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On the Existence Conditions for Balanced Fractional 2^m Factorial Designs of Resolution $\mathrm{R}^*(\{1\}|\Omega_\ell)$ with $N < \nu_\ell(m)$

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Abstract

We consider a fractional 2^m factorial design derived from a simple array (SA) such that the $(\ell + 1)$-factor and higher-order interactions are assumed to be negligible, where $2\ell \leq m$. Under these situations, if at least the main effect is estimable, then a design is said to be of resolution $\mathrm{R}^*(\{1\}|\Omega_\ell)$. In this paper, we give a necessary and sufficient condition for an SA to be a balanced fractional 2^m factorial design of resolution $\mathrm{R}^*(\{1\}|\Omega_\ell)$ for $\ell = 2, 3$, where the number of assemblies is less than the number of non-negligible factorial effects. Such a design is concretely characterized by the suffixes of the indices of an SA.

Keywords: association algebra, balanced fractional factorial design, estimable parametric function, factorial effect, resolution, simple array

1. Introduction

As a generalization of an orthogonal array, the concept of a balanced array (BA) was first introduced by Chakravarti (1956) as a partially BA. However it is a generalization of the BIB design rather than of the PBIB design. Thus Srivastava and Chopra (1971) called it by BA. A BA of strength t, size N, m constraints, two symbols and index set $\{\mu_i^{(t)} \mid 0 \leq i \leq t\}$ is briefly written by $\mathrm{BA}(N, m, 2, t; \{\mu_i^{(t)}\})$. In particular, a BA of strength $t = m$ is called a simple array (SA) (see Shirakura, 1977), and it is written by $\mathrm{SA}(m; \{\lambda_x\})$ for brevity, where $\lambda_x = \mu_x^{(m)}$. When $t < m$, a BA of strength t does not always exist for given indices $\mu_i^{(t)}$. On the other hand, an SA always exists for any λ_x and any m. The existence conditions for a BA of strength t were given by Srivastava (1972) for $m = t + 1, t + 2$, and Shirakura (1977) for $m = t + 3$. If the variance-covariance matrix of the estimators of the factorial effects to be of interest is invariant under any permutation on the factors, then a design is said to be balanced. Under certain conditions, a BA of strength 2ℓ turns out to be a balanced fractional 2^m factorial (2^m-BFF) design of resolution $2\ell + 1$ (see for $\ell = 2$, Srivastava, 1970, and for general ℓ, Yamamoto *et al.*, 1975), where $2\ell \leq m$. The characteristic roots of the information matrix of a 2^m-BFF design of resolution V, i.e., $\ell = 2$, were obtained by Srivastava and Chopra (1971). By using the triangular multidimensional partially balanced (TMDPB) association scheme and its algebra, their results were generalized by Yamamoto *et al.* (1976) and Hyodo (1992) for a resolution $2\ell + 1$ design, where $2\ell \leq m$ and $m < 2\ell \leq 2m$, respectively. The concept of the MDPB association scheme, which is a generalization of an ordinary association scheme (e.g., Bailey, 2004), was introduced by Bose and Srivastava (1964). The existence conditions for a BA of strength 2ℓ to be a 2^m-BFF design of resolution 2ℓ for general ℓ were obtained by Shirakura (1975,1980). Some algebraic properties of the information matrix of a fractional 2^m factorial (2^m-FF) design derived from an SA were investigated by Hyodo and Yamamoto (1988) and Hyodo (1989). As the extension of the concept of resolution, Yamamoto and Hyodo (1984) discussed the extended concept of resolution for 2^m fractions.

Definition 1.1. Under the assumption that the $(\ell + 1)$-factor and higher-order interactions are negligible, if the p_1-factor, the p_2-factor, $\cdots$, and the p_r-factor interactions are estimable, where $0 \leq p_1 < p_2 < \cdots < p_r \leq \ell$, and furthermore if the remaining interactions are not estimable (including the general mean and the main effect), then a design is said to be of resolution R$(\{p_1, p_2, \cdots, p_r\}|\Omega_\ell)$, where $\Omega_\ell = \{0, 1, \cdots, \ell\}$. In particular, when $p_i = i-1\,(1 \le i \le r = \ell+1)$, it is of resolution $2\ell+1$, and when $p_i = i\,(1 \le i \le r = \ell-1)$ (or $p_i = i-1\,(1 \le i \le r = \ell)$), it is of resolution 2ℓ.

By relaxing the conditions of Definition 1.1, we give the following definition of resolution:

Definition 1.2. Under the same assumptions as Definition 1.1, if at least the p_1-factor, the p_2-factor,$\cdots$, and the p_r-factor interactions are estimable, where $0 \le p_1 < p_2 < \cdots < p_r \le \ell$, then a design is said to be of resolution R$^*(\{p_1, p_2, \cdots, p_r\}|\Omega_\ell)$.

Note that the set of resolution R$(\{p_1, p_2, \cdots, p_r\}|\Omega_\ell)$ designs is a subset of resolution R$^*(\{p_1, p_2, \cdots, p_r\}|\Omega_\ell)$ designs. For example, a resolution R$^*(\{1\}|\Omega_3)$ design is of resolution R$(\omega|\Omega_3)$, where $\omega = \{1\}$, $\{0,1\}$, $\{1,2\}$, $\{1,3\}$, $\{0,1,2\}$, $\{0,1,3\}$, $\{1,2,3\}$ or $\{0,1,2,3\}$. Here when a design is derived from an SA, where the number of assemblies (or treatment combinations) is less than the number of non-negligible factorial effects, there does not exist a resolution R$(\{1,2\}|\Omega_3)$, R$(\{1,3\}|\Omega_3)$, R$(\{0,1,3\}|\Omega_3)$, R$(\{1,2,3\}|\Omega_3)$ or R$(\{0,1,2,3\}|\Omega_3)$ design (see Table 4.1 in latter).

In a practical experiment, the most interesting factorial effect may be the main effect, next may be the two-factor interaction, and so on. Using the algebraic structure of the TMDPB association scheme and the matrix equations, Kuwada *et al*. (2003) obtained a 2^m-BFF design of resolution R$^*(\{1\}|\Omega_3)$. However their results are very complex. A necessary and sufficient condition for an SA to be a 2^m-BFF design of resolution $2\ell+1$ for general ℓ has been obtained by Hyodo *et al*. (2015), where $2\ell \le m$.

In this paper, we consider a 2^m-BFF design derived from an SA such that the number of assemblies is less than the number of factorial effects up to the ℓ-factor interaction, where $\ell = 2, 3$. Under these situations, using the suffixes of the indices λ_x of an SA, we give a necessary and sufficient condition for an SA to be a 2^m-BFF design of resolution R$^*(\{1\}|\Omega_2)$, i.e., of resolution IV, and also we rewrite a necessary and sufficient condition for an SA to be a 2^m-BFF design of resolution R$^*(\{1\}|\Omega_3)$.

2. Preliminaries

We consider a 2^m-FF design T with m factors and N assemblies, where $2\ell \le m$, and the $(\ell+1)$-factor and higher-order interactions are assumed to be negligible. Then the linear model is given by $\boldsymbol{y}(T) = E_T\boldsymbol{\Theta} + \boldsymbol{e}_T$, where $\boldsymbol{y}(T)$ is an $N \times 1$ observation vector, E_T is the $N \times \nu_\ell(m)$ design matrix, $\boldsymbol{\Theta}' = (\boldsymbol{\theta}_0'; \boldsymbol{\theta}_1'; \cdots; \boldsymbol{\theta}_\ell')$, and $\boldsymbol{e}_T$ is an $N \times 1$ error vector with mean $\boldsymbol{0}_N$ and variance-covariance matrix $\sigma^2 I_N$. Here $\boldsymbol{\theta}_0, \boldsymbol{\theta}_1, \cdots$, and $\boldsymbol{\theta}_\ell$ are the general mean, the vector of the main effect,$\cdots$, and the vector of the ℓ-factor interaction, respectively, $\nu_\ell(m) = \binom{m}{0} + \binom{m}{1} + \cdots + \binom{m}{\ell}$, and I_p is the identity matrix of order p. The normal equations for estimating $\boldsymbol{\Theta}$ are given by $M_T\hat{\boldsymbol{\Theta}} = E_T'\boldsymbol{y}(T)$, where $M_T = E_T'E_T$ is the information matrix of order $\nu_\ell(m)$. If M_T is non-singular, then T is of resolution $2\ell+1$.

Let $A_\alpha^{(u,v)}(= A_\alpha^{(v,u)\prime})$ $(0 \le \alpha \le u \le v \le \ell)$ be the local association matrices of size $\binom{m}{u} \times \binom{m}{v}$ of the TMDPB association scheme, and further let $A_\beta^{\#(u,v)}(= A_\beta^{\#(v,u)\prime})$ $(0 \le \beta \le u \le v \le \ell)$ be the matrices of size $\binom{m}{u} \times \binom{m}{v}$ (see Yamamoto *et al*., 1976), where the relation between $A_\alpha^{(u,v)}$ and $A_\beta^{\#(u,v)}$ is given by

$$A_\alpha^{(u,v)} = \sum_{\beta=0}^{u} z_{\beta\alpha}^{(u,v)} A_\beta^{\#(u,v)} \quad \text{for } 0 \le \alpha \le u \le v$$

and

$$A_\beta^{\#(u,v)} = \sum_{\alpha=0}^{u} z_{(u,v)}^{\beta\alpha} A_\alpha^{(u,v)} \quad \text{for } 0 \le \beta \le u \le v.$$

Here

$$z_{\beta\alpha}^{(u,v)} = \sum_{b=0}^{\alpha} (-1)^{\alpha-b} \binom{u-\beta}{b} \binom{u-b}{u-\alpha} \binom{m-u-\beta+b}{b} \sqrt{\binom{m-u-\beta}{v-u}\binom{v-\beta}{v-u}} \Big/ \binom{v-u+b}{b} \quad \text{for } u \le v \tag{1}$$

and

$$z_{(u,v)}^{\beta\alpha} = \phi_\beta z_{\beta\alpha}^{(u,v)} \Big/ \left\{ \binom{m}{u} \binom{u}{\alpha} \binom{m-u}{v-u+\alpha} \right\} \quad \text{for } u \le v$$

(see Shirakura and Kuwada, 1976, and Yamamoto *et al.*, 1976), where $\phi_\beta = \binom{m}{\beta} - \binom{m}{\beta-1}$. Then some properties of $A_\beta^{\#(u,v)}$ are cited in the following:

$$A_\beta^{\#(u,w)} A_\gamma^{\#(w,v)} = \delta_{\beta\gamma} A_\beta^{\#(u,v)},$$
$$\sum_{\beta=0}^{u} A_\beta^{\#(u,u)} = I_{\binom{m}{u}} \tag{2}$$

and

$$\mathrm{rank}\{A_\beta^{\#(u,v)}\} = \phi_\beta,$$

where $\delta_{\beta\gamma}$ is the Kronecker delta.

Let $D_\alpha^{(u,v)} (= D_\alpha^{(v,u)\prime})$ $(0 \le \alpha \le u \le v \le \ell)$ and $D_\beta^{\#(u,v)} (= D_\beta^{\#(v,u)\prime})$ $(0 \le \beta \le u \le v \le \ell)$ be the matrices of order $\nu_\ell(m)$ such that the $(u+1)$-th row block and the $(v+1)$-th column block of $D_\alpha^{(u,v)}$ and $D_\beta^{\#(u,v)}$ are given by $A_\alpha^{(u,v)}$ and $A_\beta^{\#(u,v)}$, respectively, and zero at elsewhere. Then the information matrix M_T is given by

$$M_T = \sum_{\beta=0}^{\ell} \sum_{u=0}^{\ell-\beta} \sum_{v=0}^{\ell-\beta} \kappa_\beta^{u,v} D_\beta^{\#(u+\beta,v+\beta)}$$

(see Yamamoto *et al.*, 1976), where T is a $\mathrm{BA}(N, m, 2, 2\ell; \{\mu_i^{(2\ell)}\})$. Here the relation between $\kappa_\beta^{u,v}$ and $\mu_i^{(2\ell)}$ is given by

$$\kappa_\beta^{u,v} \left(= \kappa_\beta^{v,u}\right) = \sum_{\alpha=0}^{u+\beta} z_{\beta\alpha}^{(u+\beta,v+\beta)} \gamma_{v-u+2\alpha} \quad \text{for } 0 \le u \le v \le \ell-\beta \text{ and } 0 \le \beta \le \ell, \tag{3}$$

where

$$\gamma_j = \sum_{i=0}^{2\ell} \sum_{p=0}^{j} (-1)^p \binom{j}{p} \binom{2\ell - j}{i - j + p} \mu_i^{(2\ell)} \quad \text{for } 0 \le j \le 2\ell. \tag{4}$$

The relation between the indices $\mu_i^{(2\ell)}$ of a BA of strength 2ℓ and λ_x of an SA is given by

$$\mu_i^{(2\ell)} = \sum_{x=0}^{m} \binom{m-2\ell}{x-i} \lambda_x \quad \text{for } 0 \le i \le 2\ell. \tag{5}$$

Note that size N(=number of assemblies) of an $\mathrm{SA}(m; \{\lambda_x\})$ is given by $N = \sum_{x=0}^{m} \binom{m}{x} \lambda_x$. Furthermore M_T is isomorphic to the symmetric matrices $\|\kappa_\beta^{u,v}\| (= K_\beta,$ say) of order $(\ell - \beta + 1)$, i.e., there exists an orthogonal matrix P of order $\nu_\ell(m)$ such that

$$P' M_T P = \mathrm{diag}\left[K_0; K_1, \cdots, K_1; K_2, \cdots, K_2; \cdots; K_\ell, \cdots, K_\ell\right], \tag{6}$$

where K_β $(0 \le \beta \le \ell)$ are with multiplicities ϕ_β. From (6), the following is immediately:

Lemma 2.1. *Let T be an* $\mathrm{SA}(m; \{\lambda_x\})$. *Then the information matrix M_T is non-singular*, *i.e.*, *T is of resolution* $2\ell + 1$, *if and only if every K_β $(0 \le \beta \le \ell)$ is non-singular*, *i.e.*, $\mathrm{rank}\{K_\beta\} = \ell - \beta + 1$ *for all* β.

From (1), and (3) through (5), we have the following (see Hyodo and Yamamoto, 1988):

Lemma 2.2. *Let T be an* $\mathrm{SA}(m; \{\lambda_x\})$. *Then we have*

$$\kappa_\beta^{u,v} = \sum_{x=\beta}^{m-\beta} \left[\left\{ 2^\beta \Big/ \sqrt{\binom{m-2\beta}{u}} \right\} \left\{ \sqrt{\lambda_x} \sum_{p=0}^{u} (-1)^p \binom{x-\beta}{u-p} \binom{m-\beta-x}{p} \right\} \sqrt{\binom{m-2\beta}{x-\beta}} \right]$$
$$\times \left[\left\{ 2^\beta \Big/ \sqrt{\binom{m-2\beta}{v}} \right\} \left\{ \sqrt{\lambda_x} \sum_{q=0}^{v} (-1)^q \binom{x-\beta}{v-q} \binom{m-\beta-x}{q} \right\} \sqrt{\binom{m-2\beta}{x-\beta}} \right]$$
$$\textit{for } 0 \le u \le v \le \ell-\beta,\ 0 \le \beta \le \ell,\ \textit{and } 2\ell \le m.$$

Let $F_\beta\,(0 \le \beta \le \ell)$ be the $(\ell - \beta + 1) \times (m - 2\beta + 1)$ matrices such that the column vector corresponding to the index $\lambda_x\ (x \in V_\beta)$ is given by $\boldsymbol{F}_\beta(x)$, where the $(u + 1)$-th row of $\boldsymbol{F}_\beta(x)$ is given by

$$\sqrt{\lambda_x}\sum_{p=0}^{u}(-1)^p\binom{x-\beta}{u-p}\binom{m-\beta-x}{p} \text{ for } 0 \le u \le \ell-\beta \tag{7}$$

and $V_\beta = \{x \in N_0 \,|\, \beta \le x \le m-\beta\}$. Here N_0 is a set of non-negative integers. The $(u + 1)$-th row and the $(v + 1)$-th column of $K_\beta\,(0 \le u, v \le \ell-\beta; 0 \le \beta \le \ell)$ correspond to the $(u+\beta)$-factor interaction and the $(v+\beta)$-factor one, respectively. Thus the $(u + 1)$-th row of F_β corresponds to the $(u + \beta)$-factor interaction. Then from (7), we can easily obtain the following theorem (e.g., Hyodo *et al.*, 2015):

Theorem 2.1. *Let T be an* $\mathrm{SA}(m; \{\lambda_x\})$, *where $2\ell \le m$. Then the matrices $K_\beta\,(0 \le \beta \le \ell)$ can be expressed as $K_\beta = (D_\beta F_\beta \Lambda_\beta)(D_\beta F_\beta \Lambda_\beta)'$, where D_β and Λ_β are the diagonal matrices such that the $(u + 1)$-th element $(0 \le u \le \ell - \beta)$ of D_β and the element of Λ_β corresponding to λ_x are given by $2^\beta/\sqrt{\binom{m-2\beta}{u}}$ and $\sqrt{\binom{m-2\beta}{x-\beta}}$, respectively.*

It follows from Theorem 2.1 that $\mathrm{rank}\{K_\beta\} = \text{r-rank}\{F_\beta\}$, where $\text{r-rank}\{A\}$ denotes the row rank of a matrix A. In order to obtain the rank of a matrix A, we sometimes apply the "elementary row operations" on it. In this case, we positively use the notation "r-rank" instead of the rank. Let $SV_\beta = \{x \in V_\beta \,|\, \lambda_x \neq 0\}\,(0 \le \beta \le \ell)$, and further let NSV_β be the cardinal number of SV_β. Then the following is obtained (see Hyodo *et al.*, 2015):

Theorem 2.2. *Let T be an* $\mathrm{SA}(m; \{\lambda_x\})$, *where $2\ell \le m$. Then it holds that* $\text{r-rank}\{F_\beta(x_1, x_2, \cdots, x_{n_\beta})\} = \min(n_\beta, \ell-\beta+1)$ *for $\{x_1, x_2, \cdots, x_{n_\beta}\} \subset SV_\beta\,(0 \le \beta \le \ell)$, where $F_\beta(x_1, x_2, \cdots, x_{n_\beta}) = (\boldsymbol{F}_\beta(x_1), \boldsymbol{F}_\beta(x_2), \cdots, \boldsymbol{F}_\beta(x_{n_\beta}))$. Furthermore the first* $\min(n_\beta, \ell-\beta+1)$ *rows of $F_\beta(x_1, x_2, \cdots, x_{n_\beta})$ are linearly independent.*

The following is due to Ghosh and Kuwada (2001):

Lemma 2.3. *Let $K = \|K_{ij}\|$ and $L = \|L_{ij}\|$ $(i, j = 1, 2, 3)$ be a positive semi-definite matrix of order n with* $\mathrm{rank}\{K\} = \mathrm{rank}\left\{\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22}\end{pmatrix}\right\} = n_1 + n_2 (\ge 1)$ *and some matrix of order n such that $L_{11} = I_{n_1}$ and $L_{1j} = L'_{j1} = 0_{n_1\times n_j}$ $(j = 2, 3)$, respectively, where K_{ij} and L_{ij} are both of size $n_i \times n_j$, and $n_1 + n_2 + n_3 = n$. Then a matrix equation $XK = L$ with parameter matrix X of order n has a solution if and only if*

(i) $n_3 = 0$, *where if $n_2 \ge 1$, then L_{22} is arbitrary, or*

(ii) $n_3 \ge 1$, *and moreover*

(1) *when $n_2 = 0$, $K_{33} = 0_{n_3\times n_3}$, and furthermore $L_{33} = 0_{n_3\times n_3}$, or*

(2) *when $n_2 \ge 1$, there exists a matrix W of size $n_3 \times n_2$ such that $K_{3j} = WK_{2j}$ $(j = 1, 2, 3)$, and furthermore $L_{i3} = L_{i2}W'$ $(i = 2, 3)$, where L_{i2} are arbitrary.*

Remark 2.1. In Lemma 2.3.(i) and (ii)(2), when $n_2 \ge 1$, without loss of generality, we can put $L_{22} = I_{n_2}$ and $L_{23} = L'_{32}$ $(= W')$ (if $n_3 \ge 1$), and hence $L_{33} = WW'$. Furthermore we have $W = K_{32}K_{22}^{-1}$.

Let T be an $\mathrm{SA}(m; \{\lambda_x\})$, where $2\ell \le m$. Then a set of parametric functions $H\boldsymbol{\Theta}$ is estimable if and only if there exists a matrix X of order $v_\ell(m)$ such that $XM_T = H$, where H and X are given by

$$H = \sum_{\beta=0}^{\ell}\sum_{u=0}^{\ell-\beta}\sum_{v=0}^{\ell-\beta} h_\beta^{u,v} D_\beta^{\#(u+\beta,v+\beta)} \text{ and } X = \sum_{\beta=0}^{\ell}\sum_{u=0}^{\ell-\beta}\sum_{v=0}^{\ell-\beta} \chi_\beta^{u,v} D_\beta^{\#(u+\beta,v+\beta)},$$

where $2\ell \le m$. Thus there exist matrices X_β such that $X_\beta K_\beta = H_\beta$ for all β $(0 \le \beta \le \ell)$ if and only if T is of resolution $\mathrm{R}^*(\{1\}|\Omega_\ell)$, where

$$H_0 = \begin{pmatrix} h_0^{0,0} & 0 & h_0^{0,2} & \cdots & h_0^{0,\ell} \\ 0 & 1 & 0 & \cdots & 0 \\ h_0^{2,0} & 0 & h_0^{2,2} & \cdots & h_0^{2,\ell} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_0^{\ell,0} & 0 & h_0^{\ell,2} & \cdots & h_0^{\ell,\ell} \end{pmatrix},\ H_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & h_1^{1,1} & \cdots & h_1^{1,\ell-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & h_1^{\ell-1,1} & \cdots & h_1^{\ell-1,\ell-1} \end{pmatrix}$$

$$\text{and } H_\gamma = \begin{pmatrix} h_\gamma^{0,0} & \cdots & h_\gamma^{0,\ell-\gamma} \\ \vdots & \ddots & \vdots \\ h_\gamma^{\ell-\gamma,0} & \cdots & h_\gamma^{\ell-\gamma,\ell-\gamma} \end{pmatrix} \quad \text{for } \gamma \ge 2.$$

Let $B = \mathrm{diag}[B_1, B_2, B_3]$, $C' = (C_1'\ C_2'\ C_3')$ and $\varDelta$ be a diagonal and non-singular matrix of order $n(= n_1 + n_2 + n_3)$, a matrix of size $n \times p$ with r-rank$\{C\}$ = r-rank$\left\{\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}\right\} = n_1 + n_2$, and a diagonal and non-singular matrix of order p, respectively, where B_i and C_i $(i = 1, 2, 3)$ are of order n_i and of size $n_i \times p$, respectively. Then we have the following:

Lemma 2.4. *Let $Z = \|Z_{ij}\|$ $(i, j = 1, 2, 3)$ be a matrix of order $n(= n_1 + n_2 + n_3)$, where Z_{ij} are of size $n_i \times n_j$, and let $K(= \|K_{ij}\|) = (BC\varDelta)(BC\varDelta)'$ and $K_{ij} = (B_iC_i\varDelta)(B_jC_j\varDelta)'$, and further let $L = \|L_{ij}\|$ be a matrix given by* Lemma 2.3. *Then a matrix equation $ZK = L$ has a solution if and only if*

(i) $n_3 = 0$, *where if $n_2 \geq 1$, then L_{22} is arbitrary, or*

(ii) $n_3 \geq 1$, *and moreover*

(1) *when $n_2 = 0$, it holds $C_3 = 0_{n_3\times p}$, and furthermore $L_{33} = 0_{n_3\times n_3}$, or*

(2) *when $n_2 \geq 1$, there exists a matrix $W_2^*(= C_3C_2'(C_2C_2')^{-1})$ of size $n_3 \times n_2$ such that $C_3 = W_2^*C_2$, and furthermore $L_{i3} = L_{i2}(B_3W_2^*B_2^{-1})'$ $(i = 2, 3)$, where L_{i2} are arbitrary.*

Proof. (i) When $n_3 = 0$, if $n_2 \geq 1$, then from Lemma 2.3.(i), we have the required result.

(ii)(1) When $n_3 \geq 1$ and $n_2 = 0$, it follows from Lemma 2.3.(ii)(1) that $K_{33} = (B_3C_3\varDelta)(B_3C_3\varDelta)' = B_3C_3\varDelta\varDelta'C_3'B_3' = 0_{n_3\times n_3}$, and hence $C_3\varDelta\varDelta'C_3' = 0_{n_3\times n_3}$, which implies $C_3\varDelta = 0_{n_3\times p}$. Thus we get $C_3 = 0_{n_3\times p}$, and hence $L_{33} = 0_{n_3\times n_3}$.

(2) When $n_i \geq 1$ $(i = 2, 3)$, from Lemma 2.3.(ii)(2), there exists a matrix W_2 of size $n_3 \times n_2$ such that $K_{3j} = W_2K_{2j}$ $(j = 1, 2, 3)$. Since r-rank$\{C\}$ = r-rank$\left\{\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}\right\} = n_1 + n_2$, there exists a matrix $(W_1^*\ \ W_2^*)$ of size $n_3 \times (n_1 + n_2)$ such that $C_3 = (W_1^*\ \ W_2^*)\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$, where W_k^* $(k = 1, 2)$ are of size $n_3 \times n_k$. Then $K_{3j} = (B_3C_3\varDelta)(B_jC_j\varDelta)' = \{B_3(W_1^*\ W_2^*)\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}\varDelta\}(B_jC_j\varDelta)' = B_3\{W_1^*B_1^{-1}(B_1C_1\varDelta)\}(B_jC_j\varDelta)' + B_3\{W_2^*B_2^{-1}(B_2C_2\varDelta)\}(B_jC_j\varDelta)' = B_3W_1^*B_1^{-1}K_{1j} + B_3W_2^* \times B_2^{-1}K_{2j} = (B_3W_1^*B_1^{-1}\ \ B_3W_2^*B_2^{-1})\begin{pmatrix} K_{1j} \\ K_{2j} \end{pmatrix}$. Thus we have $(K_{31}\ \ K_{32}) = (B_3W_1^*B_1^{-1}\ \ B_3W_2^*B_2^{-1})\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} = (0_{n_3\times n_1}\ \ W_2)\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$. Since $\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$ is non-singular, we get $B_3W_1^*B_1^{-1} = 0_{n_3\times n_1}$, i.e., $W_1^* = 0_{n_3\times n_1}$, and $W_2 = B_3W_2^*B_2^{-1}$.

From Lemma 2.3, the converse is obvious, and hence the required result is obtained.

Let $\overline{T}$ be an array obtained by interchanging all of symbols 0 and 1 of T, where T is an SA$(m; \{\lambda_x\})$. Then it can be easily shown that $\overline{T}$ is also an SA$(m; \{\overline{\lambda}_x\})$, where $\overline{\lambda}_x = \lambda_{m-x}$ for $0 \leq x \leq m$ (e.g., Shirakura and Kuwada, 1975). Note that $\overline{T}$ is called a complementary SA (CSA) of T. Furthermore if T is of resolution $\mathrm{R}^*(\{1\}|\Omega_\ell)$, then $\overline{T}$ is also of resolution $\mathrm{R}^*(\{1\}|\Omega_\ell)$.

If $N \geq \nu_\ell(m)$, then there always exists a 2^m-BFF design of resolution $2\ell + 1$ (see Hyodo *et al*., 2015). Thus in the rest of this paper, we consider a 2^m-BFF design of resolution $\mathrm{R}^*(\{1\}|\Omega_\ell)$ derived from an SA with $N < \nu_\ell(m)$ for $\ell = 2, 3$.

3. Resolution $\mathrm{R}^*(\{1\}|\Omega_2)$ designs

We now consider case $\ell = 2$. Then it follows from (7) that $\boldsymbol{F}_\beta(x)$ $(0 \leq \beta \leq 2)$ are given by

$$\boldsymbol{F}_0(x) = \sqrt{\lambda_x}\begin{pmatrix} 1 \\ 2x - m \\ \{(2x - m)^2 - m\}/2 \end{pmatrix} \quad \text{for } x \in V_0, \tag{8a}$$

$$\boldsymbol{F}_1(x) = \sqrt{\lambda_x}\begin{pmatrix} 1 \\ 2x - m \end{pmatrix} \quad \text{for } x \in V_1 \tag{8b}$$

and

$$F_2(x) = \sqrt{\lambda_x}\,(1) \quad \text{for } x \in V_2. \tag{8c}$$

Let

$$f_\beta(x; m) = \{(2x - m)^2 + (2\beta - m)\}/2 \quad \text{for } x \in V_\beta\ (\beta = 0, 1). \tag{9}$$

Then we have the following:

Lemma 3.1. *If* $f_\beta(x_1; m) = f_\beta(x_2; m)$ *for* $\{x_1, x_2\} \subset V_\beta$ $(\beta = 0, 1)$, *then* $x_1 + x_2 - m = 0$.

Proof. If $f_\beta(x_1; m) = f_\beta(x_2; m)$ for $\{x_1, x_2\} \subset V_\beta$ $(\beta = 0, 1)$, i.e., $\{(2x_1 - m)^2 + (2\beta - m)\}/2 = \{(2x_2 - m)^2 + (2\beta - m)\}/2$, then we have $(2x_1 - m)^2 - (2x_2 - m)^2 = 4(x_1 - x_2)(x_1 + x_2 - m) = 0$. Thus we get the required result.

The following is the main results of this section:

Theorem 3.1. *Let* T *be an* SA$(m; \{\lambda_x\})$, *where* $SV_0 = \{x_1, x_2, \cdots, x_{\mathrm{NSV}_0}\}$, $N < \nu_2(m)$ *and* $m \geq 4$. *Then a necessary and sufficient condition for* T *to be a* 2^m-BFF *design of resolution* R$^*(\{1\}|\Omega_2)$, *i.e., of resolution* IV, *is that non-zero indices of an* SA *satisfy the following*:

(i) *When* $\mathrm{NSV}_0 = 2$, $x_1 = 1$ *and* $x_2 = m - 1$,

(ii) *when* $\mathrm{NSV}_0 = 3$,

(1) $x_1 = 0$, $x_2 = 1$ *and* $x_3 = m - 1$, *or its* CSA, *or*

(2) $x_1 = 0$, $x_2 = 2$ *and* $x_3 = 4$, *where* $m = 4$

and

(iii) *when* $\mathrm{NSV}_0 = 4$, $x_1 = 0$, $x_2 = 1$, $x_3 = m - 1$ *and* $x_4 = m$.

Proof. See Appendix.

From Theorems 2.1 and 2.2, Lemmas 2.3 and 2.4, and Remark 2.1, we have the following:

Theorem 3.2. *If* T *is a* 2^m-BFF *design of resolution* R$^*(\{1\}|\Omega_2)$, *i.e., of resolution* IV, *derived from an* SA$(m; \{\lambda_x\})$, *where* $m \geq 4$ *and* $N < \nu_2(m)$, *and moreover*

(i) *when* $\mathrm{NSV}_0 = 2$, $A_0^{\#(0,0)}\boldsymbol{\theta}_0 + \left[\left\{1/\sqrt{\binom{m}{2}}\right\} f_0(x_i; m)\right] A_0^{\#(0,2)}\boldsymbol{\theta}_2$ $(i = 1, 2)$ *and* $A_0^{\#(1,1)}\boldsymbol{\theta}_1$ *are estimable, where* $\{x_1, x_2\} = SV_0$ *and* $f_0(x; m)$ *is given by* (9), *and furthermore if* $f_0(x_i; m) = 0$ *for all* i, *then* $A_0^{\#(0,0)}\boldsymbol{\theta}_0$ *is estimable and* $A_0^{\#(2,2)}\boldsymbol{\theta}_2$ *is not estimable,*

(ii) *when* $\mathrm{NSV}_1 = 1$, $A_1^{\#(1,1)}\boldsymbol{\theta}_1$ *is estimable and* $A_1^{\#(2,2)}\boldsymbol{\theta}_2$ *is not estimable*,

(iii) *when* $\mathrm{NSV}_2 = 0$, $A_2^{\#(2,2)}\boldsymbol{\theta}_2$ *is not estimable*

and

(iv) *when* $\mathrm{NSV}_\beta \geq 3 - \beta$ $(0 \leq \beta \leq 2)$, $A_\beta^{\#(u,u)}\boldsymbol{\theta}_u$ $(\beta \leq u \leq 2)$ *are estimable*.

Note from (2) that if $A_\beta^{\#(u,u)}\boldsymbol{\theta}_u$ are estimable for all β $(0 \leq \beta \leq u \leq 2)$, then $\boldsymbol{\theta}_u$ is estimable. The results of Theorem 3.1, and estimable parametric functions and the resolution R$(\omega|\Omega_2)$ for each design are summarized in Table 3.1.

Let $K_\beta^{(0)}$ $(0 \leq \beta < \ell)$ be the matrices of order $(\ell - \beta)$ obtained from K_β by cutting off its last row and column, and further let $\boldsymbol{k}_\beta^{1\prime} = (\kappa_\beta^{0,0}\ \kappa_\beta^{0,1}\ \cdots \kappa_\beta^{0,\ell-\beta})$ and $\boldsymbol{k}_\beta^{2\prime} = (\kappa_\beta^{\ell-\beta,0}\ \kappa_\beta^{\ell-\beta,1}\ \cdots \kappa_\beta^{\ell-\beta,\ell-\beta})$. Then we have the following due to Shirakura (1980):

Proposition 3.1. *A necessary and sufficient condition for a* BA$(N, m, 2, 2\ell; \{\mu_i^{(2\ell)}\})$, T, *say, to be a* 2^m-BFF *design of resolution* 2ℓ *is that* T *satisfies the following condition*:

For r *integers* $0 \leq \beta_1 < \beta_2 < \cdots < \beta_r \leq \ell$ *with* $|K_{\beta_j}| = 0$ *and* $|K_\alpha| \neq 0$ $(\alpha \neq \beta_j$ $(1 \leq j \leq r)$; $0 \leq \alpha \leq \ell)$,

(i) *when* $\beta_1 = 0$, *there exists a scalar* d *such that* $\boldsymbol{k}_0^2 = d\boldsymbol{k}_0^1$, $|K_0^{(0)}| \neq 0$, $\kappa_{\beta_j}^{\ell-\beta_j,\ell-\beta_j} = 0$ $(1 \leq \beta_j \leq \ell)$ *and* $|K_{\beta_j}^{(0)}| \neq 0$ $(1 \leq \beta_j \leq \ell - 1)$

and

(ii) *when* $\beta_1 \geq 1$, $\kappa_{\beta_j}^{\ell-\beta_j,\ell-\beta_j} = 0$ $(1 \leq \beta_j \leq \ell)$ *and* $|K_{\beta_j}^{(0)}| \neq 0$ $(1 \leq \beta_j \leq \ell - 1)$,

where $|A|$ *denotes the determinant of a matrix* A.

In a theoretical sense, Proposition 3.1 above is a very useful result. However it is not always practical. Because the elements $\kappa_\beta^{u,v}$ of K_β $(0 \leq u \leq v \leq \ell - \beta;\ 0 \leq \beta \leq \ell)$ are given by some linear combinations of the indices $\mu_i^{(2\ell)}$ of a BA (or λ_x of an SA) (see (3) and (4)). Hence it is not always easy to obtain $\mu_i^{(2\ell)}$ (or λ_x) such that these indices satisfy some conditions. As an example, we consider case $\ell = 2$:

Let T be an SA$(m;\{\lambda_x\})$, where $m \geq 4$. Then from (3) through (5), we have

$$\kappa_0^{0,0} = \sum_{x=0}^{m} \binom{m}{x} \lambda_x (= N),$$

$$\kappa_0^{0,1} \left(= \kappa_0^{1,0}\right) = \left\{ 1 \Big/ \sqrt{\binom{m}{1}} \right\} \sum_{x=0}^{m} (2x - m) \binom{m}{x} \lambda_x,$$

$$\kappa_0^{0,2} \left(= \kappa_0^{2,0}\right) = \left[1 \Big/ \left\{ 2 \sqrt{\binom{m}{2}} \right\} \right] \sum_{x=0}^{m} \{(2x - m)^2 - m\} \binom{m}{x} \lambda_x,$$

$$\kappa_0^{1,1} = \left\{ 1 \Big/ \binom{m}{1} \right\} \sum_{x=0}^{m} (2x - m)^2 \binom{m}{x} \lambda_x,$$

$$\kappa_0^{1,2} \left(= \kappa_0^{2,1}\right) = \left[1 \Big/ \left\{ 2 \sqrt{\binom{m}{1}\binom{m}{2}} \right\} \right] \sum_{x=0}^{m} (2x - m)\{(2x - m)^2 - m\} \binom{m}{x} \lambda_x,$$

$$\kappa_0^{2,2} = \left[1 \Big/ \left\{ 4 \binom{m}{2} \right\} \right] \sum_{x=0}^{m} \{(2x - m)^2 - m\}^2 \binom{m}{x} \lambda_x,$$

$$\kappa_1^{0,0} = 2^2 \sum_{x=1}^{m-1} \binom{m-2}{x-1} \lambda_x,$$

$$\kappa_1^{0,1} \left(= \kappa_1^{1,0}\right) = \left\{ 2^2 \Big/ \sqrt{\binom{m-2}{1}} \right\} \sum_{x=1}^{m-1} (2x - m) \binom{m-2}{x-1} \lambda_x,$$

$$\kappa_1^{1,1} = \left\{ 2^2 \Big/ \binom{m-2}{1} \right\} \sum_{x=1}^{m-1} (2x - m)^2 \binom{m-2}{x-1} \lambda_x$$

and

$$\kappa_2^{0,0} = \sum_{x=2}^{m-2} \binom{m-4}{x-2} \lambda_x.$$

Thus it can be easily shown that $\kappa_1^{1,1} = 0$ if and only if there exists λ_{x^*} such that $(2x^* - m)^2 = 0$ for $x^* \in SV_1$, and $\kappa_2^{0,0} = 0$ if and only if $\lambda_{x^{**}} = 0$ for any $x^{**} \in V_2$. However it is not so easy to obtain the indices λ_x such that $\boldsymbol{k}_0^{2\prime}(= (\kappa_0^{2,0}\ \kappa_0^{2,1}\ \kappa_0^{2,2}))= d\boldsymbol{k}_0^{1\prime}(= d(\kappa_0^{0,0}\ \kappa_0^{0,1}\ \kappa_0^{0,2}))$ for $x \in SV_0$ and some d, that is to say, to obtain λ_x such that the system of the linear equations $\kappa_0^{2,u} = d\kappa_0^{0,u}$ $(u = 0, 1, 2)$ satisfies for $x \in SV_0$ and some d. On the other hand, the elements of F_β $(0 \leq \beta \leq 2)$ are given by some polynomial of x of the indices λ_x of an SA (e.g., Hyodo *et al.*, 2015) as seen from (8). In particular, the element of the first row of F_β is all one. Thus it follows from Theorem 2.2 and (8) that if r-rank$\{F_0\} = 2 < 3$, then there exist two indices λ_{x_i} $(i = 1, 2)$ such that $x_i \in SV_0$ and the elements of the last row of $F_0(x_1, x_2)$ are the same constants, i.e., $f_0(x_1; m) = f_0(x_2; m)$, where $f_0(x; m)$ is given by (9). Next if r-rank$\{F_1\} = 1 < 2$, then there exists an index λ_{x^*} such that $x^* \in SV_1$ and the last row of $\boldsymbol{F}_1(x^*)$ is 0, i.e., $2x^* - m = 0$, and if r-rank$\{F_2\} = 0 < 1$, then $F_2(x^{**}) = 0$ for any $x^{**} \in V_2$, i.e., $\lambda_{x^{**}} = 0$. Thus in order to obtain the indices λ_x of an SA such that they satisfy some conditions, the matrices F_β and Theorem 2.2 are very powerful.

Table 3.1. 2^m-BFF designs of resolution $\mathrm{R}^*(\{1\}|\Omega_2)$ with $N < \nu_2(m)$ $(m \geq 4)$

Theorem 3.1		Existence conditions ($\lambda_{x_i} \neq 0\,(i=1,2,\cdots,\mathrm{NSV}_0)$)	Estimable parametric functions	Resolution $\mathrm{R}(\omega\|\Omega_2)$
(i)		$x_1=1,\ x_2=m-1$	$A_0^{\#(0,0)}\boldsymbol{\theta}_0 + \left[(m-1)(m-4)\Big/\left\{2\sqrt{\binom{m}{2}}\right\}\right]A_0^{\#(0,2)}\boldsymbol{\theta}_2$: estimable, $A_0^{\#(1,1)}\boldsymbol{\theta}_1$: estimable, $A_0^{\#(2,2)}\boldsymbol{\theta}_2$: not estimable (if $m=4$), $A_1^{\#(u,u)}\boldsymbol{\theta}_u$: estimable ($u=1,2$), $A_2^{\#(2,2)}\boldsymbol{\theta}_2$: not estimable	ω={0, 1} (if $m=4$) {1} (if $m \geq 5$)
(ii)	(1)	$x_1=0,\ x_2=1,\ x_3=m-1$, or its CSA	$A_\gamma^{\#(u,u)}\boldsymbol{\theta}_u$: estimable ($\gamma \leq u \leq 2;\ \gamma=0,1$), $A_2^{\#(2,2)}\boldsymbol{\theta}_2$: not estimable	ω={0, 1}
	(2)	$m=4;\ \ x_1=0,\ x_2=2,\ x_3=4$	$A_\gamma^{\#(u,u)}\boldsymbol{\theta}_u$: estimable ($\gamma \leq u \leq 2;\ \gamma=0,2$), $A_1^{\#(1,1)}\boldsymbol{\theta}_1$: estimable, $A_1^{\#(2,2)}\boldsymbol{\theta}_2$: not estimable	
(iii)		$x_1=0,\ x_2=1,\ x_3=m-1,\ x_4=m$	$A_\gamma^{\#(u,u)}\boldsymbol{\theta}_u$: estimable ($\gamma \leq u \leq 2;\ \gamma=0,1$), $A_2^{\#(2,2)}\boldsymbol{\theta}_2$: not estimable	

4. Resolution $\mathrm{R}^*(\{1\}|\Omega_3)$ designs

In this section, we consider case $\ell = 3$. By use of the properties of the TMDPB association algebra and the matrix equations, a necessary and sufficient condition for an SA to be a 2^m-BFF design of resolution $\mathrm{R}^*(\{1\}|\Omega_3)$ was already given by Kuwada *et al*. (2003). However their results are very complex. On the other hand, the elements of F_β $(0 \le \beta \le 3)$ considered here are given by some polynomial of x of the indexes λ_x of an SA as in (10) below, and they are very simple. Thus using these matrices F_β and Theorem 2.2, we shall rewrite the existence conditions for a 2^m-BFF design of resolution $\mathrm{R}^*(\{1\}|\Omega_3)$ with $N < \nu_3(m)$.

From (7), $\boldsymbol{F}_\beta(x)$ $(0 \le \beta \le 3)$ are given by

$$\boldsymbol{F}_0(x) = \sqrt{\lambda_x}\begin{pmatrix} 1 \\ 2x-m \\ \{(2x-m)^2-m\}/2 \\ (2x-m)\{(2x-m)^2-(3m-2)\}/6 \end{pmatrix} \quad \text{for } x \in V_0, \tag{10a}$$

$$\boldsymbol{F}_1(x) = \sqrt{\lambda_x}\begin{pmatrix} 1 \\ 2x-m \\ \{(2x-m)^2-(m-2)\}/2 \end{pmatrix} \quad \text{for } x \in V_1, \tag{10b}$$

$$\boldsymbol{F}_2(x) = \sqrt{\lambda_x}\begin{pmatrix} 1 \\ 2x-m \end{pmatrix} \quad \text{for } x \in V_2 \tag{10c}$$

and

$$F_3(x) = \sqrt{\lambda_x}\,(1) \quad \text{for } x \in V_3. \tag{10d}$$

Let

$$g(x;m) = (2x-m)\{(2x-m)^2-(3m-2)\}/6 \quad \text{for } x \in V_0 \text{ and } m \ge 6. \tag{11}$$

Then we have the following:

Lemma 4.1. (I)(i) *If* $d_{02} = f_0(x_i;m)$ *and* $d_{03} = g(x_i;m)$ $(i = 1,2)$ *for* $\{x_1, x_2\} \subset V_0$ *and* $m \ge 6$, *where* d_{0k} $(k = 2,3)$ *are constants*, *and* $f_0(x;m)$ *and* $g(x;m)$ *are given by* (9) *and* (11), *respectively*, *then*

(1) *when* $m = 3t^2 + 2t + 1$ $(t \ge 1)$, *we get* $x_p = t(3t-1)/2(\ge 1)$ *and* $x_q = (t+1)(3t+2)/2(\ge 5)$ *for* $\{p,q\} = \{1,2\}$

and

(2) *when* $m = 3t^2 + 4t + 2$ $(t \ge 1)$, *we get* $x_p = t(3t+1)/2(\ge 2)$ *and* $x_q = (t+1)(3t+4)/2(\ge 7)$ *for* $\{p,q\} = \{1,2\}$.

Here in (1) *and* (2) *just above*, *we have* $d_{02} = m-1$ *and* $d_{03} = 0$.

(ii) *If* $d_0 + f_0(x_i;m)d_2 = g(x_i;m)$ $(i = 1,2,3)$ *for* $\{x_1,x_2,x_3\} \subset V_0$ *and* $m \ge 6$, *where* d_k $(k = 0,2)$ *are constants*, *then* $x_p + x_q - m \ne 0$ *for some* $\{p,q\} \subset \{1,2,3\}$ *and* $(2x_1-m)(2x_2-m)+(2x_2-m)(2x_3-m)+(2x_3-m)(2x_1-m)+(3m-2) = 0$. *Here*

$$d_0 = -\Big[(2x_p-m)^2(2x_q-m)^2 - m\{4(x_p-x_q)^2-(3m-2)\} - 2(2x_p-m)(2x_q-m)\Big]/\{12(x_p+x_q-m)\}$$

and

$$d_2 = \{(2x_p-m)^2 + (2x_p-m)(2x_q-m) + (2x_q-m)^2 - (3m-2)\}/\{6(x_p+x_q-m)\}.$$

(II)(i) *There does not exist a integer* $x_1^* \in V_1$ *such that* $2x_1^* - m = 0$ *and* $f_1(x_1^*;m) = 0$ *for* $m \ge 6$, *where* $f_1(x^*;m)$ *is given by* (9).

(ii) *If* $(2x_j^* - m)d^* = f_1(x_j^*;m)$ $(j = 1,2)$ *for* $\{x_1^*, x_2^*\} \subset V_1$ *and* $m \ge 6$, *where* d^* *is a constant, then* $(2x_1^*-m)(2x_2^*-m) + (m-2) = 0$. *Here* $d^* = x_1^* + x_2^* - m$.

Proof. (I)(i) If $d_{02} = f_0(x_i;m)$ $(i = 1,2)$ for $\{x_1,x_2\} \subset V_0$ and $m \ge 6$, i.e., $f_0(x_1;m) = f_0(x_2;m)$, then from Lemma 3.1, we have $x_1 + x_2 - m = 0$. In addition, if $d_{03} = g(x_i;m)$ $(i = 1,2)$, i.e., $g(x_1;m) = g(x_2;m)$, then $(2x_p-m)\{(2x_p-m)^2 - (3m-2)\} = 0$ for some $p \in \{1,2\}$. If $2x_p - m = 0$, then $x_p = m/2 = x_q$ for $q \in \{1,2\} \setminus \{p\}$, and hence $2x_p - m \ne 0$. Thus it must be $(2x_p-m)^2 - (3m-2) = 0$. Then it has solutions $x_p = (m \pm \sqrt{3m-2})/2$, which must be integers.

Thus we put $3m-2 = s^2$ $(s \geq 4)$, and hence $m = (s^2+2)/3$. If $s = 3t$ $(t \geq 2)$, then $m = 3t^2+2/3$, if $s = 3t+1$ $(t \geq 1)$, then $m = 3t^2+2t+1$, and if $s = 3t+2$ $(t \geq 1)$, then $m = 3t^2+4t+2$, and hence $s \neq 3t$. Since $(2x-m)^2-(3m-2) = 0$ for $x \in \{x_1, x_2\}$, we get $d_{02} = m-1$ and $d_{03} = 0$. Therefore (i) is proved.

(ii) It follows from Theorem 2.2 that the first three rows of $F_0(x_1, x_2, x_3)$ are linearly independent, and hence the first and the third rows of $F_0(x_1, x_2, x_3)$ are also linearly independent. Thus there exists $\{x_p, x_q\} \subset \{x_1, x_2, x_3\}$ such that $f_0(x_p; m) \neq f_0(x_q; m)$, i.e., $x_p+x_q-m \neq 0$. If $d_0+f_0(x_i; m)d_2 = g(x_i; m)$ $(i = 1, 2, 3)$, i.e., $d_0+[\{(2x_i-m)^2-m\}/2]d_2 = (2x_i-m)\{(2x_i-m)^2-(3m-2)\}/6$, where d_k $(k = 0, 2)$ are constants, then we get $d_0 = -[(2x_p-m)^2(2x_q-m)^2 - m\{4(x_p-x_q)^2-(3m-2)\} - 2(2x_p-m)(2x_q-m)]/\{12(x_p+x_q-m)\}$ and $d_2 = \{(2x_p-m)^2+(2x_p-m)(2x_q-m)+(2x_q-m)^2-(3m-2)\}/\{6(x_p+x_q-m)\}$ for some $\{p, q\} \subset \{1, 2, 3\}$, where $x_p+x_q-m \neq 0$. Substituting d_0 and d_2 into $d_0+f_0(x_r; m)d_2 = g(x_r; m)$ for $r \in \{1, 2, 3\} \setminus \{p, q\}$, we get $(2x_p-m)(2x_q-m)+(2x_q-m)(2x_r-m)+(2x_r-m)(2x_p-m)+(3m-2) = (2x_1-m)(2x_2-m)+(2x_2-m)(2x_3-m)+(2x_3-m)(2x_1-m)+(3m-2) = 0$, and hence (ii) is established.

(II)(i) If $2x_1^*-m = 0$ and $f_1(x_1^*; m) = 0$ for $x_1^* \in V_1$, i.e., $\{(2x_1^*-m)^2-(m-2)\}/2 = -(m-2)/2 = 0$, then $m = 2 < 6$. Thus the required result is obtained.

(ii) If $(2x_j^*-m)d^* = f_1(x_j^*; m)$ $(j = 1, 2)$ for $\{x_1^*, x_2^*\} \subset V_1$ and $m \geq 6$, i.e., $(2x_j^*-m)d^* = \{(2x_j^*-m)^2-(m-2)\}/2$, then $2(x_1^*-x_2^*)d^* = \{(2x_1^*-m)^2-(2x_2^*-m)^2\}/2 = 2(x_1^*-x_2^*)(x_1^*+x_2^*-m)$. Thus we get $d^* = x_1^*+x_2^*-m$, and hence $(2x_1^*-m)(2x_2^*-m)+(m-2) = 0$, which is the required result.

The following is the main theorem of this section:

Theorem 4.1. *Let T be an* SA$(m; \{\lambda_x\})$, *where* $SV_0 = \{x_1, x_2, \cdots, x_{\mathrm{NSV}_0}\}$, $N < \nu_3(m)$ *and* $m \geq 6$. *Then a necessary and sufficient condition for T to be a* 2^m-BFF *design of resolution* R$^*(\{1\}|\Omega_3)$ *is that non-zero indices of an* SA *satisfy the following*:

(i) *When* $\mathrm{NSV}_0 = 3$,

(1) $x_1 = 1$, $x_2 = 2$ *and* $x_3 = 5$, *where* $m = 6$, *or its* CSA,

(2) $x_1 = 1$, $x_2 = 2$ *and* $x_3 = 7$, *where* $m = 9$, *or its* CSA, *or*

(3) $x_1 = 1$, $x_2 = 3$ *and* $x_3 = 5$, *where* $m = 6$,

(ii) *when* $\mathrm{NSV}_0 = 4$,

(1) $x_1 = 0$, $x_2 = 1$, $x_3 = 2$ *and* $x_4 = m-1$, *or its* CSA,

(2) $x_1 = 0$, $x_2 = 1$, $x_3 = m-2$ *and* $x_4 = m-1$, *or its* CSA,

(3) $x_1 = 0$, $x_2 = 2$, $x_3 = 4$ *and* $x_4 = 6$, *where* $m = 6$,

(4) $x_1 = 0$, $x_2 = 1$, $x_3 = 2$ *and* $x_4 = m-2$, *or its* CSA,

(5) $x_1 = 0$, $x_2 = 2$, $x_3 = m-2$ *and* $x_4 = m-1$, *or its* CSA,

(6) $x_1 = 1$, $x_2 = 2$, $x_3 = m-2$ *and* $x_4 = m-1$, *where* $m \geq 7$,

(7) $x_1 = 0$, $x_2 = 1$, $x_3 = 4$ *and* $x_4 = 7$, *where* $m = 7$, *or its* CSA,

(8) $x_1 = 0$, $x_2 = 1$, $x_3 = 3$ *and* $x_4 = m-1$, *or its* CSA,

(9) $x_1 = 0$, $x_2 = 1$, $x_3 = m-3$ *and* $x_4 = m-1$, *where* $m \geq 7$, *or its* CSA, *or*

(10) $x_1 = 0$, $x_2 = 1$, $x_3 = 4$ *and* $x_4 = 7$, *where* $m = 8$, *or its* CSA,

(iii) *when* $\mathrm{NSV}_0 = 5$,

(1) $x_1 = 0$, $x_2 = 1$, $x_3 = 2$, $x_4 = m-1$ *and* $x_5 = m$, *or its* CSA,

(2) $x_1 = 0$, $x_2 = 1$, $x_3 = 2$, $x_4 = m-2$ *and* $x_5 = m$, *or its* CSA,

(3) $x_1 = 0$, $x_2 = 1$, $x_3 = 2$, $x_4 = m-2$ *and* $x_5 = m-1$, *where* $m \geq 7$, *or its* CSA,

(4) $x_1 = 0$, $x_2 = 1$, $x_3 = 3$, $x_4 = m-1$ *and* $x_5 = m$, *or its* CSA(*if* $m \geq 7$), *or*

(5) $x_1 = 0$, $x_2 = 1$, $x_3 = 4$, $x_4 = 7$ *and* $x_5 = 8$, *where* $m = 8$

and

(iv) *when* $\mathrm{NSV}_0 = 6$, $x_1 = 0$, $x_2 = 1$, $x_3 = 2$, $x_4 = m-2$, $x_5 = m-1$ *and* $x_6 = m$, *where* $m \geq 7$.

Proof. Proof is available in the Appendix.

It follows from Theorems 2.1 and 2.2, Lemmas 2.3, 2.4 and 4.1, and Remark 2.1 that we obtain the following:

Theorem 4.2. *If* T *is a* 2^m-BFF *design of resolution* $\mathrm{R}^*(\{1\}|\Omega_3)$ *derived from an* $\mathrm{SA}(m;\{\lambda_x\})$, *where* $m \geq 6$ *and* $N < \nu_3(m)$, *and furthermore*

(i) *when* $\mathrm{NSV}_0 = 3$, $A_0^{\#(0,0)}\boldsymbol{\theta}_0 + \left[\left\{1\Big/\sqrt{\binom{m}{3}}\right\}d_0\right]A_0^{\#(0,3)}\boldsymbol{\theta}_3$, $A_0^{\#(1,1)}\boldsymbol{\theta}_1$ *and* $A_0^{\#(2,2)}\boldsymbol{\theta}_2 + \left[\left\{\sqrt{\binom{m}{2}}\Big/\sqrt{\binom{m}{3}}\right\}d_2\right]A_0^{\#(2,3)}\boldsymbol{\theta}_3$ *are estimable, where* d_k $(k = 0, 2)$ *are given in* Lemma 4.1.(I)(ii). *In particular, if* $d_0 = 0$, *then* $A_0^{\#(0,0)}\boldsymbol{\theta}_0 = \boldsymbol{\theta}_0$ *is estimable and* $A_0^{\#(3,3)}\boldsymbol{\theta}_3$ *is not estimable, and also if* $d_2 = 0$, *then* $A_0^{\#(2,2)}\boldsymbol{\theta}_2$ *is estimable and* $A_0^{\#(3,3)}\boldsymbol{\theta}_3$ *is not estimable.*

(ii) *When* $\mathrm{NSV}_1 = 2$, $A_1^{\#(1,1)}\boldsymbol{\theta}_1$ *and* $A_1^{\#(2,2)}\boldsymbol{\theta}_2 + \left[\left\{\sqrt{\binom{m-2}{1}}\Big/\sqrt{\binom{m-2}{2}}\right\}d^*\right]A_1^{\#(2,3)}\boldsymbol{\theta}_3$ *are estimable, where* d^* *is given in* Lemma 4.1.(II)(ii). *Particularly if* $d^* = 0$, *then* $A_1^{\#(2,2)}\boldsymbol{\theta}_2$ *is estimable and* $A_1^{\#(3,3)}\boldsymbol{\theta}_3$ *is not estimable.*

(iii) *When* $\mathrm{NSV}_2 = 1$, $A_2^{\#(2,2)}\boldsymbol{\theta}_2 + \left[\left\{1\Big/\sqrt{\binom{m-4}{1}}\right\}d^{**}\right]A_2^{\#(2,3)}\boldsymbol{\theta}_3$ *is estimable, where* $d^{**} = 2x^{**} - m$ *for* $x^{**} \in SV_2$. *In particular, if* $d^{**} = 0$, *then* $A_2^{\#(2,2)}\boldsymbol{\theta}_2$ *is estimable and* $A_2^{\#(3,3)}\boldsymbol{\theta}_3$ *is not estimable.*

(iv) *When* $\mathrm{NSV}_3 = 0$, $A_3^{\#(3,3)}\boldsymbol{\theta}_3$ *is not estimable.*

(v) *When* $\mathrm{NSV}_\beta \geq 4-\beta$ $(0 \leq \beta \leq 3)$, $A_\beta^{\#(u,u)}\boldsymbol{\theta}_u$ $(\beta \leq u \leq 3)$ *are estimable.*

Analogously to Section 3, if $A_\beta^{\#(u,u)}\boldsymbol{\theta}_u$ are estimable for all β $(0 \leq \beta \leq u \leq 3)$, then $\boldsymbol{\theta}_u$ is estimable. In Table 4.1, the results of Theorem 4.1, and estimable parametric functions and the resolution $\mathrm{R}(\omega|\Omega_3)$ for each design are summarized.

5. Discussion

The class of BFF designs is a subset of FF designs. Thus there may exist a better FF design than a BFF design with respect to some criterion (e.g., Kuwada, 1982). However BFF designs possess the same advantage over unbalanced designs as a BIB design does over unbalanced or partially balanced designs. In this paper, we restrict our attention to the class of 2^m-BFF designs derived from SAs. Under these restrictions, we have given a necessary and sufficient condition for an SA to be a 2^m-BFF design of resolution $\mathrm{R}^*(\{1\}|\Omega_\ell)$ for $\ell = 2, 3$, where $N < \nu_\ell(m)$. As mentioned earlier, if T is an $\mathrm{SA}(m;\{\lambda_x\})$, then it is a $\mathrm{BA}(N, m, 2, t; \{\mu_i^{(t)}\})$ for any $t\,(1 \leq t \leq m)$, where the relation between $\mu_i^{(t)}$ and λ_x is given by (5). When $m = t+1$, if there exists a BA of strength t, then it is an SA. However when $m = t+2$, there exists a BA of strength t such that it is not always an SA (e.g., Kuriki and Yamamoto, 1984, and Shirakura, 1977). For example,

$$T' = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

is a $\mathrm{BA}(N = 7, m = 4, 2, t = 2; \{\mu_0^{(2)} = 1,\ \mu_1^{(2)} = \mu_2^{(2)} = 2\})$, but it is not an SA.

Let T be an $\mathrm{SA}(m;\{\lambda_1 = \lambda_{m-1} = 1,\ \lambda_x = 0\ (x \neq 1, m-1)\})$ with $N = 2m$, where $m \geq 6$. Then under the assumption that the three-factor and higher-order interactions are negligible, $A_0^{\#(0,0)}\boldsymbol{\theta}_0 + \left[(m-1)(m-4)\Big/\left\{2\sqrt{\binom{m}{2}}\right\}\right]A_0^{\#(0,2)}\boldsymbol{\theta}_2$, $A_0^{\#(1,1)}\boldsymbol{\theta}_1$ and $A_1^{\#(u,u)}\boldsymbol{\theta}_u$ $(u = 1, 2)$ are estimable, and $A_2^{\#(2,2)}\boldsymbol{\theta}_2$ is not estimable (see Table 3.1.(i)). Thus from (2), $A_0^{\#(1,1)}\boldsymbol{\theta}_1 + A_1^{\#(1,1)}\boldsymbol{\theta}_1 = \boldsymbol{\theta}_1$ is estimable, and the general mean is confounded with some of the two-factor interaction. On the other hand, under the assumption that the four-factor and higher-order interactions are negligible,

Table 4.1. 2^m-BFF designs of resolution $\mathrm{R}^*(\{1\}|\Omega_3)$ with $N < \nu_3(m)$ $(m \geq 6)$

Theorem 4.1		Existence conditions ($\lambda_{x_i} \neq 0\,(i=1,2,\cdots,\mathrm{NSV}_0)$)	Estimable parametric functions	Resolution $\mathrm{R}(\omega\|\Omega_3)$
(i)	(1)	$m=6$; $x_1=1,\ x_2=2,\ x_3=5$, or its CSA	$A_0^{\#(0,0)}\boldsymbol{\theta}_0+\left(\sqrt{5}/3\right)A_0^{\#(0,3)}\boldsymbol{\theta}_3$: estimable, $A_0^{\#(1,1)}\boldsymbol{\theta}_1$: estimable, $A_0^{\#(2,2)}\boldsymbol{\theta}_2-\left(1/\sqrt{3}\right)A_0^{\#(2,3)}\boldsymbol{\theta}_3$: estimable, $A_1^{\#(u,u)}\boldsymbol{\theta}_u$: estimable ($u = 1, 2, 3$), $A_2^{\#(2,2)}\boldsymbol{\theta}_2-\sqrt{2}A_2^{\#(2,3)}\boldsymbol{\theta}_3$: estimable, $A_3^{\#(3,3)}\boldsymbol{\theta}_3$: not estimable	$\omega=\{1\}$
	(2)	$m=9$; $x_1=1,\ x_2=2,\ x_3=7$, or its CSA	$A_0^{\#(0,0)}\boldsymbol{\theta}_0+\{28/(3\sqrt{21}\}A_0^{\#(0,3)}\boldsymbol{\theta}_3$: estimable, $A_0^{\#(1,1)}\boldsymbol{\theta}_1$: estimable, $A_0^{\#(2,2)}\boldsymbol{\theta}_2-(\sqrt{7/3})A_0^{\#(2,3)}\boldsymbol{\theta}_3$: estimable, $A_\gamma^{\#(u,u)}\boldsymbol{\theta}_u$: estimable ($\gamma\leq u\leq 3;\ \gamma=1,2$), $A_3^{\#(3,3)}\boldsymbol{\theta}_3$: not estimable	
	(3)	$m = 6$; $x_1=1,\ x_2=3,\ x_3=5$	$A_0^{\#(u,u)}\boldsymbol{\theta}_u$: estimable ($u=0,1,2$), $A_0^{\#(3,3)}\boldsymbol{\theta}_3$: not estimable, $A_\gamma^{\#(u,u)}\boldsymbol{\theta}_u$: estimable ($\gamma\leq u\leq 3;\ \gamma=1,3$), $A_2^{\#(2,2)}\boldsymbol{\theta}_2$: estimable, $A_2^{\#(3,3)}\boldsymbol{\theta}_3$: not estimable	$\omega=\{0,1,2\}$
(ii)	(1)	$x_1=0,\ x_2=1,\ x_3=2,\ x_4=m-1$, or its CSA	$A_\gamma^{\#(u,u)}\boldsymbol{\theta}_u$: estimable ($\gamma\leq u\leq 3;\ \gamma=0,1$), $A_2^{\#(2,2)}\boldsymbol{\theta}_2+\{(2x_3-m)/\sqrt{m-4}\}A_2^{\#(2,3)}\boldsymbol{\theta}_3$: estimable, $A_\gamma^{\#(3,3)}\boldsymbol{\theta}_3$: not estimable	$\omega=\{0,1\}$
	(2)	$x_1=0,\ x_2=1,\ x_3=m-2,\ x_4=m-1$, or its CSA		
	(3)	$m=6$; $x_1=0,\ x_2=2,\ x_3=4,\ x_4=6$	$A_\gamma^{\#(u,u)}\boldsymbol{\theta}_u$: estimable ($\gamma\leq u\leq 3;\ \gamma=0,2$), $A_1^{\#(u,u)}\boldsymbol{\theta}_u$: estimable ($u=1,2$), $A_\gamma^{\#(3,3)}\boldsymbol{\theta}_3$: not estimable ($\gamma = 1, 3$)	$\omega=\{0,1,2\}$

Table 4.1. (Continued)

Theorem 4.1		Existence conditions ($\lambda_{x_i} \neq 0\,(i=1,2,\cdots,\mathrm{NSV}_0)$)	Estimable parametric functions	Resolution $\mathrm{R}(\omega\vert\Omega_3)$
	(4)	$x_1=0,\ x_2=1,\ x_3=2,\ x_4=m-2$, or its CSA	$A_\gamma^{\#(u,u)}\boldsymbol{\theta}_u$: estimable ($\gamma \le u \le 3;\ \gamma=0,1,2$), $A_3^{\#(3,3)}\boldsymbol{\theta}_3$: not estimable	$\omega=\{0,1,2\}$
	(5)	$x_1=0,\ x_2=2,\ x_3=m-2,\ x_4=m-1$, or its CSA		
	(6)	$m\ge 7;\ \ x_1=1,\ x_2=2,\ x_3=m-2,\ x_4=m-1$		
	(7)	$m=7;\ \ x_1=0,\ x_2=1,\ x_3=4,\ x_4=7$, or its CSA	$A_\gamma^{\#(u,u)}\boldsymbol{\theta}_u$: estimable ($\gamma \le u \le 3;\ \gamma=0,3$), $A_1^{\#(1,1)}\boldsymbol{\theta}_1$: estimable, $A_1^{\#(2,2)}\boldsymbol{\theta}_2-\sqrt{2}A_1^{\#(2,3)}\boldsymbol{\theta}_3$: estimable, $A_2^{\#(2,2)}\boldsymbol{\theta}_2+\sqrt{1/3}A_2^{\#(2,3)}\boldsymbol{\theta}_3$: estimable	$\omega=\{0,1\}$
	(8)	$x_1=0,\ x_2=1,\ x_3=3,\ x_4=m-1$, or its CSA	$A_\gamma^{\#(u,u)}\boldsymbol{\theta}_u$: estimable ($\gamma \le u \le 3;\ \gamma=0,1,3$), $A_2^{\#(2,2)}\boldsymbol{\theta}_2+\left\{(2x_3-m)/\sqrt{m-4}\right\}A_2^{\#(2,3)}\boldsymbol{\theta}_3$: estimable, $A_2^{\#(3,3)}\boldsymbol{\theta}_3$: not estimable (if $2x_3-m=0$)	$\omega=\{0,1,2\}$ (if $m=6$) $\{0,1\}$ (if $m\ge 7$)
	(9)	$m\ge 7;\ \ x_1=0,\ x_2=1,\ x_3=m-3,\ x_4=m-1$, or its CSA		$\omega=\{0,1\}$
	(10)	$m=8;\ \ x_1=0,\ x_2=1,\ x_3=4,\ x_4=7$, or its CSA		$\omega=\{0,1,2\}$
(iii)	(1)	$x_1=0,\ x_2=1,\ x_3=2,\ x_4=m-1,\ x_5=m$, or its CSA	$A_\gamma^{\#(u,u)}\boldsymbol{\theta}_u$: estimable ($\gamma \le u \le 3;\ \gamma=0,1$), $A_2^{\#(2,2)}\boldsymbol{\theta}_2+\left\{(2x_3-m)/\sqrt{m-4}\right\}A_2^{\#(2,3)}\boldsymbol{\theta}_3$: estimable, $A_3^{\#(3,3)}\boldsymbol{\theta}_3$: not estimable	$\omega=\{0,1\}$
	(2)	$x_1=0,\ x_2=1,\ x_3=2,\ x_4=m-2,\ x_5=m$, or its CSA	$A_\gamma^{\#(u,u)}\boldsymbol{\theta}_u$: estimable ($\gamma \le u \le 3;\ \gamma=0,1,2$), $A_3^{\#(3,3)}\boldsymbol{\theta}_3$: not estimable	$\omega=\{0,1,2\}$
	(3)	$m\ge 7;\ \ x_1=0,\ x_2=1,\ x_3=2,\ x_4=m-2,\ x_5=m-1$, or its CSA		
	(4)	$x_1=0,\ x_2=1,\ x_3=3,\ x_4=m-1,\ x_5=m$, or its CSA (if $m\ge 7$)	$A_\gamma^{\#(u,u)}\boldsymbol{\theta}_u$: estimable ($\gamma \le u \le 3;\ \gamma=0,1,3$), $A_2^{\#(2,2)}\boldsymbol{\theta}_2+\left\{(2x_3-m)/\sqrt{m-4}\right\}A_2^{\#(2,3)}\boldsymbol{\theta}_3$: estimable, $A_2^{\#(3,3)}\boldsymbol{\theta}_3$: not estimable (if $2x_3-m=0$)	$\omega=\{0,1,2\}$ (if $m=6$) $\{0,1\}$ (if $m\ge 7$)
	(5)	$m=8;\ \ x_1=0,\ x_2=1,\ x_3=4,\ x_4=7,\ x_5=8$		$\omega=\{0,1,2\}$
(iv)		$m\ge 7;\ \ x_1=0,\ x_2=1,\ x_3=2,\ x_4=m-2,\ x_5=m-1$, $x_6=m$	$A_\gamma^{\#(u,u)}\boldsymbol{\theta}_u$: estimable ($\gamma \le u \le 3;\ \gamma=0,1,2$), $A_3^{\#(3,3)}\boldsymbol{\theta}_3$: not estimable	

$$\begin{aligned}
&\text{r-rank}\{F_0(1, m-1)\}\\
&= \text{r-rank}\left\{\begin{pmatrix} 1 & 1 \\ -(m-2) & m-2 \\ (m-1)(m-4)/2 & (m-1)(m-4)/2 \\ -(m-1)(m-2)(m-6)/6 & (m-1)(m-2)(m-6)/6 \end{pmatrix}\right\}\\
&= 2 < 4,\\
&\text{r-rank}\{F_1(1, m-1)\}\\
&= \text{r-rank}\left\{\begin{pmatrix} 1 & 1 \\ -(m-2) & m-2 \\ (m-2)(m-3)/2 & (m-2)(m-3)/2 \end{pmatrix}\right\}\\
&= 2 < 3
\end{aligned}$$

and

$$\text{r-rank}\{F_2\} = \text{r-rank}\{F_3\} = 0.$$

Thus the third row of $F_0(1, m-1)$ equals $(m-1)(m-4)/2$ times the first and its last row equals $(m-1)(m-6)/6$ times the second, and the last row of $F_1(1, m-1)$ equals $(m-2)(m-3)/2$ times the first. Hence from Lemma 2.4, $A_0^{\#(0,0)}\boldsymbol{\theta}_0 + \{(m-4)\sqrt{(m-1)/(2m)}\}A_0^{\#(0,2)}\boldsymbol{\theta}_2$, $A_0^{\#(1,1)}\boldsymbol{\theta}_1 + [(m-6)\sqrt{(m-1)/\{6(m-2)\}}]A_0^{\#(1,3)}\boldsymbol{\theta}_3$, $A_1^{\#(1,1)}\boldsymbol{\theta}_1 + \left\{\sqrt{\binom{m-2}{2}}\right\}A_1^{\#(1,3)}\boldsymbol{\theta}_3$ and $A_1^{\#(2,2)}\boldsymbol{\theta}_2$ are estimable, and $A_\gamma^{\#(u,u)}\boldsymbol{\theta}_u$ $(2 \le \gamma \le u \le 3)$ are not estimable. This implies that the main effect is confounded with some of the three-factor interaction. Therefore if the three-factor and higher-order interactions are negligible, then T is of resolution $\text{R}^*(\{1\}|\Omega_2)$, and hence it is also of resolution $\text{R}(\{1\}|\Omega_2)$. However if the three-factor interaction is not negligible, then the main effect is not estimable, and hence it is not of resolution $\text{R}^*(\{1\}|\Omega_3)$.

Appendix

In this Appendix, we provide the proofs of Theorems 3.1 and 4.1.

Proof of Theorem 3.1. We shall prove the claim by listing out all possible cases. In these cases, since $SV_0 \supset SV_1 \supset SV_2$, we have $\text{NSV}_0 \ge \text{NSV}_1 \ge \text{NSV}_2$, and $\text{NSV}_\gamma - \text{NSV}_{\gamma+1} \le 2$ for $\gamma = 0, 1$. Furthermore when $\text{NSV}_0 \ge 5$, we have $N \ge 1 + 1 + m + m + \binom{m}{2} > \nu_2(m)$ for $m \ge 4$, and hence $\text{NSV}_0 \le 4$. Then the proof starts with case $\text{NSV}_2 = 0$.

[A] When $\text{NSV}_2 = 0$, i.e., $\lambda_{x^{**}} = 0$ for any $x^{**} \in V_2$, from Theorem 2.2, it must be that $1 \le \text{NSV}_1 \le 2$ and $\text{NSV}_0 \ge 2$. In addition,

[a] when $\text{NSV}_1 = 1$, i.e., $x_1^* = 1$ or $m-1$, we have $\text{r-rank}\{F_1(x_1^*)\} = 1 < 2$. Thus from (8b), the last row of $F_1(x_1^*)$ must be 0, i.e., $2x_1^* - m = 0$, where $m = 2s \ge 4$. However $2x_1^* - m = -(m-2) < 0$ and $m - 2 > 0$ for $m \ge 4$ according as $x_1^* = 1$ and $m-1$, respectively. Therefore in this case, there does not exist a 2^m-BFF design of resolution $\text{R}^*(\{1\}|\Omega_2)$.

[b] When $\text{NSV}_1 = 2$, i.e., $x_1^* = 1$ and $x_2^* = m-1$, we have $\text{r-rank}\{F_1(x_1^*, x_2^*)\} = 2$.

[1] When NSV_0=2, i.e., $x_i = x_i^*$ $(i = 1, 2)$, we have $\text{r-rank}\{F_0(x_1, x_2)\} = 2 < 3$. Thus from (8a), the elements of the last row of $F_0(x_1, x_2)$ must be the same, i.e., $f_0(x_1; m) = f_0(x_2; m)$, where $f_0(x; m)$ is given by (9). Since $x_1 + x_2 - m = 0$ and $\nu_2(m) - N \le \nu_2(m) - 2m = (m-4)(m+1)/2 + 3 > 0$ for $m \ge 4$, it follows from Lemma 3.1 that case (i) is established.

[2] When $\text{NSV}_0 = 2 + p$ $(p = 1, 2)$, i.e., $x_1 = 0$ or m (if $p = 1$) ($x_1 = 0$ (if $p = 2$)) and $x_{i+1} = x_i^*$ $(i = 1, 2)$ (and $x_4 = m$ (if $p = 2$)), we have $\text{r-rank}\{F_0(x_1, x_2, \cdots, x_{2+p})\} = 3$. Then $\nu_2(m) - N \le (m-4)(m+1)/2 + (3-p) > 0$ for $m \ge 4$ and $p = 1, 2$. Thus cases (ii)(1) and (iii) are proved.

[B] When $\text{NSV}_2 = 1$, i.e., $2 \le x_1^{**} \le m-2$, we have $\text{r-rank}\{F_2(x_1^{**})\} = 1$. When $\text{NSV}_1 = 3$, we have $N \ge 2m + \binom{m}{2} > \nu_2(m)$ for $m \ge 4$, and hence $(\text{NSV}_2 = 1 \le)\text{NSV}_1 \le 2$. Furthermore from Theorem 2.2, it must be $\text{NSV}_0 \ge 2$.

[a] When $\text{NSV}_1 = 1$, i.e., $x_1^* = x_1^{**}$, we have $\text{r-rank}\{F_1(x_1^*)\} = 1 < 2$. Thus from (8b), it must be $2x_1^* - m = 0$, and hence $x_1^* = m/2$, where $m = 2s \ge 4$. Furthermore

[1] when $\text{NSV}_0 = 2$, i.e., $x_1 = 0$ or m and $x_2 = x_1^*$, we have $\text{r-rank}\{F_0(x_1, x_2)\} = 2 < 3$. Thus it must be $f_0(x_1; m) = f_0(x_2; m)$ for $m \ge 4$. However $x_1 + x_2 - m \ne 0$ for $m \ge 4$, and hence, from Lemma 3.1, there does not exist a 2^m-BFF design of resolution $\text{R}^*(\{1\}|\Omega_2)$.

[2] When $\mathrm{NSV}_0 = 3$, i.e., $x_1 = 0$, $x_2 = x_1^*$ and $x_3 = m$, we have r-rank$\{F_0(x_1, x_2, x_3)\} = 3$. When $m = 4$, we have $8 \leq N < \nu_2(4) = 11$, and when $m = 2s \geq 6$, $\nu_2(m) - N \leq \nu_2(m) - \{2 + \binom{m}{s}\} \leq \binom{m}{2} + m - \binom{m}{3} - 1 = -(m-6)\{(m-6)(m+6)+35\}/6 \leq 0$ for $m = 2s \geq 6$. Thus we get $m = 4$, and hence $x_1 = 0$, $x_2 = m/2 = 2$ and $x_3 = 4$, which is case (ii)(2).

[b] When $\mathrm{NSV}_1 = 2$, i.e., $x_1^* = 1$ or $m-1$ and $x_2^* = x_1^{**}$, we have r-rank$\{F_1(x_1^*, x_2^*)\} = 2$. In this case, $N \geq m + \binom{m}{2} = \nu_2(m) - 1$, and hence it must be $\mathrm{NSV}_0 = 2$, i.e., $x_i = x_i^*$ $(i = 1, 2)$. Then we have r-rank$\{F_0(x_1, x_2)\} = 2 < 3$. Thus from (8a), it must be $f_0(x_1; m) = f_0(x_2; m)$ for $m \geq 4$. However $x_1 + x_2 - m \neq 0$ for $m \geq 4$, and hence, from Lemma 3.1, a 2^m-BFF design of resolution $\mathrm{R}^*(\{1\}|\Omega_2)$ does not exist.

[C] When $\mathrm{NSV}_2 = q \geq 2$, where $(m-2) - 1 \geq q$, we have $N \geq q\binom{m}{2} \geq 2\binom{m}{2} > \nu_2(m)$ for $m \geq 3 + q$. Thus in this case, there does not exist a 2^m-BFF design of resolution $\mathrm{R}^*(\{1\}|\Omega_2)$ with $N < \nu_2(m)$ for $m \geq 3 + q$.

Proof of Theorem 4.1. Similarly to the proof of Theorem 3.1, it will be done by listing out all possible cases. In these cases, we have $SV_0 \supset SV_1 \supset SV_2 \supset SV_3$, and hence $\mathrm{NSV}_0 \geq \mathrm{NSV}_1 \geq \mathrm{NSV}_2 \geq \mathrm{NSV}_3$, and $\mathrm{NSV}_\gamma - \mathrm{NSV}_{\gamma+1} \leq 2$ for $\gamma = 0, 1, 2$. Moreover when $\mathrm{NSV}_0 \geq 7$, $N \geq 1 + 1 + m + m + \binom{m}{2} + \binom{m}{2} + \binom{m}{3} > \nu_3(m)$ for $m \geq 6$, and hence $\mathrm{NSV}_0 \leq 6$. We also begin the proof with $\mathrm{NSV}_3 = 0$.

[A] When $\mathrm{NSV}_3 = 0$ (and hence $\mathrm{NSV}_2 \leq 2$), i.e., $\lambda_{x^{***}} = 0$ for any $x^{***} \in V_3$, it follows from Theorem 2.2 that $\mathrm{NSV}_1 \geq 1$ and $\mathrm{NSV}_0 \geq 2$. Moreover

[a] when $\mathrm{NSV}_2 = 0$ (and hence $\mathrm{NSV}_1 \leq 2$), and furthermore

[1] when $\mathrm{NSV}_1 = 1$, i.e., $x_1^* = 1$ or $m-1$, we have r-rank$\{F_1(x_1^*)\} = 1 < 3$. Thus from Lemma 4.1.(II)(i), there does not exist a 2^m-BFF design of resolution $\mathrm{R}^*(\{1\}|\Omega_3)$.

[2] When $\mathrm{NSV}_1 = 2$, i.e., $x_1^* = 1$ and $x_2^* = m-1$, r-rank$\{F_1(x_1^*, x_2^*)\} = 2 < 3$. Then we have $(2x_1^* - m)(2x_2^* - m) + (m-2) = -(m-2)(m-3) < 0$ for $m \geq 6$. Therefore from Lemma 4.1.(II)(ii), a 2^m-BFF design of resolution $\mathrm{R}^*(\{1\}|\Omega_3)$ does not exist.

[b] When $\mathrm{NSV}_2 = 1$ (and hence $\mathrm{NSV}_1 \leq 3$), i.e., $x_1^{**} = 2$ or $m-2$, we have r-rank$\{F_2(x_1^{**})\} = 1 < 2$.

[1] When $\mathrm{NSV}_1 = 1$, i.e., $x_1^* = x_1^{**}$, we have r-rank$\{F_1(x_1^*)\} = 1 < 3$. Then from Lemma 4.1.(II)(i), there does not exist a 2^m-BFF design of resolution $\mathrm{R}^*(\{1\}|\Omega_3)$.

[2] When $\mathrm{NSV}_1 = 2$, i.e., $x_1^* = 1$ or $m-1$ and $x_2^* = x_1^{**}$, we have r-rank$\{F_1(x_1^*, x_2^*)\} = 2 < 3$. When $x_1^* = 1$, $(2x_1^* - m)(2x_2^* - m) + (m-2) = (m-2)(m-3) > 0$ and $-(m-2)(m-5) < 0$ for $m \geq 6$ according as $x_2^* = 2$ and $m-2$, respectively. Thus, it follows from Lemma 4.1.(II)(ii) and the relation of the CSA that there does not exist a 2^m-BFF design of resolution $\mathrm{R}^*(\{1\}|\Omega_3)$.

[3] When $\mathrm{NSV}_1 = 3$ (and hence $\mathrm{NSV}_0 \leq 5$), i.e., $x_1^* = 1$, $x_2^* = x_1^{**}$ and $x_3^* = m-1$, r-rank$\{F_1(x_1^*, x_2^*, x_3^*)\} = 3$. In addition,

[3.1] when $\mathrm{NSV}_0 = 3$, i.e., $x_i = x_i^*$ $(i = 1, 2, 3)$, r-rank$\{F_0(x_1, x_2, x_3)\} = 3 < 4$. Then $x_1 + x_2 - m \neq 0$ for $x_2 = 2$ or $m-2$, and $m \geq 6$. Furthermore from Lemma 4.1.(I)(ii), we have $(2x_1 - m)(2x_2 - m) + (2x_2 - m)(2x_3 - m) + (2x_3 - m)(2x_1 - m) + (3m-2) = -(m-1)(m-6) = 0$ for $x_2 = 2$ or $m-2$. Thus we get $m = 6$, and hence $x_3 = m - 1 = 5$. In this case, $27 \leq N < \nu_3(6) = 42$. Therefore case (i)(1) is established. Here if $x_2 = 2$, then $d_0 = 10/3$ and $d_2 = -2/3$, and if $x_2 = m-2 = 4$, then $d_0 = -10/3$ and $d_2 = 2/3$, where d_k $(k = 0, 2)$ are constants given in Lemma 4.1.(I)(ii).

[3.2] When $\mathrm{NSV}_0 = 3 + p$ $(p = 1, 2)$, i.e., $x_1 = 0$ or m (if $p = 1$) ($x_1 = 0$ (if $p = 2$)), $x_2 = 1$, $x_3 = 2$ or $m-2$, and $x_4 = m-1$ (and $x_5 = m$ (if $p = 2$)), we have r-rank$\{F_0(x_1, x_2, \cdots, x_{3+p})\} = 4$. Then $\binom{m}{2} + 2m + p \leq N < \nu_3(m)$ for $m \geq 6$ and $p = 1, 2$. Thus we obtain cases (ii)(1) and (2), and (iii)(1).

[c] When $\mathrm{NSV}_2 = 2$ (and hence $\mathrm{NSV}_1 \leq 4$), i.e., $x_1^{**} = 2$ and $x_2^{**} = m-2$, r-rank$\{F_2(x_1^{**}, x_2^{**})\} = 2$.

[1] When $\mathrm{NSV}_1 = 2$ (and hence $\mathrm{NSV}_0 \leq 4$), i.e., $x_j^* = x_j^{**}$ $(j = 1, 2)$, r-rank$\{F_1(x_1^*, x_2^*)\} = 2 < 3$. Thus from Lemma 4.1.(II)(ii), $(2x_1^* - m)(2x_2^* - m) + (m-2) = -(m-3)(m-6) = 0$ for $m \geq 6$, and hence we get $m = 6$ and $x_2^* = m - 2 = 4$. In this case, $d^* = 0$, where d^* is a constant given in Lemma 4.1.(II)(ii). Furthermore

[1.1] when $\mathrm{NSV}_0 = 2$, i.e., $x_i = x_i^*$ $(i = 1, 2)$, r-rank$\{F_0(x_1, x_2)\} = 2 < 4$. Since $m = 6$, from Lemma 4.1.(I)(i)(1), we get $t = 1$, and hence $x_1 = 1$ and $x_2 = 5$, which contradict $x_1 = 2$ and $x_2 = 4$. Thus in this case, there does not exist a 2^m-BFF design of resolution $\mathrm{R}^*(\{1\}|\Omega_3)$.

[1.2] When $\text{NSV}_0 = 3$, i.e., $x_1 = 0$ or $m(= 6)$ and $x_{i+1} = x_i^*$ $(i = 1, 2)$, r-rank$\{F_0(x_1, x_2, x_3)\} = 3 < 4$. Since $m = 6$, we have $x_1 + x_2 - m = -4 \neq 0$ and $(2x_1 - m)(2x_2 - m) + (2x_2 - m)(2x_3 - m) + (2x_3 - m)(2x_1 - m) + (3m - 2) = 12 \neq 0$ for $x_1 = 0$. Therefore from Lemma 4.1.(I)(ii) and the relation of the CSA, a 2^m-BFF design of resolution $\text{R}^*(\{1\}|\Omega_3)$ does not exist.

[1.3] When $\text{NSV}_0 = 4$, i.e., $x_1 = 0$, $x_{i+1} = x_i^*$ $(i = 1, 2)$ and $x_4 = m(= 6)$, r-rank$\{F_0(x_1, x_2, \cdots, x_4)\} = 4$. In this case, $32 \leq N < \nu_3(6) = 42$, and hence case (ii)(3) is proved.

[2] When $\text{NSV}_1 = 3$ (and hence $\text{NSV}_0 \leq 5$), i.e., $x_1^* = 1$ or $m - 1$ and $x_{j+1}^* = x_j^{**}$ $(j = 1, 2)$, we have r-rank$\{F_1(x_1^*, x_2^*, x_3^*)\}$ =3.

[2.1] When $\text{NSV}_0 = 3$, i.e., $x_i = x_i^*$ $(i = 1, 2, 3)$, r-rank$\{F_0(x_1, x_2, x_3)\} = 3 < 4$. Then $x_1+x_2-m = -(m-3)(\neq 0)$ and $1(\neq 0)$ for $m \geq 6$ according as $x_1 = 1$ and $m - 1$, respectively. Furthermore $(2x_1 - m)(2x_2 - m) + (2x_2 - m)(2x_3 - m) + (2x_3 - m)(2x_1 - m) + (3m - 2) = -(m - 2)(m - 9)$ for $x_1 = 1$ or $m - 1$. Thus from Lemma 4.1.(I)(ii), we get $m = 9$, and hence $x_1 = 1$ or $m - 1 = 8$ and $x_3 = m - 2 = 7$. In this case, $81 \leq N < \nu_3(9) = 130$. Thus case (i)(2) is established. If $x_1 = 1$, $x_2 = 2$ and $x_3 = 7$, then we get $d_0 = 56/3$ and $d_2 = -7/3$, and if $x_1 = 8$, $x_2 = 2$ and $x_3 = 7$, then $d_0 = -56/3$ and $d_2 = 7/3$.

[2.2] When $\text{NSV}_0 = 3 + p$ $(p = 1, 2)$, i.e., $x_1 = 0$ or m (if $p = 1$) ($x_1 = 0$ (if $p = 2$)) and $x_{i+1} = x_i^*$ $(i = 1, 2, 3)$(and $x_5 = m$ (if $p = 2$)), we have r-rank$\{F_0(x_1, x_2, \cdots, x_{3+p})\} = 4$. Furthermore $\nu_3(m) - N \leq (m - 6)(m^2 + 5)/6 + (6 - p) > 0$ for $m \geq 6$ and $p = 1, 2$. Thus cases (ii)(4) and (5), and (iii)(2) are proved.

[3] When $\text{NSV}_1 = 4$, i.e., $x_1^* = 1, x_2^* = 2, x_3^* = m - 2$ and $x_4^* = m - 1$, r-rank$\{F_1(x_1^*, x_2^*, \cdots, x_4^*)\} = 3$. Furthermore when $\text{NSV}_0 = 4 + p$ $(p = 0, 1, 2)$, i.e., when $p = 0$, we have $x_i = x_i^*$ $(i = 1, 2, \cdots, 4)$, when $p = 1$, $x_1 = 0$ or m and $x_{i+1} = x_i^*$ $(i = 1, 2, \cdots, 4)$, and when $p = 2$, $x_1 = 0$, $x_{i+1} = x_i^*$ $(i = 1, 2, \cdots, 4)$ and $x_6 = m$, we have r-rank$\{F_0(x_1, x_2, \cdots, x_{4+p})\} = 4$. Then $\nu_3(m) - N \leq (m - 6)(m^2 - 1)/6 - p = (m - 7)(m^2 + m + 6)/6 + (8 - p)$ for $m \geq 6$ and $p = 0, 1, 2$. Thus when $m = 6$, we have $\nu_3(6) - N \leq 0$ for $p = 0, 1, 2$, and when $m \geq 7$, $\nu_3(m) - N > 0$ for $p = 0, 1, 2$. Therefore we establish cases (ii)(6), (iii)(3) and (iv).

[B] When $\text{NSV}_3 = 1$ (and hence $\text{NSV}_2 \leq 3$), i.e., $3 \leq x_1^{***} \leq m - 3$, it follows from Theorem 2.2 that r-rank$\{F_3(x_1^{***})\} = 1$, and it must be $\text{NSV}_0 \geq 2$. In addition,

[a] when $\text{NSV}_2 = 1$ (and hence $\text{NSV}_1 \leq 3$), i.e., $x_1^{**} = x_1^{***}$, we have r-rank$\{F_2(x_1^{**})\} = 1 < 2$.

[1] When NSV_1= 1, i.e., $x_1^* = x_1^{**}$, r-rank$\{F_1(x_1^*)\} = 1 < 3$. Thus from Lemma 4.1.(II)(i), there does not exist a 2^m-BFF design of resolution $\text{R}^*(\{1\}|\Omega_3)$.

[2] When $\text{NSV}_1 = 2$ (and hence $\text{NSV}_0 \leq 4$), i.e., $x_1^* = 1$ or $m - 1$ and $x_2^* = x_1^{**}$, r-rank$\{F_1(x_1^*, x_2^*)\} = 2 < 3$. Then from Lemma 4.1.(II)(ii), it must be $(2x_1^* - m)(2x_2^* - m) + (m - 2) = 0$ for $m \geq 6$. We consider case $x_1^* = 1$. Then we have $(2x_1^* - m)(2x_2^* - m) + (m - 2) = -(m - 2)\{2x_2^* - (m + 1)\} = 0$ for $m \geq 6$, and hence $x_2^* = (m + 1)/2$, where $m = 2s + 1 \geq 7$. Moreover

[2.1] when $\text{NSV}_0 = 2$, i.e., $x_i = x_i^*$ $(i = 1, 2)$, we have r-rank$\{F_0(x_1, x_2)\} = 2 < 4$. Since $x_1 = 1$, from Lemma 4.1.(I)(i)(1), we get $t = 1$, and hence $m = 6$ and $x_2 = 5$. However $m = 6 < 7$ and $5 \notin SV_3$ for $m = 6$. Thus from Lemma 4.1.(I)(i)(1) and the relation of the CSA, a 2^m-BFF design of resolution $\text{R}^*(\{1\}|\Omega_3)$ does not exist.

[2.2] When $\text{NSV}_0 = 3$, i.e., $x_1 = 0$ or m and $x_{i+1} = x_i^*$ $(i = 1, 2)$, we have r-rank$\{F_0(x_1, x_2, x_3)\} = 3 < 4$. Then it holds $x_p + x_q - m \neq 0$ for any $\{p, q\} \subset \{1, 2, 3\}$ and $m \geq 7$. However when $x_2 = 1$, $(2x_1 - m)(2x_2 - m) + (2x_2 - m)(2x_3 - m) + (2x_3 - m)(2x_1 - m) + (3m - 2) = m(m - 1) > 0$ and $-m(m - 5) < 0$ for $m \geq 7$ according as $x_1 = 0$ and m, respectively. Therefore it follows from Lemma 4.1.(I)(ii) and the relation of the CSA that there does not exist a 2^m-BFF design of resolution $\text{R}^*(\{1\}|\Omega_3)$.

[2.3] When $\text{NSV}_0 = 4$, i.e., $x_1 = 0$ and $x_{i+1} = x_i^*$ $(i = 1, 2)$ and $x_4 = m$, r-rank$\{F_0(x_1, x_2, \cdots, x_4)\} = 4$. In this case, when $m = 7$, we have $44 \leq N < \nu_3(7) = 64$, and when $m = 2s + 1 \geq 9$, $\nu_3(m) - N \leq \binom{m}{2} + \binom{m}{3} - \binom{m}{4} - 1 < \binom{m}{2} + \binom{m}{3} - \binom{m}{4} = -[(m - 9)\{(m - 9)(m^2 + 8m + 74) + 682\}/24 + 6] < 0$ for $m \geq 9$. Thus we get $m = 7$, and hence $x_1 = 0$, $x_2 = 1$ or $m - 1 = 6$, $x_3 = (m + 1)/2 = 4$ and $x_4 = m = 7$. Therefore case (ii)(7) is proved.

[3] When $\text{NSV}_1 = 3$ (and hence $\text{NSV}_0 \leq 5$), i.e., $x_1^* = 1$, $x_2^* = x_1^{**}$ and $x_3^* = m - 1$, we have r-rank$\{F_1(x_1^*, x_2^*, x_3^*)\}$ = 3.

[3.1] When NSV_0=3, i.e., $x_i = x_i^*$ $(i = 1, 2, 3)$, r-rank$\{F_0(x_1, x_2, x_3)\} = 3 < 4$. Then $x_1 + x_2 - m < 0$ for $m \geq 6$. Furthermore from Lemma 4.1.(I)(ii), it must be $(2x_1 - m)(2x_2 - m)$+$(2x_2 - m)(2x_3 - m)$+$(2x_3 - m)(2x_1 - m)$+$(3m - 2) = -(m - 1)(m - 6) = 0$ for $m \geq 6$. Thus we get $m = 6$, and hence, $x_1 = 1$, $x_2 = 3$ and $x_3 = m - 1 = 5$. In this case, $32 \leq N < \nu_3(6) = 42$, and hence we have case (i)(3), and $d_0 = d_2 = 0$.

[3.2] When $\mathrm{NSV}_0 = 3 + p$ ($p = 1, 2$), i.e., $x_1 = 0$ or m (if $p = 1$) ($x_1 = 0$ (if $p = 2$)), $x_{i+1} = x_i^*$ ($i = 1, 2$) and $x_4 = m - 1$ (and $x_5 = m$ (if $p = 2$)), r-rank$\{F_0(x_1, x_2, \cdots, x_{3+p})\} = 4$. When $x_3 = 3$ or $m - 3$ (if $m \geq 7$), $\nu_3(m) - N \leq (m - 6)(m + 3)/2 + (10 - p) > 0$ for $m \geq 6$ and $p = 1, 2$, and hence we obtain cases (ii)(8) and (9), and (iii)(4). When $x_3 = 4$ and $m = 8$, $\nu_3(8) - N \leq 7 - p > 0$ for $p = 1, 2$. Thus cases (ii)(10) and (iii)(5) are obtained. Furthermore when $4 \leq x_3 \leq m-4$ and $m \geq 9$, then $N \geq p + 2m + \binom{m}{x_3} \geq p + 2m + \binom{m}{4}$, and hence $\nu_3(m) - N \leq \binom{m}{2} + \binom{m}{3} - m - \binom{m}{4} - (p - 1) < \binom{m}{2} + \binom{m}{3} - \binom{m}{4} < 0$ for $m \geq 9$ and $p = 1, 2$ (see [B][a][2][2.3] above). Thus there does not exist a 2^m-BFF design of resolution $\mathrm{R}^*(\{1\}|\Omega_3)$ with $N < \nu_3(m)$ for $m \geq 9$.

[b] When NSV_2=2 (and hence $\mathrm{NSV}_1 \leq 4$), i.e., $x_1^{**} = 2$ or $m - 2$ and $x_2^{**} = x_1^{***}$, we have r-rank$\{F_2(x_1^{**}, x_2^{**})\} = 2$.

[1] When NSV_1=2, i.e., $x_j^* = x_j^{**}$ ($j = 1, 2$), we have r-rank$\{F_1(x_1^*, x_2^*)\} = 2 < 3$. Thus from Lemma 4.1.(II)(ii), it must be $(2x_1^* - m)(2x_2^* - m) + (m - 2) = 0$ for $m \geq 6$. When $x_1^* = 2$, we have $x_2^* = (m + 1)/2 + 1/(m - 4)$, which is an integer for $m = 6$ only, and hence $x_2^* = 4$. However 4 does not belong to SV_3 for $m = 6$. Therefore from Lemma 4.1.(II)(ii) and the relation of the CSA, there does not exist a 2^m-BFF design of resolution $\mathrm{R}^*(\{1\}|\Omega_3)$.

[2] When $\mathrm{NSV}_1 = 3$ (and hence $\mathrm{NSV}_0 \leq 5$), i.e., $x_1^* = 1$ or $m - 1$ and $x_{j+1}^* = x_j^{**}(j = 1, 2)$, we have r-rank$\{F_1(x_1^*, x_2^*, x_3^*)\} = 3$. In this case, $N \geq m + \binom{m}{2} + \binom{m}{x_3^*}$ for $m \geq 6$. Thus when $4 \leq x_3^* \leq m - 4$ and $m \geq 8$, $N \geq m + \binom{m}{2} + \binom{m}{4} > \nu_3(m)$, and hence we only consider case $x_3^* = 3$ or $m - 3$ (if $m \geq 7$) for $m \geq 6$.

[2.1] When $\mathrm{NSV}_0 = 3$, i.e., $x_i = x_i^*$ ($i = 1, 2, 3$), we have r-rank$\{F_0(x_1, x_2, x_3)\} = 3 < 4$. Then $x_s + x_t - m \neq 0$ for any $\{s, t\} \subset \{1, 2, 3\}$. Moreover when $x_1 = 1$, $(2x_1 - m)(2x_2 - m) + (2x_2 - m)(2x_3 - m) + (2x_3 - m)(2x_1 - m) + (3m - 2) = 3(m^2 - 7m + 14)$, $-(m^2 - 15m + 30)$, $-(m^2 - 11m + 22)$ and $-(m^2 - 7m - 2)$ according as $x_2 = 2$ and $x_3 = 3$, $x_2 = 2$ and $x_3 = m - 3$, $x_2 = m - 2$ and $x_3 = 3$, and $x_2 = m - 2$ and $x_3 = m - 3$, respectively. However $m^2 - 7m + 14 = (m - 6)(m - 1) + 8 > 0$ for $m \geq 6$, and furthermore three quadratic equations $m^2 - 15m + 30 = 0$, $m^2 - 11m + 22 = 0$ and $m^2 - 7m - 2 = 0$ do not have an integer solution for $m \geq 6$. Therefore from Lemma 4.1.(I)(ii) and the relation of the CSA, there does not exist a 2^m-BFF design of resolution $\mathrm{R}^*(\{1\}|\Omega_3)$.

[2.2] When $\mathrm{NSV}_0 = 3 + p$ ($p = 1, 2$), we have $N \geq p + m + \binom{m}{2} + \binom{m}{3} \geq \nu_3(m)$ for $m \geq 6$ and $p = 1, 2$. Thus there does not exist a 2^m-BFF design of resolution $\mathrm{R}^*(\{1\}|\Omega_3)$ with $N < \nu_3(m)$.

[3] When $\mathrm{NSV}_1 = 4$, we have $N \geq 2m + \binom{m}{2} + \binom{m}{3} > \nu_3(m)$ for $m \geq 6$. Hence there does not exist a 2^m-BFF design of resolution $\mathrm{R}^*(\{1\}|\Omega_3)$ with $N < \nu_3(m)$.

[c] When $\mathrm{NSV}_2 = 3$, we have $N \geq 2\binom{m}{2} + \binom{m}{3} > \nu_3(m)$, and hence a 2^m-BFF design of resolution $\mathrm{R}^*(\{1\}|\Omega_3)$ with $N < \nu_3(m)$ does not exist.

[C] When $\mathrm{NSV}_3 = q \geq 2$, where $(m - 3) - 2 \geq q$, it holds that $\binom{m}{3} > 1 + m + \binom{m}{2}$ for $m \geq 5 + q \geq 7$. Thus we have $N \geq q\binom{m}{3} \geq 2\binom{m}{3} > \nu_3(m)$. Therefore there does not exist a 2^m-BFF design of resolution $\mathrm{R}^*(\{1\}|\Omega_3)$ with $N < \nu_3(m)$ for $m \geq 5 + q$.

Therefore the proof is complete.

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