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The Principle of Indifference Does Not Lead to Contradictions

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Abstract

The Principle of Indifference says that if there are a finite number of propositions and a state of knowledge according to which none of the propositions is more plausible than any other, then, conditional on that knowledge, all of them have the same probability. Most researchers reject the principle because there exist counterexamples believed to prove that it leads to contradictions. We analyse three examples representative of the objections to the principle and show that, rather than disproving it, they suffer from a common error in applying it. From this and the fact that the Principle of Indifference complies with plausible reasoning we conclude that it does not lead to contradictions.

Keywords: Principle of Indifference, Principle of Insufficient Reason

1. Introduction

The Principle of Indifference, which dates back to Jakob Bernoulli, can be expressed as follows:

> If, given some proposition C, one of the propositions $A_1, \cdots, A_n$ must be true and the others must be false and none of the A_i is more plausible than any other, then, conditional on C, all A_i have the same probability $p(A_i|C) = 1/n$.

C is often referred to as someone's (prior) knowledge, which may or may not contain reasons to favour some proposition over some other. The rule became known as the *Principle of Insufficient Reason*. In 1921, Keynes introduced the term *Principle of Indifference* (Keynes, 1921). Keynes raised doubt whether the principle was valid and presented counterexamples intended to prove that it leads to contradictions and is therefore invalid. Other authors have commented on these examples and added their own.

Due to the presumed contradictions, most researchers have rejected and still reject the principle (Bartelborth, 2012; Carnap, 1966; Howie, 2002; Robert, 2007; van Fraassen, 1989). However, in the 1950s, Jaynes derived it from a set of desiderata for plausible reasoning (Jaynes 1958, 2003). In his probability theory, the principle is a proven theorem; it cannot be contradictory there, unless the whole theory were contradictory. If it is not, it must be possible to eliminate the contradictions on the ground of Jaynes' desiderata. This is what we are going to undertake.

2. Counterexamples to the Principle

We start with three examples Keynes gave in order to demonstrate where the Principle of Indifference fails. For ease of presentation, we adapt the wording without substantially altering the content.

1. *The colour of the book*: A person who knows nothing about the colour of a certain book must, according to the Principle of Indifference, assign the proposition R := *The book is red* the same probability as its contrary $\overline{R}$ = *The book is not red*, namely $p(R) = 1/2$. For the same reason the person must assign the proposition B := *The book is blue* the probability $p(B) = 1/2$ and the proposition G := *The book is green* the probability $p(G) = 1/2$. But now the proposition $R \vee B \vee G$ = *The book is red or blue or green* would have the probability $p(R \vee B \vee G) = p(R) + p(B) + p(G) = 3/2$, which is impossible as probabilities do not exceed unity.

2. *Sizes of countries*: A person who knows nothing about the sizes of countries must, according to the Principle of Indifference, assign the proposition E := *England is bigger than France* the same probability as its contrary $\overline{E}$ = *England is not bigger than France*, namely $p(E) = 1/2$. For the same reason the person must assign the proposition B := *The British Isles are bigger than France* the probability $p(B) = 1/2$. But this is impossible, since England is only a part of the British Isles; the Isles must surpass France in size with higher probability than England alone.

3. *Specific volume vs. specific density*: A person who only knows that the specific volume of some substance lies between 1 and 3 (measured in some unit) must, according to the Principle of Indifference, assign the propositions V := *The specific volume lies between 1 and 2* and its contrary $\overline{V}$ = *The specific volume lies between 2 and 3* the probabilities $p(V) = p(\overline{V}) = 1/2$. The reciprocal of the specific volume is the specific density. The person only knows that it lies between 1/3 and 1, and must therefore assign the proposition D := *The specific density lies between 1/3 and 2/3* the same probability as its contrary $\overline{D}$ = *The specific density lies between 2/3 and 1*, namely $p(D) = 1/2$. But this is impossible, since D says (though in other words) that the specific volume lies between 3/2 and 3 and must be more probable than $\overline{V}$.

As far as we know, every serious objection to the Principle of Indifference is a modification of one of these examples, corresponding to one of the following patterns:

1. The pattern of *The colour of the book*: If one out of two equally probable propositions (*The book is red* and *The book is not red*) is dissected (as *The book is not red* is dissected into *The book is blue*, *The book is green*, and maybe others, too), the resulting ones (*The book is red*, *The book is blue*, *The book is green*, etc.) cannot be equally probable; yet the Principle of Indifference says they are equally probable.
2. The pattern of *Sizes of countries*: A proposition (*England is bigger than France*) cannot have the same probability as an obviously more probable one (*The British Isles are bigger than France*); yet the Principle of Indifference says it has the same probability.
3. The pattern of *Specific volume vs. specific density*: Uniform distribution on one scale (equal probabilities for equally large intervals on the specific volume scale) leads to non-uniform distribution on a different scale (non-equal probabilities for equally large intervals on the specific density scale); yet the Principle of Indifference requires uniform distributions on both scales.

Among the objections that correspond to these patterns are: the *partitioning incoherence of Laplace's equiprobability* claimed by Robert (2007), corresponding to pattern 1; Carnap's *life on Mars* (Carnap, 1966), corresponding to pattern 2; Keynes' *urn with black and white balls* (Keynes, 1921), van Fraassen's *cube factory* (van Fraassen, 1989) and von Mises' *wine/water-paradox* (Mikkelson, 2004), all corresponding to pattern 3.

At first sight, the examples seem to disprove the Principle of Indifference; however, we will find that they suffer from a common error in applying it.

3. Desiderata for Plausible Reasoning

We base our analysis on Jaynes' desiderata for plausible reasoning, i.e. for assigning degrees of plausibility to propositions (in Jaynes' terminology, such plausibility assignment is called a *conclusion*):

(I) Degrees of plausibility are represented by real numbers.

(II) Plausible reasoning qualitatively corresponds with common sense.

(IIIa) If a conclusion can be reasoned in more than one way, every way leads to the same result.

(IIIb) Every conclusion is based on all available knowledge.

(IIIc) Equivalent states of knowledge lead to equivalent conclusions.

Desideratum (I), together with the convention that a greater plausibility shall correspond to a greater number, guarantees that 1) any two propositions A and B can be compared with respect to plausibility such that either A is more plausible than B, or B is more plausible than A, or A and B are equally plausible, and 2) if A is more plausible than B, and B is more plausible than C, then A is more plausible than C.

From desideratum (II), the following rule, which we call *implication rule*, can be derived:

> If, given C, A implies B and B does not imply A, then, given C, B is more plausible than A.

It corresponds with common sense because, given C, B is true whenever A is true, but B can even be true when A is false.

Desiderata (IIIa)–(IIIc) ensure that reasoning is consistent.

4. Analysis of the Counterexamples

4.1 Example 1: The colour of the book

We start the examination by asking for the knowledge based on which the probabilities are assigned. Let C_R be some knowledge according to which R is equally plausible as $\overline{R}$; then, following the Principle of Indifference, the probability of R, given C_R, is $1/2$: $p(R|C_R) = 1/2$. Let C_B be some knowledge which makes B equally plausible as $\overline{B}$, and C_G some knowledge which makes G equally plausible as $\overline{G}$, then $p(B|C_B) = 1/2$ and $p(G|C_G) = 1/2$. Now each of the colours has a probability of $1/2$ to be the colour of the book, but these probabilities are conditional on *different* knowledge; and

$$p(R|C_R) + p(B|C_B) + p(G|C_G) = 3/2$$

does not violate any rule of probability (as C_R, C_B and C_G are different, the sum on the left side of the equation is not a probability). This has already been observed by Jeffreys in his review of Keynes' work (Jeffreys, 1922).

As long as C_R, C_B and C_G are not identical, Example 1 does not disprove the Principle of Indifference. The principle would fail only if there existed a state of knowledge which made R equally plausible as $\overline{R}$ and, at the same time, B equally plausible as $\overline{B}$ and G equally plausible as $\overline{G}$. Now we prove by contradiction that such knowledge cannot exist.

Assume that there exists some knowledge C according to which each of the propositions R, B and G is equally plausible as its respective contrary. First we note that C allows red, blue and green to be possible colours of the book (if, for instance, red were impossible, then R would be less plausible than $\overline{R}$). Then, given C, R implies $\overline{B}$ but $\overline{B}$ does not imply R (a red book is clearly non-blue, whereas a non-blue book does not have to be red); following the implication rule, $\overline{B}$ is more plausible than R, conditional on C. Using Jaynes' notation, where $A|B$ stands for the plausibility of A, given B, we have thus found

$$\overline{B}|C > R|C\,.$$

An analogous reasoning shows that $\overline{R}$ is more plausible than B, conditional on C:

$$\overline{R}|C > B|C\,.$$

Now remember that C is assumed to make R and $\overline{R}$ equally plausible:

$$R|C = \overline{R}|C\,.$$

Putting these relations together, we arrive at

$$\overline{B}|C > R|C = \overline{R}|C > B|C\,,$$

which contradicts the assumption that C makes B and $\overline{B}$ equally plausible. Therefore, a state of knowledge according to which each of the propositions R, B and G were equally plausible as its respective contrary does not exist.

From the above follows that Example 1 does not disprove the Principle of Indifference; its paradox results from the assumption of prior knowledge which cannot exist.

4.2 Example 2: Sizes of countries

As in Example 1, we start by asking for the knowledge based on which the probabilities are assigned. Let C_E be some knowledge according to which E is equally plausible as $\overline{E}$; then $p(E|C_E) = 1/2$. Let C_B be some knowledge which makes B equally plausible as $\overline{B}$; then $p(B|C_B) = 1/2$. We thus obtain

$$p(E|C_E) = p(B|C_B)\,;$$

but although England is a part of the British Isles, this equality is not impossible since the probabilities are conditional on different knowledge.

The Principle of Indifference would fail on Example 2 only if there existed a state of knowledge which contained the information that England is a part of the British Isles and, at the same time, made E equally plausible as $\overline{E}$ and B equally plausible as $\overline{B}$. We prove by contradiction that such knowledge cannot exist.

Assume that there exists some knowledge C according to which England is a part of the British Isles and each of the propositions E and B is equally plausible as its respective contrary. Then, given C, E implies B but B does not imply E (if England is bigger than France, then the British Isles also are; however, from the premise that the British Isles are bigger than France, one cannot conclude that England also is); following the implication rule, B is more plausible than E, conditional on C:

$$B|C > E|C\,.$$

On the other hand, given C, $\overline{B}$ implies $\overline{E}$ but $\overline{E}$ does not imply $\overline{B}$ (if the British Isles are not bigger than France, then England is also not; however, from the premise that England is not bigger than France, one cannot conclude that the British Isles are not); following the implication rule, $\overline{E}$ is more plausible than $\overline{B}$, conditional on C:

$$\overline{E}|C > \overline{B}|C\,.$$

Now remember that C is assumed to make E and $\overline{E}$ equally plausible:

$$E|C = \overline{E}|C\,.$$

Putting these relations together, we arrive at

$$B|C > E|C = \overline{E}|C > \overline{B}|C\,,$$

which contradicts the assumption that C makes B and $\overline{B}$ equally plausible. Therefore, a state of knowledge according to which England is a part of the British Isles and each of the propositions E and B were equally plausible as its respective contrary does not exist.

It follows that Example 2 does not disprove the Principle of Indifference; its paradox results from the assumption of prior knowledge which cannot exist.

4.3 Example 3: Specific volume vs. specific density

Again we ask for the knowledge based on which the probabilities are assigned. Let C_V be some knowledge according to which V is equally plausible as $\overline{V}$; then $p(\overline{V}|C_V) = 1/2$. Let C_D be some knowledge which makes D equally plausible as $\overline{D}$; then $p(D|C_D) = 1/2$. We thus obtain

$$p(\overline{V}|C_V) = p(D|C_D)\,;$$

but although specific volume and specific density are reciprocals of each other, this equality is not impossible since the probabilities are conditional on different knowledge.

The Principle of Indifference would fail on Example 3 only if there existed a state of knowledge which contained the information that specific volume and specific density are reciprocals of each other and, at the same time, made V equally plausible as $\overline{V}$ and D equally plausible as $\overline{D}$. We prove by contradiction that such knowledge cannot exist.

Assume that there exists some knowledge C according to which specific volume and specific density are reciprocals of each other and each of the propositions V and D is equally plausible as its respective contrary. Then, given C, $\overline{V}$ (*The specific volume lies between 2 and 3*) implies D (equivalent to *The specific volume lies between 3/2 and 3*) but D does not imply $\overline{V}$; following the implication rule, D is more plausible than $\overline{V}$, conditional on C:

$$D|C > \overline{V}|C\,.$$

On the other hand, given C, $\overline{D}$ (*The specific density lies between 2/3 and 1*) implies V (equivalent to *The specific density lies between 1/2 and 1*) but V does not imply $\overline{D}$; following the implication rule, V is more plausible than $\overline{D}$, conditional on C:

$$V|C > \overline{D}|C\,.$$

Now remember that C is assumed to make V and $\overline{V}$ equally plausible:

$$V|C = \overline{V}|C\,.$$

Putting these relations together, we arrive at

$$D|C > \overline{V}|C = V|C > \overline{D}|C\,,$$

which contradicts the assumption that C makes D and $\overline{D}$ equally plausible. Therefore, a state of knowledge according to which specific volume and specific density are reciprocals of each other and each of the propositions V and D were equally plausible as its respective contrary does not exist.

It follows that Example 3 does not disprove the Principle of Indifference; its paradox results from the assumption of prior knowledge which cannot exist.

5. Results

We have analysed three examples representative of the objections to the Principle of Indifference and found that, rather than disproving the principle, they suffer from a common error in applying it. The error can be described as follows: First, a state of knowledge is assumed which contains no reason to favour any proposition (out of a certain set of propositions) over its contrary; second, based on this knowledge, each proposition is assigned a probability in accordance with the Principle of Indifference; third, the probability assignment is shown to be contradictory and the contradiction is attributed to the Principle of Indifference. However, as we have proven for each example, a state of knowledge which contains no reason to favour any of said propositions over its contrary and, at the same time, renders the resulting probability assignment inconsistent is impossible.

Therefore, none of the examples disproves the Principle of Indifference; the contradiction is always caused by the assumption of prior knowledge which cannot exist.

6. Discussion and Conclusions

The Principle of Indifference is a proven theorem in a probability theory that emerged from a set of desiderata for plausible reasoning. On the other hand, there exist counterexamples believed to show that it leads to contradictions. Being proven, the principle cannot be contradictory, unless the whole theory were contradictory. If it is not, it must be possible to eliminate the contradictions on the ground of the desiderata the theory is built upon.

Based on these desiderata, we have analysed three counterexamples and found that in none of them the principle fails. Whenever a contradiction arises, it is caused by the erroneous assumption that there be no reason to favour any relevant proposition over its contrary. In each of the examples we have found such reason; it has never been an *empirical* reason, it was always a *logical* one: Whatever prior knowledge one has with respect to the colour of a certain book and whether or not it contains reasons to favour or disfavour red, to favour or disfavour blue or to favour or disfavour green – it is logically impossible that each of the propositions *The book is red*, *The book is blue* and *The book is green* be exactly as plausible as its negation; and analogous arguments hold with respect to the other examples.

Some opponents to the Principle of Indifference consider only empirical knowledge as relevant knowledge. In his *Sizes of countries* example, Keynes explicitely rejects a way out of the contradiction based on logic. However, the use of logical evidence in plausibility assignments is mandatory because of two reasons: First, it is required by desideratum (IIIb); second, consistent reasoning would be impossible if the rules of logic could be arbitrarily ignored, in particular, fulfillment of the desiderata (IIIa) and (IIIc) would not be guaranteed.

As far as we know, every serious objection to the Principle of Indifference corresponds to the pattern of one of the examples we have analysed. If this is true, then the principle does not lead to contradictions at all.

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