ

Proof of lemma 1

$$\mathbb{P}\left(Y(t_i) \leq x_i, Y(t_j) \leq x_j, \sup_{t_i \leq t \leq t_j} Y(t) \leq h\right)$$

$$= \int_{-\infty}^{x_i}\int_{-\infty}^{x_j} \mathbb{P}\left(Y(t_i) \in dy, Y(t_j) \in dz\right) \mathbb{P}\left(\sup_{t_i \leq t \leq t_j} Y(t) \leq h \,\middle|\, Y(t_i) \in dy, Y(t_j) \in dz\right) dy dz \tag{2.29}$$

The pair $(Y(t_i), Y(t_j))$ is bivariate normal with correlation coefficient equal to $\sqrt{t_i / t_j}$. The conditional cumulative distribution function of $\sup_{t_i \leq t \leq t_j} Y(t)$ is given by Wang and Pötzelberger (1997) and can be written as follows :

$$\mathbb{P}\left(\sup_{t_i \leq t \leq t_j} Y(t) \leq h \,\middle|\, Y(t_i) \in dy, Y(t_j) \in dz\right) = 1 - \exp\left(\frac{2(h-y)(z-h)}{\sigma^2(t_j - t_i)}\right) \tag{2.30}$$

One can then solve the integration problem in (2.29) to obtain (2.28).

□

Differentiating the right-hand side of (2.29) and dividing by the density function of $Y(t_i)$, one can obtain :

$$\phi(x_i, x_j, h, \mu, \sigma, t_i, t_j) = \mathbb{P}\left(Y(t_j) \in dx_j, \sup_{t_i \leq t \leq t_j} Y(t) \leq h \,\middle|\, Y(t_i) \in dx_i\right)$$

$$= \exp\left(-\frac{1}{2}\left(\frac{x_j - x_i - \mu(t_j - t_i)}{\sigma\sqrt{(t_j - t_i)}}\right)^2\right) / \left(\sigma\sqrt{2\pi(t_j - t_i)}\right) \tag{2.31}$$

$$-\exp\left(\frac{2\mu(h - x_i)}{\sigma^2}\right)\left(\exp\left(-\frac{1}{2}\left(\frac{x_j - 2h + x_i - \mu(t_j - t_i)}{\sigma\sqrt{(t_j - t_i)}}\right)^2\right) / \left(\sigma\sqrt{2\pi(t_j - t_i)}\right)\right)$$

Plugging :

$$f_2(x_1, x_2) = \phi(x_1, x_2, a_2 - b_2 t_1, \mu_2, \sigma, t_1, t_2) \tag{2.32}$$

$$f_3(x_2, x_3) = \phi(x_2, x_3, a_3 - b_3 t_2, \mu_3, \sigma, t_2, t_3) \tag{2.33}$$

into (2.23), the rest of the proof, whose details are omitted, then consists in performing the necessary calculations to solve the triple integral in (2.22) and obtain the linear combination of eight trivariate cumulative distribution functions given by Formula 1. Elementary modifications provide the survival probability when the boundary is defined by the function $g_2(t)$ in (2.4). A basic application of Ito's lemma to $\ln(X_2(t) / X_2(0))$ shows that the survival probability of the process X_2 is given by the formula for the survival probability of the process X_1 with the two following adjustments : the drift coefficients become $\mu_i = \mu - b_i - \sigma^2 / 2$, $i \in \{1,2,3\}$ and k_i becomes $\ln(k_i / X_2(0))$.

□

2.4 Generalization to Higher Dimension

Similar exact formulae can be derived for $n > 3$ but they become more and more cumbersome. In general, for any $n \in \mathbb{N}$, they will involve a number 2^n of the $n-$ variate cumulative distribution functions of Gaussian type given by (2.16). For an arithmetic Brownian motion subject to the absorbing boundary $g_1(t)$, the integration problem to solve is the following :

$$\int_{D^n} \prod_{i=0}^{n-1} \phi(x_i, x_{i+1}, a_{i+1} - b_{i+1} t_i, \mu_{i+1}, \sigma, t_i, t_{i+1})\, dx_n dx_{n-1} ... dx_1 \tag{2.35}$$

where $x_0 = 0$ and

$$D^n = \left]-\infty, \min(a_1 + b_1 t_1, k_1, a_2)\right] \times \left]-\infty, \min(a_2 + b_2 t_2, k_2, a_3)\right] \times ... \times \left]-\infty, \min[a_{n-1} + b_{n-1} t_{n-1}, k_n]\right]$$

The main issue is numerical rather than analytical : evaluating the Gaussian integral given by (2.16) in high dimension is not easy. Rewriting it in terms of the standard normal cumulative distribution function of order n, as was done in (2.18) – (2.21) for $n = 3$, does not solve the numerical issue, as there does not exist an algorithm capable of evaluating the

n − variate standard normal cumulative distribution function with arbitrary precision in "reasonable" time as soon as $n = 4$. For more background on this topic, the reader may refer to Genz and Bretz (2009).
However, for $n = 4$, it can be verified that the following integration rule holds :

$$\Phi_4\left[x_1, x_2, x_3, x_4; \rho_1, \rho_2, \rho_3\right] \tag{2.36}$$

$$= \int_{y_2=-\infty}^{x_2} \int_{y_3=-\infty}^{\frac{x_3-\rho_2 y_2}{\sqrt{1-\rho_2^2}}} \frac{1}{2\pi}\exp\left(-\frac{\left(y_2^2+y_3^2\right)}{2}\right) N\left[\frac{x_1-\rho_1 y_2}{\sqrt{1-\rho_1^2}}\right] N\left[\frac{x_4-\rho_3\sqrt{1-\rho_2^2}\,y_3-\rho_3\rho_2 y_2}{\sqrt{1-\rho_3^2}}\right] dy_2 dy_3$$

More generally, the actual numerical dimension of the function Φ_n can always be reduced by a factor of 2 by using :

$$\Phi_n\left[x_1, x_2, \dots, x_{n-1}, x_n; \rho_1, \dots, \rho_{n-2}, \rho_{n-1}\right]$$

$$= \int_{y_2=-\infty}^{x_2} \int_{y_3=-\infty}^{x_3} \dots \int_{y_{n-1}=-\infty}^{x_{n-1}} N\left[\frac{x_1-\rho_1 y_2}{\sqrt{1-\rho_1^2}}\right] N\left[\frac{x_n-\rho_{n-1} y_{n-1}}{\sqrt{1-\rho_{n-1}^2}}\right] \tag{2.37}$$

$$\frac{\exp\left(-\frac{y_2^2}{2}-\frac{1}{2\left(1-\rho_2^2\right)}\left(y_3-\rho_2 y_2\right)^2 \dots -\frac{1}{2\left(1-\rho_{n-2}^2\right)}\left(y_{n-1}-\rho_{n-2} y_{n-2}\right)^2\right)}{\prod_{i=2}^{n-2}\sqrt{\left(1-\rho_i^2\right)}\left(2\pi\right)^{\frac{n-2}{2}}} dy_2 dy_3 \dots dy_{n-1}$$

Given the smoothness of the integrand in (2.37), it should be possible to attain a combination of accuracy and efficiency that would be satisfactory for all practical purposes in "moderate" dimension, roughly speaking, by applying adaptive Gauss-Legendre quadrature combined with a Kronrod rule (Kronrod, 1964; Calvetti et al., 2000) to reduce the number of required iterations. These are standard numerical techniques and it is easy to find available code or built-in functions in the usual scientific computing software. The dimension n at which the use of a closed form formula analogous to Formula 1 ceases to be "competitive" with regard to a conditional Monte Carlo scheme should be numerically investigated. It must be emphasized that, even in "high" dimension, where Monte Carlo simulation becomes the method of last resort, exact formulae valid in lower dimension remain useful in two ways : they provide benchmarks with respect to which the accuracy of the numerical algorithms can be checked, and they can be used as control variates that substantially reduce the variance of the Monte Carlo estimates.

3. Survival Probability of an Arithmetic or a Geometric Brownian Motion under and above a Sequence of Two-sided affine or Exponential Boundaries over a Finite Time Interval

3.1 Definitions

Let us consider a finite time interval $\left[t_0, t_2\right]$ divided in two subintervals $\left[t_0, t_1\right]$ and $\left[t_1, t_2\right]$, $t_2 \geq t_1 \geq t_0 = 0$. The absorbing boundary now consists of two parallel upper and lower curves in each time interval, these curves being line segments when dealing with process X_1, or exponential curves when dealing with process X_2. More specifically, let $\mathrm{P}_{[\dots]}\left(\mu, \sigma, a_1, a_2, a_3, a_4, b_1, b_2, k_1, k_2, t_1, t_2\right)$ be defined as one of the following four cumulative distribution functions, where k_1 and k_2 are real constants :

$$\mathrm{P}_{[\mathrm{AUL1}]}\left(\mu,\sigma,a_1,a_2,a_3,a_4,b_1,b_2,k_1,k_2,t_1,t_2\right),\ a_1\in\mathbb{R}_+, a_2\in\mathbb{R}_-, \left(a_3,a_4,b_1,b_2,k_1,k_2\right)\in\mathbb{R}^8,$$
$$a_3>a_4, a_3>a_2+b_1t_1,\ t_2\geq t_1\geq t_0=0 \tag{3.1}$$

$$=\mathbb{P}\begin{pmatrix}\left(X_1(t)<a_1+b_1t,\forall 0\leq t\leq t_1\right)\cap\left(X_1(t)>a_2+b_1t,\forall 0\leq t\leq t_1\right)\cap X_1(t_1)<k_1\\ \cap\left(X_1(t)<a_3+b_2(t-t_1),\forall t_1\leq t\leq t_2\right)\cap\left(X_1(t)>a_4+b_2(t-t_1),\forall t_1\leq t\leq t_2\right)\cap X_1(t_2)<k_2\end{pmatrix}$$

$$\mathrm{P}_{[\mathrm{AUL2}]}\left(\mu,\sigma,a_1,a_2,a_3,a_4,b_1,b_2,k_1,k_2,t_1,t_2\right),\ a_1\in\mathbb{R}_+, a_2\in\mathbb{R}_-, \left(a_3,a_4,b_1,b_2,k_1,k_2\right)\in\mathbb{R}^8,$$
$$a_3>a_4, a_3+b_2t_1>a_2+b_1t_1,\ t_2\geq t_1\geq t_0=0 \tag{3.2}$$

$$=\mathbb{P}\begin{pmatrix}\left(X_1(t)<a_1+b_1t,\forall 0\leq t\leq t_1\right)\cap\left(X_1(t)>a_2+b_1t,\forall 0\leq t\leq t_1\right)\cap X_1(t_1)<k_1\\ \cap\left(X_1(t)<a_3+b_2t,\forall t_1\leq t\leq t_2\right)\cap\left(X_1(t)>a_4+b_2t,\forall t_1\leq t\leq t_2\right)\cap X_1(t_2)<k_2\end{pmatrix}$$

$$\mathrm{P}_{[\mathrm{GUL1}]}\left(\mu,\sigma,a_1,a_2,a_3,a_4,b_1,b_2,k_1,k_2,t_1,t_2\right),\ a_1\in\mathbb{R}_+, a_2\in\mathbb{R}_-, \left(a_3,a_4,b_1,b_2,k_1,k_2\right)\in\mathbb{R}^8,$$
$$a_3>a_4, a_3>a_2+b_1t_1,\ t_2\geq t_1\geq t_0=0 \tag{3.3}$$

$$=\mathbb{P}\begin{pmatrix}\left(X_2(t)<X_2(0)\exp\left(a_1+b_1t\right),\forall 0\leq t\leq t_1\right)\cap\left(X_2(t)>X_2(0)\exp\left(a_2+b_1t\right),\forall 0\leq t\leq t_1\right)\\ \cap X_2(t_1)<k_1\cap\left(X_2(t)<X_2(0)\exp\left(a_3+b_2(t-t_1)\right),\forall t_1\leq t\leq t_2\right)\\ \cap\left(X_2(t)>X_2(0)\exp\left(a_4+b_2(t-t_1)\right),\forall t_1\leq t\leq t_2\right)\cap X_2(t_2)<k_2\end{pmatrix}$$

$$\mathrm{P}_{[\mathrm{GUL2}]}\left(\mu,\sigma,a_1,a_2,a_3,a_4,b_1,b_2,k_1,k_2,t_1,t_2\right),\ a_1\in\mathbb{R}_+, a_2\in\mathbb{R}_-, \left(a_3,a_4,b_1,b_2,k_1,k_2\right)\in\mathbb{R}^8,$$
$$a_3>a_4, a_3+b_2t_1>a_2+b_1t_1,\ t_2\geq t_1\geq t_0=0 \tag{3.4}$$

$$=\mathbb{P}\begin{pmatrix}\left(X_2(t)<X_2(0)\exp\left(a_1+b_1t\right),\forall 0\leq t\leq t_1\right)\cap\left(X_2(t)>X_2(0)\exp\left(a_2+b_1t\right),\forall 0\leq t\leq t_1\right)\\ \cap X_2(t_1)<k_1\cap\left(X_2(t)<X_2(0)\exp\left(a_3+b_2t\right),\forall t_1\leq t\leq t_2\right)\\ \cap\left(X_2(t)>X_2(0)\exp\left(a_4+b_2t\right),\forall t_1\leq t\leq t_2\right)\cap X_2(t_2)<k_2\end{pmatrix}$$

3.2 Statement of Formula 2

Formula 2 *Let* $\mathrm{P}_{[\ldots]}\left(\mu,\sigma,a_1,a_2,a_3,a_4,b_1,b_2,k_1,k_2,t_1,t_2\right)$ *be defined as in Subsection 3.1. Then,*

$$\mathrm{P}_{[\ldots]}\left(\mu,\sigma,a_1,a_2,a_3,a_4,b_1,b_2,k_1,k_2,t_1,t_2\right)$$
$$=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\exp\left(\frac{2\mu_1}{\sigma^2}m\theta+\frac{2\mu_2}{\sigma^2}n\phi\right) \tag{3.5}$$

$$\begin{Bmatrix} N_2\left[\dfrac{\beta_1-2m\theta+\lambda_1}{\sigma\sqrt{t_1}},\dfrac{\beta_3-2m\theta-2n\phi+\lambda_2}{\sigma\sqrt{t_2}};\sqrt{\dfrac{t_1}{t_2}}\right] \\ -N_2\left[\dfrac{\beta_2-2m\theta+\lambda_1}{\sigma\sqrt{t_1}},\dfrac{\beta_3-2m\theta-2n\phi+\lambda_2}{\sigma\sqrt{t_2}};\sqrt{\dfrac{t_1}{t_2}}\right] \\ -N_2\left[\dfrac{\beta_1-2m\theta+\lambda_1}{\sigma\sqrt{t_1}},\dfrac{\beta_4-2m\theta-2n\phi+\lambda_2}{\sigma\sqrt{t_2}};\sqrt{\dfrac{t_1}{t_2}}\right] \\ +N_2\left[\dfrac{\beta_2-2m\theta+\lambda_1}{\sigma\sqrt{t_1}},\dfrac{\beta_4-2m\theta-2n\phi+\lambda_2}{\sigma\sqrt{t_2}};\sqrt{\dfrac{t_1}{t_2}}\right] \end{Bmatrix}$$

$$-\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\exp\left(\frac{2\mu_1}{\sigma^2}m\theta+\frac{2\mu_2}{\sigma^2}\left(\beta_4-n\phi-2m\theta\right)+\frac{2}{\sigma^2}\left(\mu_2^2-\mu_1\mu_2\right)t_1\right)$$

$$\begin{Bmatrix} N_2\left[\dfrac{\beta_1-2m\theta+\lambda_3}{\sigma\sqrt{t_1}},\dfrac{\beta_3-2\beta_4+2m\theta+2n\phi+\lambda_4}{\sigma\sqrt{t_2}};-\sqrt{\dfrac{t_1}{t_2}}\right] \\ -N_2\left[\dfrac{\beta_2-2m\theta+\lambda_3}{\sigma\sqrt{t_1}},\dfrac{\beta_3-2\beta_4+2m\theta+2n\phi+\lambda_4}{\sigma\sqrt{t_2}};-\sqrt{\dfrac{t_1}{t_2}}\right] \\ -N_2\left[\dfrac{\beta_1-2m\theta+\lambda_3}{\sigma\sqrt{t_1}},\dfrac{-\beta_4+2m\theta+2n\phi+\lambda_4}{\sigma\sqrt{t_2}};-\sqrt{\dfrac{t_1}{t_2}}\right] \\ +N_2\left[\dfrac{\beta_2-2m\theta+\lambda_3}{\sigma\sqrt{t_1}},\dfrac{-\beta_4+2m\theta+2n\phi+\lambda_4}{\sigma\sqrt{t_2}};-\sqrt{\dfrac{t_1}{t_2}}\right] \end{Bmatrix}$$

$$-\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\exp\left(\frac{2\mu_1}{\sigma^2}\left[a_2-m\theta\right]+\frac{2\mu_2}{\sigma^2}\left[n\phi\right]\right)$$

$$\begin{Bmatrix} N_2\left[\dfrac{\beta_1-2a_2+2m\theta+\lambda_1}{\sigma\sqrt{t_1}},\dfrac{\beta_3-2a_2+2m\theta-2n\phi+\lambda_2}{\sigma\sqrt{t_2}};\sqrt{\dfrac{t_1}{t_2}}\right] \\ -N_2\left[\dfrac{\beta_2-2a_2+2m\theta+\lambda_1}{\sigma\sqrt{t_1}},\dfrac{\beta_3-2a_2+2m\theta-2n\phi+\lambda_2}{\sigma\sqrt{t_2}};\sqrt{\dfrac{t_1}{t_2}}\right] \\ -N_2\left[\dfrac{\beta_1-2a_2+2m\theta+\lambda_1}{\sigma\sqrt{t_1}},\dfrac{\beta_4-2a_2+2m\theta-2n\phi+\lambda_2}{\sigma\sqrt{t_2}};\sqrt{\dfrac{t_1}{t_2}}\right] \\ +N_2\left[\dfrac{\beta_2-2a_2+2m\theta+\lambda_1}{\sigma\sqrt{t_1}},\dfrac{\beta_4-2a_2+2m\theta-2n\phi+\lambda_2}{\sigma\sqrt{t_2}};\sqrt{\dfrac{t_1}{t_2}}\right] \end{Bmatrix}$$

$$+\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\exp\left(\frac{2\mu_1}{\sigma^2}\left(a_2-m\theta\right)+\frac{2\mu_2}{\sigma^2}\left(\beta_4-n\phi+2m\theta-2a_2\right)+\frac{2}{\sigma^2}\left(\mu_2^2-\mu_1\mu_2\right)t_1\right)$$

$$\left\{\begin{array}{l} N_2\left[\dfrac{\beta_1-2a_2+2m\theta+\lambda_3}{\sigma\sqrt{t_1}},\dfrac{\beta_3-2\beta_4+2a_2+2n\phi-2m\theta+\lambda_4}{\sigma\sqrt{t_2}};-\sqrt{\dfrac{t_1}{t_2}}\right] \\ -N_2\left[\dfrac{\beta_2-2a_2+2m\theta+\lambda_3}{\sigma\sqrt{t_1}},\dfrac{\beta_3-2\beta_4+2a_2+2n\phi-2m\theta+\lambda_4}{\sigma\sqrt{t_2}};-\sqrt{\dfrac{t_1}{t_2}}\right] \\ -N_2\left[\dfrac{\beta_1-2a_2+2m\theta+\lambda_3}{\sigma\sqrt{t_1}},\dfrac{-\beta_4+2a_2+2n\phi-2m\theta+\lambda_4}{\sigma\sqrt{t_2}};-\sqrt{\dfrac{t_1}{t_2}}\right] \\ +N_2\left[\dfrac{\beta_2-2a_2+2m\theta+\lambda_3}{\sigma\sqrt{t_1}},\dfrac{-\beta_4+2a_2+2n\phi-2m\theta+\lambda_4}{\sigma\sqrt{t_2}};-\sqrt{\dfrac{t_1}{t_2}}\right] \end{array}\right\}$$

where the following notations hold :

$$\mu_1=\left(\mu-b_1\right)\left(\mathbb{I}_{\left\{P_{[\ldots]}=P_{[AUL1]}\right\}}+\mathbb{I}_{\left\{P_{[\ldots]}=P_{[AUL2]}\right\}}\right)+\left(\mu-\frac{\sigma^2}{2}-b_1\right)\left(\mathbb{I}_{\left\{P_{[\ldots]}=P_{[GUL1]}\right\}}+\mathbb{I}_{\left\{P_{[\ldots]}=P_{[GUL2]}\right\}}\right)$$

$$\mu_2=\left(\mu-b_2\right)\left(\mathbb{I}_{\left\{P_{[\ldots]}=P_{[AUL1]}\right\}}+\mathbb{I}_{\left\{P_{[\ldots]}=P_{[AUL2]}\right\}}\right)+\left(\mu-\frac{\sigma^2}{2}-b_2\right)\left(\mathbb{I}_{\left\{P_{[\ldots]}=P_{[GUL1]}\right\}}+\mathbb{I}_{\left\{P_{[\ldots]}=P_{[GUL2]}\right\}}\right)$$

$$\theta=a_1-a_2$$

$$\phi=a_3-a_4$$

$$\beta_1=\left(\min\left(a_1+b_1t_1,k_1,a_3\right)-b_1t_1\right)\mathbb{I}_{\left\{P_{[\ldots]}=P_{[AUL1]}\right\}}+\left(\min\left(a_1+b_1t_1,k_1,a_3+b_2t_1\right)-b_1t_1\right)\mathbb{I}_{\left\{P_{[\ldots]}=P_{[AUL2]}\right\}}$$

$$+\left(\min\left(a_1+b_1t_1,\ln\left(k_1/X_2(0)\right),a_3\right)-b_1t_1\right)\mathbb{I}_{\left\{P_{[\ldots]}=P_{[GUL1]}\right\}}$$

$$+\left(\min\left(a_1+b_1t_1,\ln\left(k_1/X_2(0)\right),a_3+b_2t_1\right)-b_1t_1\right)\mathbb{I}_{\left\{P_{[\ldots]}=P_{[GUL2]}\right\}}$$

$$\beta_2=\left(\max\left(a_2+b_1t_1,a_4\right)-b_1t_1\right)\left(\mathbb{I}_{\left\{P_{[\ldots]}=P_{[AUL1]}\right\}}+\mathbb{I}_{\left\{P_{[\ldots]}=P_{[GUL1]}\right\}}\right)$$

$$+\left(\max\left(a_2+b_1t_1,a_4+b_2t_1\right)-b_1t_1\right)\left(\mathbb{I}_{\left\{P_{[\ldots]}=P_{[AUL2]}\right\}}+\mathbb{I}_{\left\{P_{[\ldots]}=P_{[GUL2]}\right\}}\right)$$

$$\beta_3=\left(\min\left(a_3+b_2\left(t_2-t_1\right),k_2\right)-b_2t_2\right)\mathbb{I}_{\left\{P_{[\ldots]}=P_{[AUL1]}\right\}}+\left(\min\left(a_3+b_2t_2,k_2\right)-b_2t_2\right)\mathbb{I}_{\left\{P_{[\ldots]}=P_{[AUL2]}\right\}}$$

$$+\left(\min\left(a_3+b_2\left(t_2-t_1\right),\ln\left(k_2/X_2(0)\right)\right)-b_2t_2\right)\mathbb{I}_{\left\{P_{[\ldots]}=P_{[GUL1]}\right\}}$$

$$+\left(\min\left(a_3+b_2t_2,\ln\left(k_2/X_2(0)\right)\right)-b_2t_2\right)\mathbb{I}_{\left\{P_{[\ldots]}=P_{[GUL2]}\right\}}$$

$$\beta_4=\left(a_4-b_2t_1\right)\left(\mathbb{I}_{\left\{P_{[\ldots]}=P_{[AUL1]}\right\}}+\mathbb{I}_{\left\{P_{[\ldots]}=P_{[GUL1]}\right\}}\right)+a_4\left(\mathbb{I}_{\left\{P_{[\ldots]}=P_{[AUL2]}\right\}}+\mathbb{I}_{\left\{P_{[\ldots]}=P_{[GUL2]}\right\}}\right)$$

$$\lambda_1=-\mu_1t_1,\ \lambda_2=-\mu_1t_1-\mu_2\left(t_2-t_1\right),\ \lambda_3=-\mu_1t_1+2\mu_2t_1,\ \lambda_4=\mu_1t_1-2\mu_2t_1-\mu_2\left(t_2-t_1\right)$$

End of Formula 2.

A few numerical values are reported in Table 2 for various levels of volatility and other parameters fixed as follows :

$\mu=0.01$, $t_1=0.25$, $t_2=0.5$, $t_3=1$, $k_1=0$, $k_2=0.02$, $a_1=0.36$, $b_1=0.15$, $a_2=-0.42$,

$b_2 = 0.15$, $a_3 = a_1 + b_1 t_1$, $b_3 = -0.12$, $a_4 = a_2 + b_2 t_1$, $b_4 = -0.12$. A comparison is made with results obtained using the algorithm by Pötzelberger and Wang (2001), denoted by PW, specifically designed for two-sided boundaries. The infinite double series in Formula 2 is truncated to summation operators ranging from $m = -4$ to $m = 4$ and from $n = -4$ to $n = 4$, since adding more terms does not modify the obtained numerical results at least up to the 8th digit. Computational time is approximately 0.3 second. In general, the infinite double series can be truncated in a simple manner by setting a convergence threshold such that no further terms are added once the difference between two successive finite sums becomes smaller than that prespecified level.

Table 2. Numerical evaluation of the survival probability of an arithmetic Brownian under a two-sided piecewise affine, time-homogeneous, absorbing boundary, as a function of volatility

	Formula 2	PW simulation algorithm 10,000,000 runs	PW simulation algorithm 100,000,000 runs
Volatility = 20%	0.377958716	0.377923023	0.377971427
Volatility = 50%	0.12468422	0.124786191	0.124625324
Volatility = 80%	0.02140513	0.021276911	0.021382631

3.3 Proof of Formula 2

Let us consider the calculation of $\mathrm{P}_{[\mathrm{AUL1}]}$. Since the upper and the lower sides of the boundary grow at the same rate in each time interval, i.e. at the rate b_1 both from below and from above in $[t_0, t_1]$ and at the rate b_2 both from below and from above in $[t_1, t_2]$, the same technique can be applied as in the beginning of the proof of Formula 1, i.e. the initial boundary crossing problem is turned into one where the boundary and the drift of the process become piecewise constant. Hence, denoting by p the sought probability, the problem can be formulated as follows :

$$p = \int_{\beta_2}^{\beta_1} \int_{\beta_4}^{\beta_3} f_1(x_1) f_2(x_1, x_2) dx_2 dx_1 \tag{3.6}$$

where the functions $f_1(x_1)$ and $f_2(x_1, x_2)$ are defined by :

$$f_1(x_1) = \mathbb{P}\left(\sup_{0 \le t \le t_1} Y(t) < a_1, \inf_{0 \le t \le t_1} Y(t) > a_2 \,\middle|\, Y(t_1) \in dx_1 \right) \tag{3.7}$$

$$f_2(x_1, x_2) = \mathbb{P}\left(\sup_{t_1 \le t \le t_2} Y(t) < a_3 - b_2 t_1, \inf_{t_1 \le t \le t_2} Y(t) > a_4 - b_2 t_1, Y(t_2) \le x_2 \,\middle|\, Y(t_1) \in dx_1 \right) \tag{3.8}$$

and the process $\{Y(t), t \ge 0\}$ is defined by :

$$dY(t) = \begin{cases} \mu_1 dt + \sigma dB(t), \forall 0 \le t < t_1 \\ \mu_2 dt + \sigma dB(t), \forall t_1 \le t \le t_2 \end{cases} \tag{3.9}$$

$$\mu_i = \mu - b_i, \; i \in \{1,2\}$$

The function $f_1(x_1)$ results from the differentiation of the classical formula for the joint distribution of the maximum, the minimum and the endpoint of a Brownian motion (see, e.g., Cox & Miller, 1965). To obtain $f_2(x_1, x_2)$, the following lemma is introduced.

Lemma 2 *Let $Y(t)$ be an arithmetic Brownian motion with constant drift $\mu \in \mathbb{R}$ and volatility $\sigma \in \mathbb{R}_+$ under a given probability measure $\mathbb{P}$. Let q be the conditional probability defined, at time $t_0 = 0$, by :*

$$q = \mathbb{P}\left(\sup_{t_i \le t \le t_j} Y(t) \le b, \inf_{t_i \le t \le t_j} Y(t) \ge a, Y(t_j) \le x_j \middle| Y(t_i) \in dx_i \right) \tag{3.10}$$

where x_i, x_j, a and b are real constants such that : $b > a$, $b \ge x_i > a$, $b \ge x_j > a$, and t_i and t_j are two non-random times such that : $t_j > t_j \ge 0$. Then,

$$q = \sum_{n=-\infty}^{\infty} \exp\left(\frac{2n\mu(b-a)}{\sigma^2} \right) \left[\begin{array}{l} N\left(\dfrac{x_j - x_i - \mu(t_j - t_i) - 2n(b-a)}{\sigma\sqrt{t_j - t_i}} \right) \\ -N\left(\dfrac{a - x_i - \mu(t_j - t_i) - 2n(b-a)}{\sigma\sqrt{t_j - t_i}} \right) \end{array} \right] \tag{3.11}$$

$$- \sum_{n=-\infty}^{\infty} \exp\left(\frac{2\mu(a - x_i - n(b-a))}{\sigma^2} \right) \left[\begin{array}{l} N\left(\dfrac{x_j - 2a + x_i - \mu(t_j - t_i) + 2n(b-a)}{\sigma\sqrt{t_j - t_i}} \right) \\ -N\left(\dfrac{-a + x_i - \mu(t_j - t_i) + 2n(b-a)}{\sigma\sqrt{t_j - t_i}} \right) \end{array} \right]$$

Proof of lemma 2

$$\mathbb{P}\left(Y(t_i) \le x_i, \sup_{t_i \le t \le t_j} Y(t) \le b, \inf_{t_i \le t \le t_j} Y(t) \ge a, Y(t_j) \le x_j \right) \tag{3.12}$$

$$= \int_{y=a}^{x_i} \int_{z=a}^{x_j} \mathbb{P}\left(Y(t_i) \in dy, Y(t_j) \in dz\right) \mathbb{P}\left(\sup_{t_i \le t \le t_j} Y(t) \le b, \inf_{t_i \le t \le t_j} Y(t) \ge a \middle| Y(t_i) \in dy, Y(t_j) \in dz \right) dzdy$$

The following result can be found in Guillaume (2010) :

$$\mathbb{P}\left(\sup_{t_i \le t \le t_j} Y(t) \le b, \inf_{t_i \le t \le t_j} Y(t) \ge a \middle| Y(t_i) \in dy, Y(t_j) \in dz \right) \tag{3.13}$$

$$= \sum_{n=-\infty}^{\infty} \exp\left(\frac{2n(b-a)(z - y - n(b-a))}{\sigma^2 (t_j - t_i)} \right) - \exp\left(\frac{2(b - y - n(b-a))(z - b + n(b-a))}{\sigma^2 (t_j - t_i)} \right)$$

Plugging (3.13) into (3.12) yields :

$$\mathbb{P}\left(Y\left(t_i\right) \le x_i, \sup_{t_i \le t \le t_j} Y\left(t\right) \le b, \inf_{t_i \le t \le t_j} Y\left(t\right) \ge a, Y\left(t_j\right) \le x_j\right)$$

$$= \sum_{n=-\infty}^{\infty} \exp\left(\frac{2n\mu\left(b-a\right)}{\sigma^2}\right)\left\{N_2\left[\frac{x_i - \mu t_i}{\sigma\sqrt{t_i}}, \frac{x_j - 2n\left(b-a\right) - \mu t_j}{\sigma\sqrt{t_j}}; \sqrt{\frac{t_i}{t_j}}\right]\right. \tag{3.14}$$

$$-N_2\left[\frac{x_i - \mu t_i}{\sigma\sqrt{t_i}}, \frac{a - 2n\left(b-a\right) - \mu t_j}{\sigma\sqrt{t_j}}; \sqrt{\frac{t_i}{t_j}}\right] - N_2\left[\frac{a - \mu t_i}{\sigma\sqrt{t_i}}, \frac{x_j - 2n\left(b-a\right) - \mu t_j}{\sigma\sqrt{t_j}}; \sqrt{\frac{t_i}{t_j}}\right]$$

$$\left. + N_2\left[\frac{a - \mu t_i}{\sigma\sqrt{t_i}}, \frac{a - 2n\left(b-a\right) - \mu t_j}{\sigma\sqrt{t_j}}; \sqrt{\frac{t_i}{t_j}}\right]\right\}$$

$$- \sum_{n=-\infty}^{\infty} \exp\left(\frac{2\mu\left(a - n\left(b-a\right)\right)}{\sigma^2}\right)\left\{N_2\left[\frac{x_i + \mu t_i}{\sigma\sqrt{t_i}}, \frac{x_j - 2a + 2n\left(b-a\right) - \mu t_j}{\sigma\sqrt{t_j}}; -\sqrt{\frac{t_i}{t_j}}\right]\right.$$

$$-N_2\left[\frac{x_j + \mu t_i}{\sigma\sqrt{t_i}}, \frac{-a + 2n\left(b-a\right) - \mu t_j}{\sigma\sqrt{t_j}}; -\sqrt{\frac{t_i}{t_j}}\right] - N_2\left[\frac{a + \mu t_i}{\sigma\sqrt{t_i}}, \frac{x_j - 2a + 2n\left(b-a\right) - \mu t_j}{\sigma\sqrt{t_j}}; -\sqrt{\frac{t_i}{t_j}}\right]$$

$$\left. + N_2\left[\frac{a + \mu t_i}{\sigma\sqrt{t_i}}, \frac{-a + 2n\left(b-a\right) - \mu t_j}{\sigma\sqrt{t_j}}; -\sqrt{\frac{t_i}{t_j}}\right]\right\}$$

The interchange between summation and integral is a straightforward application of Tonelli's theorem to non-negative measurable functions, where the measures are the counting measure on $\mathbb{Z}$ and the Lebesgue measure on $\mathbb{R}$. Lemma 2 ensues by differentiating (3.14) and dividing by the density function of $Y\left(t_i\right)$.

□

Applying Lemma 2, the function $f_2\left(x_1, x_2\right)$ can be plugged in (3.6). Then, performing the necessary calculations, Formula 2 can be obtained.

□

4. Conclusion

In this paper, new formulae were obtained for the probability of absorption of generalised Brownian motion through sequences of affine or exponential one-sided or two-sided boundaries. It was shown that the method could be applied to higher numbers of successive one-sided boundaries. However, such an extension may not be commendable in the case of two-sided boundaries, as the resulting analytical formulae will involve a quickly increasing number of summation operators, thus slowing down the process of numerical convergence.

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