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A Strong Law of Large Numbers for Set-Valued Negatively Dependent Random Variables

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Abstract

In this paper, we shall represent a strong law of large numbers (SLLN) for weighted sums of negatively dependent set-valued random variables in the sense of the Hausdorff metric d_H, based on the result of single-valued random variable obtained by Taylor (Taylor, 1978).

Keywords: set-valued random variable, the strong laws of large numbers, negatively dependent.

1. Introduction

We all know that the laws of large numbers (LLN) are one of the most important theories and play an important role in probability and statistics. There are a lot of researches for independent single-valued random variables, and many beautiful results have been obtained (Taylor,1978, Billingsley, 1999). But it is not always plausible to assume that the sequence of random variables $\{X_k : k \geq 1\}$ are independent in many stochastic models. Sometimes the increasing of one random variable will induce the decreasing of another random variable. Then the concept of dependent was useful. Lehmann (Lehmann, 1966) provided an extensive introductory overview of various concepts of positive and negative dependence in the bivariate case. Multivariate generalizations of conceptions of dependence were initiated by Harris (Harris, 1970), Brindley and Thompson (Brindley & Thompson, 1972). Asadian et al. proved the Rosenthal's type inequalities for negatively dependent single-valued random variables in (Asadian, Fakoor & Bozorgnia, 2006). Negative dependence has been particularly useful in obtaining strong laws of large numbers. Bozorgnia, Patterson and Taylor discussed the properties for negatively dependent random variables in (Bozorgnia, Patterson & Taylor, 1993), and proved the laws of large number for negative dependence random variables in (Bozorgnia, Patterson & Taylor, 1992). The limit theorems of single-valued negative dependence random variables have been extensively studied and got very interesting results (Mi & Tae, 2005) (Valentin, 1995), but all the results are limited to single-valued random variables.

The theory of set-valued random variables and their applications have become one of new and active branches in probability theory. And the limit theory of set-valued random variables has been developed quite extensively. In 1975, Artstein and Vitale used an embedding theorem to prove a strong law of large numbers for independent and identically distributed set-valued random variables whose basic space is a d-dimensional Euclidean space $\mathbb{R}^d$ (Artstein, 1975), and Hiai extended it to the case that basic space is a separable Banach space $\mathfrak{X}$ (Hiai, 1984). Taylor and Inoue proved SLLN's for only independent case in Banach space (Taylor & Inoue, 1985). Many other authors such as Giné, Hahn and Zinn (Giné, Hahn & Zinn, 1983), Hess (Hess,1979), Puri and Ralescu (Puri & Ralescu, 1983) discussed SLLN's under different settings for set-valued random variables where the underlying space is a separable Banach space. And all the above limit theories are under the independent condition.

For set-valued random variables, there is not too much research for negatively dependent sequences. Guan and Sun gived the definition of negatively dependent set-valued random variables and proved the weak laws of large numbers for negatively dependent set-valued random variables in $\mathbb{R}^+$ space (Guan & Sun, 2014). In this paper, what we concerned is SLLN for weighted sums of rowwise negatively dependent set-valued random variables, where the underlying space is $\mathbb{R}^+$ and the convergence is in the sense of the Hausdorff metric. The results are both the extension of the single-valued's case and also the extension of the set-valued's case.

This paper is organized as follows. In section 2, we shall briefly introduce some definitions and basic results of set-valued random variables and single-valued negatively dependent random variables. In section 3, we shall give basic definition and results on set-valued negatively dependent random variables. In section 4, we shall prove a strong law of large numbers for weighted sums of set-valued negatively dependent random variables.

2. Preliminaries on Set-valued Random Variables

Throughout this paper, we assume that $(\Omega, \mathcal{A}, \mu)$ is a nonatomic complete probability space, $(\mathbb{R}, |\cdot|)$ is a real space, $\mathbb{R}^+$ denote the nonnegative real numbers, $\mathbf{K}_k(\mathbb{R})$ is the family of all nonempty compact subsets of $\mathbb{R}$, and $\mathbf{K}_{kc}(\mathbb{R})$. ($\mathbf{K}_{bc}(\mathbb{R})$) is the family of all nonempty compact convex (bounded convex) subsets of $\mathbb{R}$.

Let A and B be two nonempty subsets of $\mathbb{R}$ and let $\lambda \in \mathbb{R}$, the set of all real numbers. We define addition and scalar multiplication as

$$A + B = \{a + b : a \in A, b \in B\},$$

$$\lambda A = \{\lambda a : a \in A\}.$$

The Hausdorff metric on $\mathbf{K}_k(\mathbb{R})$ is defined by

$$d_H(A, B) = \max\{\sup_{a\in A} \inf_{b\in B} |a - b|, \sup_{b\in B} \inf_{a\in A} |a - b|\},$$

for $A,\ B \in \mathbf{K}_k(\mathbb{R})$. For an A in $\mathbf{K}_k(\mathbb{R})$, let $\|A\|_{\mathbf{K}} = d_H(\{0\}, A)$. The metric space $(\mathbf{K}_k(\mathbb{R}), d_H)$ is complete and separable, and $\mathbf{K}_{kc}(\mathbb{R})$ is a closed subset of $(\mathbf{K}_k(\mathbb{R}), d_H)$ (Li, Ogura & Kreinovich, 2002). For more general hyperspaces, more topological properties of hyperspaces, readers may refer to a good book (Beer, 1993).

For each $A \in \mathbf{k}_{kc}(\mathbb{R})$, define the support function by

$$s(k, A) = \sup_{a\in A} ka,$$

where $k \in \mathbb{R}$.

The following are equivalent definition of Hausdorff metric. For $A,\ B \in \mathbf{K}_k(\mathbb{R})$,

$$d_H(A, B) = \sup_{k\in\{1,-1\}} |s(k, A) - s(k, B)|.$$

A set-valued mapping $F : \Omega \to \mathbf{K}_k(\mathbb{R})$ is called *a set-valued random variable (or a random set, or a multifunction)* if, for each open subset O of $\mathbb{R}$, $F^{-1}(O) = \{\omega \in \Omega : F(\omega) \cap O \neq \emptyset\} \in \mathcal{A}$.

In fact, set-valued random variables can be defined as a mapping from Ω to the family of all closed subsets of $\mathbb{R}$. Since our main results shall be only related to compact set-valued random variables, we limit the definition above in the compact case. Concerning its equivalent definitions, please refer to (Castaing & Valadier, 1977), (Hiai & Umegaki, 1977), (Li, Ogura & Kreinovich, 2002).

Concerning operations, it is well known that $\mathcal{P}_0(\mathbb{R})$(the family of all the subsets of $\mathbb{R}$) is not a linear space, in general, $A + (-1)A \neq \{0\}$. Thus , adding -1 times a set does not constitute a natural operation of subtraction. Instead, Hukuhara defined *Hukuhara difference* as follows in (Hukuhara, 1967) and many authors used this definition in their work (Puri & Ralescu, 1983).

Let $A, B \in \mathbf{K}(\mathbb{R})$, If there exist a $W \in \mathbf{K}(\mathbb{R})$, such that $A = B + W$, then W is called the Hukuhara difference of A and B, denote by $A \ominus B$.

A set-valued random variable F is called integrably bounded , if $\int_\Omega \|F(\omega)\|_{\mathbf{K}} d\mu < \infty$ (Hiai & Umegaki, 1977), (Li, Ogura & Kreinovich, 2002).

Let $L^1[\Omega, \mathcal{A}, \mu; \mathbf{K_k}(\mathbb{R})]$ denote the space of all integrably bounded random variables, and $L^1[\Omega, \mathcal{A}, \mu; \mathbf{K_{kc}}(\mathbb{R})]$ denote the space of all integrably bounded random variables taking values in $\mathbf{K_{kc}}(\mathbb{R})$. For $F, G \in L^1[\Omega, \mathcal{A}, \mu; \mathbf{K_k}(\mathbb{R})]$, $F = G$ if and only if $F(\omega) = G(\omega)\ a.e.(\mu)$.

For each set-valued random variable F, *the expectation of* F, denoted by $E[F]$, is defined as

$$E[F] = \Big\{ \int_\Omega f d\mu : f \in S_F \Big\},$$

where $\int_\Omega f d\mu$ is the usual Bochner integral in $L^1[\Omega, \mathbb{R}]$, the family of integrable $\mathbb{R}$-valued random variables, and $S_F = \{f \in L^1[\Omega; \mathbb{R}] : f(\omega) \in F(\omega), a.e.(\mu)\}$. This integral was first introduced by Aumann (Aumann, 1965), called Aumann integral in literature.

A sequence of set-valued random variables $\{F_n : n \in \mathbb{N}\}$ is called to be stochastically dominated by a set-valued random variable F if

$$\mu\{\|F_n\|_{\mathbf{K}} > t\} \leq \mu\{\|F\|_{\mathbf{K}} > t\},\ \ t \geq 0,\ n \geq 1.$$

A triangle sequence of set-valued random variables $\{F_{nk} : n \geq 1\}$ is said to be stochastic dominated by a set-valued random variable F, if for each $t > 0$, for all n and k,

$$P\{\|F_{nk}\|_{\mathbf{K}} > t\} \leq P\{\|F\|_{\mathbf{K}} > t\}.$$

In the following, we will recall some contents about real-valued negatively dependent random variables, which will be used later.

Definition 2.1 A finite family of real-valued random variables $X_1, \cdots, X_n$ is said to be negatively dependent if for all real $x_1, x_2, \cdots, x_n$,

$$P\{X_1 > x_1, \cdots, X_n > x_n\} \leq \prod_{i=1}^{n} P\{X_i > x_i\}$$

and

$$P\{X_1 \leq x_1, \cdots, X_n \leq x_n\} \leq \prod_{i=1}^{n} P\{X_i \leq x_i\}.$$

An infinite family of random variables is negatively dependent if every finite subfamily is negatively dependent. The following results are very useful and we will use them in the later.

Lemma 2.1 (Bozorgnia, Patterson & Taylor, 1993) Let real-valued random variables $\{x_i : 1 \leq i \leq n\}$ be negatively dependent. Then the following are true:

(i)$E[\prod_{i=1}^{n} x_i] \leq \prod_{i=1}^{n} E[x_i]$;

(ii)$Cov(x_i, x_j) \leq 0, \quad i \neq j$;

(iii) If $\{g_i : 1 \leq i \leq n\}$ be all nondecreasing (or all nonincreasing) Borel functions, then random variables $g_1(x_1), g_2(x_2), \cdots, g_n(x_n)$ are negatively dependent random variables.

Lemma 2.2 (Taylor & Patterson, 1997) Let $x_1, x_2, \cdots, x_n$ be real-valued negatively dependent random variables, then for any real numbers $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ such that $a_i < b_i$,

i) $\{I_{\{-\infty < x_i < b_i\}}, 1 \leq i \leq n\}$ are negatively dependent random variables;

ii) $\{y_i : 1 \leq i \leq n\}$ are negatively dependent random variables, where $y_i = x_i I_{\{a_i \leq x_i \leq b_i\}} + b_i I_{\{x_i > b_i\}} + a_i I_{\{x_i < a_i\}}$.

Lemma 2.3 (Taylor & Patterson, 1997) If x is a real-valued random variable with $|x| \leq M$, and $E[x] = 0$, then

$$1 \leq E[e^{tx}] \leq e^{t^2 E[x^2]} \quad \text{for all} \quad |t| \leq \frac{1}{M}.$$

For more contents about negatively dependent random variables, readers can refer to (Bozorgnia, Patterson & Taylor, 1992), (Bozorgnia, Patterson & Taylor, 1993), (Bozorgnia, Patterson & Taylor, 1997), (Taylor & Patterson, 1997).

3. Set-Valued Negatively Dependent Random Variables

In this section, we shall introduce the definition of set-valued negatively dependent random variables and prove some results which will be used later. In the following, We consider the results in $\mathbb{R}^+ = \{x \geq 0 : x \in \mathbb{R}\}$ space.

Definition 3.1 (Guan & Sun, 2014) A finite family of set-valued random variables $F_1, F_2, \cdots, F_n$ is said to be negatively dependent if $s(\cdot, F_1), \cdots, s(\cdot, F_n)$ is single-valued negatively dependent random variables.

Remark 1 From the definition we obviously can know that if specially $F_1 = \{f_1\}, \cdots, F_n = \{f_n\}$, where $f_1, \cdots, f_n$ are single-valued random variables, then it degenerate to the single-valued negatively dependent random variables.

Remark 2 Specially for $F_1 = [\xi_1, \eta_1], \cdots, F_n = [\xi_n, \eta_n]$ are interval valued random variables, it is easy to prove that $\xi_1, \cdots, \xi_n$ and $\eta_1, \cdots, \eta_n$ are all real-valued negatively dependent random variables.

Remark 3 Any infinite sequence $\{F_n : n \geq 1\}$ is said to be negatively dependent if and only if every finite subset $\{F_1, \cdots, F_n\}$ is negatively dependent.

Theorem 3.1 A finite family of set-valued random variables $F_1, \cdots, F_n$ in $\mathbf{K}_k(\mathbb{R}^+)$ is negatively dependent, then

$\|F_1\|_{\mathbf{K}}, \|F_2\|_{\mathbf{K}}, \cdots, \|F_n\|_{\mathbf{K}}$ are single-valued negatively dependent random variables.

Proof. Since $F_1, \cdots, F_n$ in $\mathbf{K}_k(\mathbb{R}^+)$ is negatively dependent, by the definition of negatively dependent, we know that for any $k \in \mathbb{R}$, $s(k, F_1), \cdots, s(k, F_n)$ are real-valued negatively dependent random variables. For each $1 \le i \le n$, we have

$$\|F_i\|_{\mathbf{K}} = \sup F_i = s(1, F_i).$$

If we take $k = 1$, then $s(1, F_1), \cdots, s(1, F_n)$ are real-valued negatively dependent random variables. That is $\sup F_1, \sup F_2, \cdots, \sup F_n$ are real-valued negatively dependent random variables. Then $\|F_1\|_{\mathbf{K}}, \|F_2\|_{\mathbf{K}}, \cdots, \|F_n\|_{\mathbf{K}}$ are real-valued negatively dependent random variables. □

Theorem 3.2 A finite family of set-valued random variables $F_1, \cdots, F_n$ in $\mathbf{K}_k(\mathbb{R}^+)$ is negatively dependent, then for any real numbers $b_1, \cdots, b_n$ such that $b_i > 0$,

(i)$\{I_{\{\|F_i\|_{\mathbf{K}} < b_i\}}, 1 \le i \le n\}$ are negatively dependent random variables;

(ii) $\{Y_i : 1 \le i \le n\}$ are negatively dependent set-valued random variables, where $Y_i = F_i I_{\{\|F_i\|_{\mathbf{K}} \le b_i\}} + b_i I_{\{\|F_i\|_{\mathbf{K}} > b_i\}}$.

Proof. (i) By theorem 3.1, we know that $\|F_1\|_{\mathbf{K}}, \|F_2\|_{\mathbf{K}}, \cdots, \|F_n\|_{\mathbf{K}}$ are single-valued negatively dependent random variables. Then by lemma 2.2, we can get the result.

(ii)Since $F_1, \cdots, F_n$ in $\mathbf{K}_k(\mathbb{R}^+)$ is negatively dependent, by definition we know for any $k \in \mathbb{R}$, $s(k, F_1), \cdots, s(k, F_n)$ is negatively dependent random variables. Then by lemma 2.2, we can know that for any $k \in \mathbb{R}$, $\{s(k, F_i) I_{\{0 \le s(k,F_i) \le b_i\}} + b_i I_{\{s(k,F_i) > b_i\}} : 1 \le i \le n\}$ is negatively dependent random variables. Since $F_1, \cdots, F_n$ in $\mathbf{K}_k(\mathbb{R}^+)$, then

$$\|F_i\|_{\mathbf{K}} = \sup_{k \in \{1,-1\}} |s(k, F_i)| = s(1, F_i).$$

Then $\{s(1, F_i) I_{\{0 \le \|F_i\|_{\mathbf{K}} \le b_i\}} + b_i I_{\{\|F_i\|_{\mathbf{K}} > b_i\}} : 1 \le i \le n\}$ is negatively dependent random variables. That means $\{s(1, Y_i) : 1 \le i \le n\}$ is negatively dependent random variables. Then for any $k \in \mathbb{R}$, $\{s(k, Y_i) : 1 \le i \le n\}$ is negatively dependent random variables. Then $\{Y_i : 1 \le i \le n\}$ are negatively dependent set-valued random variables. □

4. Main Results

Throughout this section, $\{a_{nk}\}$ will be a Toeplitz sequence of nonnegative real numbers. And a strong law of large numbers for weighted sums of rowwise negative dependent set-valued random variables will be obtained.

Definition 4.1 A double array $\{a_{nk} : n, k = 1, 2, \cdots\}$ of real numbers is said to be a Toeplitz sequence, if

(i)$\lim\limits_{n\to\infty} a_{nk} = 0$ for each k;

(ii) $\sum\limits_{k=1}^{\infty} |a_{nk}| \le M$ for each n.

Before we prove the strong law of large numbers for negatively dependent weighted sums of set-valued random variables, we need the following three lemmas.

Lemma 4.1 Let $\{F_{nk}\} \in \mathbf{K}_k(\mathbb{R}^+)$ be a rowwise negatively dependent random variables and stochastic dominated by F, $E[\|F\|_{\mathbf{K}}^{1+1/r}] < \infty$, where $r > 0$, and $\max\limits_k a_{nk} \le B n^{-r}$, then for every $\varepsilon > 0$

$$\sum_{n=1}^{\infty} P\{\|a_{nk} F_{nk}\|_{\mathbf{K}} > \varepsilon \ \text{ for some } k\} < \infty.$$

Proof. The proof is similar to the proof of Lemma 1 of (William, 1966), here we omit it. □

Lemma 4.2 If $\{F_{nk}\} \in \mathbf{K}_k(\mathbb{R}^+)$ is a rowwise negatively dependent random variables and stochastic dominated by F, $E[\|F\|_{\mathbf{K}}^{1+1/r}] < \infty$, where $r > 0$, and $\max\limits_k a_{nk} \le B n^{-r}$, then for $\alpha < \frac{r}{2(1+r)}$,

$$\sum_{n=1}^{\infty} P\{\|a_{nk} F_{nk}\|_{\mathbf{K}} > n^{-\alpha} \ \text{ for at least 2 } values\ of\ k\} < \infty.$$

Proof. By Markov's inequality and the stochastic dominated by F of $\{F_{nk}\}$,

$$P\{\|a_{nk} F_{nk}\|_{\mathbf{K}} > n^{-\alpha}\} \le a_{nk}^{1+1/r} E[\|F\|_{\mathbf{K}}^{1+1/r}] n^{\alpha(1+1/r)}. \tag{4.1}$$

Thus by the negative dependence of F_{ni} and F_{nk} and (4.1)

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\{\|a_{nk}F_{nk}\|_{\mathbf{K}} > n^{-\alpha} \text{ for at least 2 values of } k\} \\
&\leq \sum_{n=1}^{\infty}\sum_{i\neq k} P\{\|a_{ni}F_{ni}\|_{\mathbf{K}} > n^{-\alpha}, \|a_{nk}F_{nk}\|_{\mathbf{K}} > n^{-\alpha}\} \\
&\leq \sum_{n=1}^{\infty}\sum_{i\neq k} P\{\|a_{ni}F_{ni}\|_{\mathbf{K}} > n^{-\alpha}\}P\{\|a_{nk}F_{nk}\|_{\mathbf{K}} > n^{-\alpha}\} \\
&\leq \sum_{n=1}^{\infty} E^2[\|F\|_{\mathbf{K}}^{1+1/r}]n^{2\alpha(1+1/r)}\sum_{i\neq k}|a_{ni}|^{1+1/r}\|a_{nk}\|_{\mathbf{K}}^{1+1/r} \\
&\leq \sum_{n=1}^{\infty} E^2[\|F\|_{\mathbf{K}}^{1+1/r}]B^{2/r}M^2 n^{2[-1+\alpha(1+1/r)]} \\
&< \infty
\end{aligned}$$

for $\alpha < \frac{r}{2(1+r)}$, the result follows. □

Lemma 4.3 If $\{F_{nk}\} \in \mathbf{K}_k(\mathbb{R}^+)$ be a rowwise negatively dependent random variables and stochastic dominated by F, $E[\|F\|_{\mathbf{K}}^{1+1/r}] < \infty$, where $r > 0$. $E[F_{nk}] = \{0\}$ and $\max_k a_{nk} \leq Bn^{-r}$ for all n and k, then for each $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\Big\{\|\sum_{k} a_{nk}F_{nk}I_{\{\|a_{nk}F_{nk}\|_{\mathbf{K}}<n^{-\alpha}\}}\|_{\mathbf{K}} > \varepsilon\Big\} < \infty,$$

where $\alpha < \frac{r}{2(1+r)}$.

Proof. *Step 1* Let $Y_{nk} = a_{nk}F_{nk}I_{\{\|a_{nk}F_{nk}\|_{\mathbf{K}}<n^{-\alpha}\}} + n^{-\alpha}I_{\{\|a_{nk}F_{nk}\|_{\mathbf{K}}\geq n^{-\alpha}\}}$. Since $E[F_{nk}] = \{0\}$, we have

$$E[F_{nk}] = E\Big[F_{nk}I_{\{\|a_{nk}F_{nk}\|_{\mathbf{K}}<n^{-\alpha}\}} + F_{nk}I_{\{\|a_{nk}F_{nk}\|_{\mathbf{K}}\geq n^{-\alpha}\}}\Big] = \{0\}.$$

Thus we have

$$\Big\|E[F_{nk}I_{\{\|a_{nk}F_{nk}\|_{\mathbf{K}}<n^{-\alpha}\}}]\Big\|_{\mathbf{K}} = \Big\|E[F_{nk}I_{\{\|a_{nk}F_{nk}\|_{\mathbf{K}}\geq n^{-\alpha}\}}]\Big\|_{\mathbf{K}}. \tag{4.2}$$

Since $E[F_{nk}] = \{0\}$, we have

$$F_{nk}I_{\{\|a_{nk}F_{nk}\|_{\mathbf{K}}<n^{-\alpha}\}} = E[F_{nk}I_{\{\|a_{nk}F_{nk}\|_{\mathbf{K}}<n^{-\alpha}\}}] + E[F_{nk}I_{\{\|a_{nk}F_{nk}\|_{\mathbf{K}}\geq n^{-\alpha}\}} + F_{nk}I_{\{\|a_{nk}F_{nk}\|_{\mathbf{K}}<n^{-\alpha}\}},$$

That means $F_{nk}I_{\{\|a_{nk}F_{nk}\|_{\mathbf{K}}<n^{-\alpha}\}} \ominus E[F_{nk}I_{\{\|a_{nk}F_{nk}\|_{\mathbf{K}}<n^{-\alpha}\}}]$ exists. Furthermore, we can know that $Y_{nk} \ominus E[Y_{nk}]$ exists. Thus we can have

$$\begin{aligned}
a_{nk}F_{nk}I_{\{\|a_{nk}F_{nk}\|_{\mathbf{K}}<n^{-\alpha}\}} &= Y_{nk} \ominus E[Y_{nk}] + E[a_{nk}F_{nk}I_{\{\|a_{nk}F_{nk}\|_{\mathbf{K}}<n^{-\alpha}\}}] \\
&\quad + n^{-\alpha}E[I_{\{\|a_{nk}F_{nk}\|_{\mathbf{K}}\geq n^{-\alpha}\}}] - n^{-\alpha}I_{\{\|a_{nk}F_{nk}\|_{\mathbf{K}}\geq n^{-\alpha}\}}.
\end{aligned}$$

Then we have

$$\begin{aligned}
\sum_{k=1}^{n} a_{nk}F_{nk}I_{\{\|a_{nk}F_{nk}\|_{\mathbf{K}}<n^{-\alpha}\}} &= \sum_{k=1}^{n}\Big[Y_{nk} \ominus E[Y_{nk}]\Big] + \sum_{k=1}^{n} E\Big[a_{nk}F_{nk}I_{\{\|a_{nk}F_{nk}\|_{\mathbf{K}}<n^{-\alpha}\}}\Big] \\
&\quad + \sum_{k=1}^{n} n^{-\alpha}\Big\{E[I_{\{\|a_{nk}F_{nk}\|_{\mathbf{K}}\geq n^{-\alpha}\}}] - I_{\{\|a_{nk}F_{nk}\|_{\mathbf{K}}\geq n^{-\alpha}\}}\Big\}.
\end{aligned}$$

Then

$$\begin{aligned}
\Big\|\sum_{k=1}^{n} a_{nk}F_{nk}I_{\{\|a_{nk}F_{nk}\|_{\mathbf{K}}<n^{-\alpha}\}}\Big\|_{\mathbf{K}} &\leq \Big\|\sum_{k=1}^{n}[Y_{nk} \ominus E[Y_{nk}]]\Big\|_{\mathbf{K}} + \Big\|\sum_{k=1}^{n} E[a_{nk}F_{nk}I_{\{\|a_{nk}F_{nk}\|_{\mathbf{K}}<n^{-\alpha}\}}]\Big\|_{\mathbf{K}} \\
&\quad + \sum_{k=1}^{\infty} n^{-\alpha}\Big|P\{\|a_{nk}F_{nk}\|_{\mathbf{K}} \geq n^{-\alpha}\} - I_{\{\|a_{nk}F_{nk}\|_{\mathbf{K}}\geq n^{-\alpha}\}}\Big| \\
&= I + II + III.
\end{aligned}$$

Step 2 For I, using integration by parts, we have

$$
\begin{aligned}
&\sum_{k=1}^{\infty} E\Big[\|a_{nk}F_{nk}\|_{\mathbf{K}}^{2} I_{\{\|a_{nk}F_{nk}\|_{\mathbf{K}}<n^{-\alpha}\}}\Big]\\
&= \sum_{k=1}^{\infty} a_{nk}^{2} \int_{0}^{a_{nk}^{-1}n^{-\alpha}} x^{2} dP\{\|F_{nk}\|_{\mathbf{K}} \le x\}\\
&= \sum_{k=1}^{\infty} n^{-2\alpha} P\{\|F_{nk}\|_{\mathbf{K}} \le a_{nk}^{-1}n^{-\alpha}\} - \sum_{k=1}^{\infty} a_{nk}^{2} \int_{0}^{a_{nk}^{-1}n^{-\alpha}} 2xP\{\|F_{nk}\|_{\mathbf{K}} \le x\}dx\\
&= \sum_{k=1}^{\infty} n^{-2\alpha} P\{\|F_{nk}\|_{\mathbf{K}} \le a_{nk}^{-1}n^{-\alpha}\} - n^{-2\alpha} + a_{nk}^{2} \int_{0}^{a_{nk}^{-1}n^{-\alpha}} 2xP\{\|F_{nk}\|_{\mathbf{K}} > x\}dx\\
&\le \sum_{k=1}^{\infty} a_{nk}^{2} \int_{0}^{a_{nk}^{-1}n^{-\alpha}} 2xP\{\|F\|_{\mathbf{K}} > x\}dx\\
&\le \sum_{k=1}^{\infty} a_{nk}^{2} \int_{0}^{a_{nk}^{-1}n^{-\alpha}} 2xE[\|F\|_{\mathbf{K}}]^{1+1/r}x^{-1-1/r}dx\\
&\le M\sum_{k=1}^{\infty} a_{nk}^{2} \int_{0}^{a_{nk}^{-1}n^{-\alpha}} x^{-1/r}dx\\
&= M\sum_{k=1}^{\infty} a_{nk}^{1+1/r}n^{-\alpha+\alpha/r}\\
&\le MB^{1/r}n^{-1-\alpha+\alpha/r} \qquad\qquad (4.3)\\
&\longrightarrow 0, \ \ \text{as } n \to \infty.
\end{aligned}
$$

By theorem 3.1, $\{Y_{nk}\}$ is rowwise negatively dependent set-valued random variables, and

$$E[s(k, Y_{nk}) - s(k, E[Y_{nk}])] = 0, |s(k, Y_{nk}) - s(k, E[Y_{nk})| \le M$$

for fixed k and constant M.

So we have

$$
\begin{aligned}
P\{I > \varepsilon\} &= P\Big\{\|\sum_{k=1}^{\infty}[Y_{nk} \ominus E[Y_{nk}]]\|_{\mathbf{K}} > \epsilon\Big\}\\
&= P\Big\{\|n^{\lambda}\sum_{k=1}^{\infty}[Y_{nk} \ominus E[Y_{nk}]]\|_{\mathbf{K}} > n^{\lambda}\epsilon\Big\}\\
&= P\Big\{n^{\lambda}\sup_{j\in\{1,-1\}} |\sum_{k=1}^{\infty} s(j, Y_{nk}) - s(j, E[Y_{nk}])| > n^{\lambda}\epsilon\Big\}\\
&\le P\Big\{n^{\lambda}\sup_{j\in\{1,-1\}} \sum_{k=1}^{\infty}\Big[s(j, Y_{nk}) - s(j, E[Y_{nk}])\Big] > n^{\lambda}\epsilon\Big\}\\
&\quad +P\Big\{n^{\lambda}\sup_{j\in\{1,-1\}} \sum_{k=1}^{\infty}\Big[s(j, E[Y_{nk}]) - s(j, Y_{nk})\Big] > n^{\lambda}\epsilon\Big\}\\
&\le e^{-n^{\lambda}\epsilon}E\Big[\exp\Big\{n^{\lambda}\sup_{j\in\{1,-1\}} \sum_{k=1}^{\infty}\Big[s(j, Y_{nk}) - s(j, E[Y_{nk}])\Big]\Big\}\Big]\\
&\quad +e^{-n^{\lambda}\epsilon}E\Big[\exp\Big\{n^{\lambda}\sup_{j\in\{1,-1\}} \sum_{k=1}^{\infty}\Big[s(j, E[Y_{nk}]) - s(j, Y_{nk})\Big]\Big\}\Big] \ \text{(by Markov inequality)}
\end{aligned}
$$

$$
\begin{aligned}
&\leq e^{-n^{\lambda}\epsilon}E\Big[\exp\Big\{n^{\lambda}\sum_{k=1}^{\infty}\Big[s(1,Y_{nk})-s(1,E[Y_{nk}])\Big]\Big\}\Big] \\
&\quad +e^{-n^{\lambda}\epsilon}E\Big[\exp\Big\{n^{\lambda}\sum_{k=1}^{\infty}\Big[s(-1,Y_{nk})-s(-1,E[Y_{nk}])\Big]\Big\}\Big] \\
&\quad +e^{-n^{\lambda}\epsilon}E\Big[\exp\Big\{n^{\lambda}\sum_{k=1}^{\infty}\Big[s(1,E[Y_{nk}])-s(1,Y_{nk})\Big]\Big\}\Big] \\
&\quad +e^{-n^{\lambda}\epsilon}E\Big[\exp\Big\{n^{\lambda}\sum_{k=1}^{\infty}\Big[s(-1,E[Y_{nk}])-s(-1,Y_{nk})\Big]\Big\}\Big] \\
&\leq e^{-n^{\lambda}\epsilon}\prod_{k=1}^{\infty}E\Big[\exp\Big\{n^{\lambda}[s(1,Y_{nk})-s(1,E[Y_{nk}])]\Big\}\Big] \\
&\quad +e^{-n^{\lambda}\epsilon}\prod_{k=1}^{\infty}E\Big[\exp\Big\{n^{\lambda}[s(-1,Y_{nk})-s(-1,E[Y_{nk}])]\Big\}\Big] \\
&\quad +e^{-n^{\lambda}\epsilon}\prod_{k=1}^{\infty}E\Big[\exp\Big\{n^{\lambda}[s(1,E[Y_{nk}])-s(1,Y_{nk})]\Big\}\Big] \\
&\quad +e^{-n^{\lambda}\epsilon}\prod_{k=1}^{\infty}E\Big[\exp\Big\{n^{\lambda}[s(-1,E[Y_{nk}])-s(-1,Y_{nk})]\Big\}\Big] \quad \text{(by lemma 2.1)} \\
&\leq 4e^{-n^{\lambda}\epsilon}\prod_{k=1}^{\infty}\exp\Big\{n^{2\lambda}E\Big[\Big[s(1,Y_{nk})-s(1,E[Y_{nk}])\Big]^2\Big]\Big\} \quad \text{(by lemma 2.3)} \\
&\leq 4e^{-n^{\lambda}\epsilon}\prod_{k=1}^{\infty}\exp\Big\{n^{2\lambda}E\Big[\sup_{j\in\{1,-1\}}\Big[s(j,Y_{nk})-s(j,E[Y_{nk}])\Big]^2\Big]\Big\} \\
&= 4e^{-n^{\lambda}\epsilon}\prod_{k=1}^{\infty}\exp\Big\{n^{2\lambda}E\Big[\|Y_{nk}\ominus E[Y_{nk}]\|_{\mathbf{K}}^2\Big]\Big\} \\
&\leq 4e^{-n^{\lambda}\epsilon}\prod_{k=1}^{\infty}\exp\Big\{n^{2\lambda}4E\Big[\|Y_{nk}\|_{\mathbf{K}}^2\Big]\Big\} \\
&= 4e^{-n^{\lambda}\epsilon}\exp\Big\{\sum_{k=1}^{\infty}4n^{2\lambda}E\Big[\|Y_{nk}\|_{\mathbf{K}}^2\Big]\Big\} \\
&\leq 4e^{-\varepsilon n^{\lambda}}\exp\Big\{4n^{2\lambda}\sum_{k=1}^{\infty}E[\|a_{nk}F_{nk}\|_{\mathbf{K}}^2 I_{\{\|a_{nk}F_{nk}\|_{\mathbf{K}}<n^{-\alpha}\}}] \\
&\qquad +4n^{2\lambda}\sum_{k=1}^{\infty}n^{-2\alpha}P\{\|a_{nk}F_{nk}\|_{\mathbf{K}}\geq n^{-\alpha}\}\Big\} \\
&\leq 4e^{-\varepsilon n^{\lambda}}\exp\Big\{4n^{2\lambda}MB^{1/r}n^{-1-\alpha+\alpha/r} \\
&\qquad +4n^{2\lambda}\sum_{k=1}^{\infty}n^{-2\alpha}P\{\|a_{nk}F_{nk}\|_{\mathbf{K}}\geq n^{-\alpha}\}\Big\} \quad \text{(by (4.3))} \\
&\leq 4e^{-\varepsilon n^{\lambda}}\exp\Big\{4n^{2\lambda}MB^{1/r}n^{-1-\alpha+\alpha/r}+4n^{2\lambda}\sum_{k=1}^{\infty}n^{-2\alpha}P\{a_{nk}\|F\|_{\mathbf{K}}\geq n^{-\alpha}\}\Big\} \\
&\leq 4e^{-\varepsilon n^{\lambda}}\exp\Big\{4n^{2\lambda}MB^{1/r}n^{-1-\alpha+\alpha/r}+4n^{2\lambda}\sum_{k=1}^{\infty}n^{-2\alpha}E[\|a_{nk}F\|_{\mathbf{K}}]^{1+1/r}n^{\alpha(1+1/r)}\Big\} \\
&\leq 4e^{-\varepsilon n^{\lambda}}\exp\Big\{4n^{2\lambda}MB^{1/r}n^{-1-\alpha+\alpha/r}+4n^{2\lambda}n^{-2\alpha}n^{\alpha(1+1/r)}E[\|F\|_{\mathbf{K}}]^{1+1/r}\sum_{k=1}^{\infty}a_{nk}^{1+1/r}\Big\} \\
&\leq 4e^{-\varepsilon n^{\lambda}}\exp\Big\{4n^{2\lambda}MB^{1/r}n^{-1-\alpha+\alpha/r}+4n^{2\lambda}MB^{1/r}n^{-1}n^{-2\alpha}n^{\alpha(1+1/r)}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&\le 4e^{-\varepsilon n^{\lambda}} \exp\left\{Mn^{2\lambda-1-\alpha+\alpha/r}\right\}\\
&= 4e^{-\varepsilon n^{\lambda}} \exp\left\{Mn^{-1+\alpha/r}\right\} \quad (\text{by taking } \ \lambda = \frac{\alpha}{2})\\
&\le Me^{-\varepsilon n^{\lambda}}.
\end{aligned}
$$

Here the M is constant and may not be the same. The last term in the above is summable with respect to n.

Step 3 For II, we have

$$
\begin{aligned}
II &= \|\sum_{k=1}^{\infty} E[a_{nk}F_{nk}I_{\{\|a_{nk}F_{nk}\|_{\mathbf{K}}<n^{-\alpha}\}}]\|_{\mathbf{K}}\\
&= \|\sum_{k=1}^{\infty} E[a_{nk}F_{nk}I_{\{\|a_{nk}F_{nk}\|_{\mathbf{K}}\ge n^{-\alpha}\}}]\|_{\mathbf{K}} \quad (\text{by } \ (4.2))\\
&\le \sum_{k=1}^{\infty} E[\|a_{nk}F_{nk}\|_{\mathbf{K}}I_{\{\|a_{nk}F_{nk}\|_{\mathbf{K}}\ge n^{-\alpha}\}}]\\
&\le \sum_{k=1}^{\infty} a_{nk}\int_0^{\infty} P\{\|F_{nk}\|_{\mathbf{K}}I_{\{\|F_{nk}\|_{\mathbf{K}}\ge n^{-\alpha}a_{nk}^{-1}\}} > t\}dt\\
&= \sum_{k=1}^{\infty} a_{nk}\int_0^{n^{-\alpha}a_{nk}^{-1}} P\{\|F_{nk}\|_{\mathbf{K}} \ge n^{-\alpha}a_{nk}^{-1}\}dt + \sum_{k=1}^{\infty} a_{nk}\int_{n^{-\alpha}a_{nk}^{-1}}^{\infty} P\{\|F_{nk}\|_{\mathbf{K}} \ge t\}dt\\
&\le \sum_{k=1}^{\infty} a_{nk}\int_0^{n^{-\alpha}a_{nk}^{-1}} P\{\|F\|_{\mathbf{K}} \ge n^{-\alpha}a_{nk}^{-1}\}dt + \sum_{k=1}^{\infty} a_{nk}\int_{n^{-\alpha}a_{nk}^{-1}}^{\infty} P\{\|F\|_{\mathbf{K}} \ge t\}dt\\
&\le \sum_{k=1}^{\infty} a_{nk}n^{-\alpha}a_{nk}^{-1}E[\|F\|_{\mathbf{K}}^{1+1/r}](n^{\alpha}a_{nk})^{1+1/r} + \sum_{k=1}^{\infty} a_{nk}\int_{n^{-\alpha}a_{nk}^{-1}}^{\infty} E[\|F\|_{\mathbf{K}}^{1+1/r}]t^{-1-1/r}dt\\
&\le M\sum_{k=1}^{\infty} n^{\alpha/r}a_{nk}^{1+1/r} + M\sum_{k=1}^{\infty} a_{nk}\int_{n^{-\alpha}a_{nk}^{-1}}^{\infty} t^{-1-1/r}dt\\
&= M\sum_{k=1}^{\infty} n^{\alpha/r}a_{nk}^{1+1/r} + Mr\sum_{k=1}^{\infty} n^{\alpha/r}a_{nk}^{1+1/r}\\
&\le M_2\sum_{k=1}^{\infty} n^{\alpha/r}a_{nk}(Bn^{-r})^{1/r}\\
&\le M_3n^{\frac{\alpha}{r}-1}\\
&\longrightarrow 0, \quad (n\to\infty).
\end{aligned}
$$

step 4 For III, let $Z_{nk} = I_{\{\|a_{nk}F_{nk}\|_{\mathbf{K}}\ge n^{-\alpha}\}} - P\{\|a_{nk}F_{nk}\|_{\mathbf{K}} \ge n^{-\alpha}\}$. Then $\{Z_{nk}\}$ are rowwise negatively dependent, $\|Z_{nk}\|_{\mathbf{K}} \le 1$ and $E[Z_{nk}] = 0$ for all n and k. Hence, we have

$$
\begin{aligned}
P\{III > \varepsilon\} &= P\Big\{n^{-\alpha}\sum_{k=1}^{\infty}\Big|Z_{nk}\Big| > \varepsilon\Big\}\\
&= P\Big\{n^{-\alpha}\sum_{k=1}^{\infty} Z_{nk} > \varepsilon\Big\} + P\Big\{n^{-\alpha}\sum_{k=1}^{\infty} -Z_{nk} > \varepsilon\Big\}\\
&\le e^{-\varepsilon n^{\alpha}}E\Big[\exp\Big\{\sum_{k=1}^{\infty} Z_{nk}\Big\}\Big] + e^{-\varepsilon n^{\alpha}}E\Big[\exp\Big\{\sum_{k=1}^{\infty} -Z_{nk}\Big\}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\le 2e^{-\varepsilon n^{\alpha}} E\Big[\exp\Big\{\sum_{k=1}^{\infty} Z_{nk}^2\Big\}\Big] \\
&\le 2e^{-\varepsilon n^{\alpha}} \exp\Big\{\sum_{k=1}^{\infty} 2P\{\|a_{nk}F_{nk}\|_{\mathbf{K}} \ge n^{-\alpha}\}\Big\} \\
&\le 2e^{-\varepsilon n^{\alpha}} \exp\Big\{\sum_{k=1}^{\infty} 2P\{\|a_{nk}F\|_{\mathbf{K}} \ge n^{-\alpha}\}\Big\} \\
&\le 2e^{-\varepsilon n^{\alpha}} \exp\Big\{\sum_{k=1}^{\infty} 2E[\|F\|_{\mathbf{K}}^{1+1/r}]a_{nk}^{1+1/r} n^{\alpha(1+1/r)}\Big\} \\
&\le 2e^{-\varepsilon n^{\alpha}} \exp\Big\{2MB^{1/r}n^{-1}n^{\alpha(1+1/r)}\Big\} \\
&= 2e^{-\varepsilon n^{\alpha}} \exp\Big\{2Kn^{-1+\alpha(1+1/r)}\Big\} \\
&\le K' e^{-\varepsilon n^{\alpha}}.
\end{aligned}
$$

And $K' e^{-\varepsilon n^{\alpha}}$ is summable with respect to n.

Then combine the above steps, we can have

$$
\begin{aligned}
\sum_{n=1}^{\infty} P\Big\{\|\sum_k a_{nk}F_{nk}I_{\{\|a_{nk}F_{nk}\|_{\mathbf{K}}<n^{-\alpha}\}}\|_{\mathbf{K}} > \varepsilon\Big\} &\le \sum_{n=1}^{\infty} P\{I > \frac{\varepsilon}{3}\} + P\{II > \frac{\varepsilon}{3}\} + P\{III > \frac{\varepsilon}{3}\} \\
&< \infty.
\end{aligned}
$$

The result was proved. □

Theorem 4.1 Let $\{F_{nk}\} \in \mathbf{K}_k(\mathbb{R}^+)$ be an array of rowwise negatively dependent set-valued random variables with $E[F_{nk}] = \{0\}$ and stochastic dominated by a set-valued random variable F. If $\max_k a_{nk} = O(n^{-r}),\ r > 0$, then $E[\|F\|_{\mathbf{K}}^{1+1/r}] < \infty$ implies that

$$
\sum_{k=1}^{\infty} a_{nk}F_{nk} \to 0 \ \ a.e.
$$

with respect to the Hausdorff metric d_H.

Proof. It suffices to show that for every $\varepsilon > 0$ that

$$
\sum_{n=1}^{\infty} P\Big\{\|\sum_{k=1}^{\infty} a_{nk}F_{nk}\|_{\mathbf{K}} \ge \varepsilon\Big\} < \infty.
$$

Now

$$
\begin{aligned}
\Big\{\|\sum_{k=1}^{\infty} a_{nk}F_{nk}\|_{\mathbf{K}} \ge \varepsilon\Big\} &\subseteq \Big\{\|\sum_{k=1}^{\infty} a_{nk}F_{nk}I_{\{\|a_{nk}F_{nk}\|_{\mathbf{K}}<n^{-\alpha}\}}\|_{\mathbf{K}} \ge \frac{\varepsilon}{2}\Big\} \\
&\quad \bigcup \Big\{\|a_{nk}F_{nk}\|_{\mathbf{K}} \ge \frac{\varepsilon}{2} \ \text{ for some k}\Big\} \\
&\quad \bigcup \Big\{\|a_{nk}F_{nk}\|_{\mathbf{K}} > n^{-\alpha} \ \text{ for at least two k}\Big\}.
\end{aligned}
$$

Thus by Lemma 4.1, Lemma 4.2 and Lemma 4.3, we have

$$
\sum_{n=1}^{\infty} P\Big\{\|\sum_{k=1}^{\infty} a_{nk}F_{nk}\|_{\mathbf{K}} \ge \varepsilon\Big\} < \infty.
$$

The result was proved. □

Remark All the above results are obtained in the space $\mathbb{R}^+$. From the proof, we know that they are also true for $\{x \in \mathbb{R} : x \le 0\}$.

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