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Estimation for Wrapped Zero Inflated Poisson and Wrapped Poisson Distributions

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Abstract

There has been a growing interest in discrete circular models such as wrapped zero inflated Poisson and wrapped Poisson distributions and the trigonometric moments (see Brobbey et al., 2016 and Girija et al., 2014). Also, characteristic functions of stable processes have been used to study the estimation of the model parameters using estimating function approach (see Thavaneswaran et al., 2013). One difficulty in estimating the circular mean and the resultant mean length parameter of wrapped Poisson (WP) or wrapped zero inflated Poisson (WZIP) is that neither the likelihood of WP/WZIP random variable nor the score function is available in closed form, which leads one to use either trigonometric method of moment estimation (TMME) or an estimating function approach. In this paper, we study the estimation of WZIP distribution and WP distribution using estimating functions and obtain the closed form expression of the information matrix. We also derive the asymptotic distribution of the tangent of the mean direction for both the WZIP and WP distributions.

Keywords: Circular distribution, zero-inflated Poisson, characteristic function, trigonometric moments, skewness, kurtosis, wrapped stable distributions.

1. Introduction

Directional statistics is an emerging area of statistics and is being used as a tool for practitioners in many scientific fields such as astronomy, biology, earth science, meteorology, medicine and physics. Directional data is a set of observations measured on directions. The sample space may be a circle, a sphere or an hypersphere. The directions may be in two or three dimensions. More specifically, the directions may be regarded as points on the circumference of a circle (in two dimensions) or on the surface of a sphere (in three dimensions). For further details, see Jammalamadaka and SenGupta (2001).

Various circular models have been studied by many researchers. Mardia and Jupp (2000) published a comprehensive text on directional statistics. For discrete circular models, Girija et al. (2014) derived characteristics functions of the wrapped Poisson distribution. Brobbey et al. (2016) introduced a new discrete circular distribution, the wrapped zero-inflated Poisson (WZIP) distribution and derived its population characteristics. In this paper, we study the estimation of the parameters of the WZIP and WP distributions using estimating functions, and also derive asymptotic distribution of the tangent of the mean direction. We first briefly discuss the wrapped discrete distribution and WZIP distribution as given in Brobbey et al. (2016).

1.1 Wrapped Discrete Distribution

If X is a linear random variable on the real line with density $f(x)$, the corresponding wrapped random variable Θ is defined by $\Theta = x \pmod{2\pi}$. The operation corresponds to wrapping the real line around the unit circle accumulating probability over all the overlapping points $x = \Theta, \Theta \pm 2\pi, \Theta \pm 3\pi, \cdots$. In particular, if X has a distribution concentrated on the points $x = \frac{k}{2\pi m}$, $k = 0, \pm 1, \pm 2, \cdots$ and m is an integer, we have a wrapped discrete circular random variable Θ, such that the probability mass function of θ is

$$p_w(\Theta = \frac{2\pi r}{m}) = \sum_{k=-\infty}^{\infty} p(r + km), \qquad r = 0, 1, 2, \cdots, m-1.$$

The probability mass function satisfies the following properties

1. $p_w(\Theta = \frac{2\pi r}{m}) \geq 0$

2. $\sum\limits_{r=0}^{m-1} p_w(\Theta = \frac{2\pi r}{m}) = 1$

3. The random variable Θ has the same distribution as $(\Theta + 2\pi k)$. That is, for any integer k, $p_w(\theta) = p_w(\theta + 2\pi k)$.

1.2 Wrapped Zero-inflated Poisson Distribution

Data with excess zeros are often observed in applied science and public health studies. Zero-inflated Poisson (ZIP) regression model is a common statistical tool for analyzing such data. The *zero-inflated Poisson* distribution is given by

$$p(x) = \begin{cases} w + (1-w)e^{-\lambda} & x = 0 \\ (1-w)\frac{e^{-\lambda}\lambda^x}{x!} & x = 1, \cdots, \infty. \end{cases}$$

As we mentioned earlier, Brobbey et al. (2016) introduced the wrapped zero-inflated Poisson distribution and derived its population characteristics. The results are summarized in the following Theorem 1 and Corollary 1.

Theorem 1. *(a) The pth trigonometric moments of the wrapped zero-inflated Poisson distribution are given by*

$$\alpha_p = w + (1-w)e^{-\lambda(1-\cos\frac{2\pi p}{m})}\cos\left(\lambda\sin\frac{2\pi p}{m}\right)$$

$$\beta_p = (1-w)e^{-\lambda(1-\cos\frac{2\pi p}{m})}\sin\left(\lambda\sin\frac{2\pi p}{m}\right).$$

(b) The pth circular mean is

$$\mu_p = \tan^{-1}\left[\frac{(1-w)e^{-\lambda(1-\cos\frac{2\pi p}{m})}\sin\left(\lambda\sin\frac{2\pi p}{m}\right)}{w + (1-w)e^{-\lambda(1-\cos\frac{2\pi p}{m})}\cos\left(\lambda\sin\frac{2\pi p}{m}\right)}\right].$$

so that the mean of the wrapped zero inflated Poisson is given by μ_1.

(c) The pth circular mean resultant length is

$$\rho_p = \sqrt{w^2 + (1-w)^2 e^{-2\lambda(1-\cos\frac{2\pi p}{m})} + 2w(1-w)e^{-\lambda(1-\cos\frac{2\pi p}{m})}\cos\left(\lambda\sin\frac{2\pi p}{m}\right)}.$$

and the mean resultant length, ρ, variance, V_0, and the standard deviation, σ_0, of the wrapped zero-inflated Poisson are given by

$$\begin{aligned} \rho &= \sqrt{a_1^2 + b_1^2 + 2a_1 b_1 \cos\left(\lambda\sin\frac{2\pi}{m}\right)} \\ V_0 &= 1 - \rho \\ \sigma_0 &= \sqrt{\log\left(\frac{1}{a_1^2 + b_1^2 + 2a_1 b_1\cos\left(\lambda\sin\frac{2\pi}{m}\right)}\right)}. \end{aligned}$$

where $a_1 = w$ and $b_1 = (1-w)e^{-\lambda(1-\cos\frac{2\pi}{m})}$.

Corollary 1. *The pth central trigonometric moments are*

$$\bar{\alpha_p} = w\cos p\mu + (1-w)e^{-\lambda(1-\cos\frac{2\pi p}{m})}\cos\left(\lambda\sin\frac{2\pi p}{m} - p\mu\right)$$

$$\bar{\beta_p} = w\sin p\mu + (1-w)e^{-\lambda(1-\cos\frac{2\pi p}{m})}\sin\left(\lambda\sin\frac{2\pi p}{m} - p\mu\right).$$

The circular skewness, s, and circular kurtosis, k, of the wrapped zero-inflated Poisson distribution are given by

$$
\begin{aligned}
s &= \frac{\left[w\sin 2\mu + (1-w)e^{-\lambda(1-\cos\frac{4\pi}{m})}\sin\left(\lambda\sin\frac{4\pi}{m} - 2\mu\right)\right]}{\left[1-\sqrt{a_1^2+b_1^2+2a_1b_1\cos\left(\lambda\sin\frac{2\pi}{m}\right)}\right]^{3/2}} \\
k &= \frac{\left[w\cos 2\mu + (1-w)e^{-\lambda(1-\cos\frac{4\pi}{m})}\cos\left(\lambda\sin\frac{4\pi}{m} - 2\mu\right) - \left(a_1^2+b_1^2+2a_1b_1\cos\left(\lambda\sin\frac{2\pi}{m}\right)\right)^2\right]}{\left[1-\sqrt{a_1^2+b_1^2+2a_1b_1\cos\left(\lambda\sin\frac{2\pi}{m}\right)}\right]^{2}}.
\end{aligned}
$$

2. Estimating Function based on Characteristic Function

Small and McLeish (1994) were the first to use estimating functions based on the characteristic function to study inference for the stable distributions based on independent observations. For models with heavy tailed distributions, Thavaneswaran and Heyde (1999) discussed the superiority of the LAD estimating function over the least squares estimating function. For models with stable errors, estimating functions based on sines and cosines are natural, since closed form expressions are available for $E\cos(uy_t)$ and $E\sin(uy_t)$ in the characteristic function $E[\exp(iuy_t)] = E[\cos(uy_t) + i\sin(uy_t)]$. Suppose first that $y_1, \cdots, y_n$ is a sequence of independent, identically distributed (i.i.d.) random variables following a symmetric stable distribution with real-valued location parameter θ, positive scale parameter c, and stability index α with $0 < \alpha \leq 2$. In Thavaneswaran et al. (2013), it is assumed that α is known. In practice, we estimate α by the quantile estimation method proposed by McCulloch (1986). The characteristic function of y_t takes the form $E[\exp(iuy_t)] = \exp(iu\theta - |cu|^\alpha)$, for $u > 0$. The cases $\alpha = 1$ and $\alpha = 2$ correspond respectively to the Cauchy and normal distributions.

Theorem 2. *If $y_1, \cdots, y_n$ is a sequence of independent, identically distributed (i.i.d.) real-valued random variables having a symmetric stable distribution with the characteristic function $E[\exp(iu\phi_t)] = \exp(iu\theta - |cu|^\alpha)$ then based on the elementary estimating functions*

$$\mathbf{h}_t(\theta, c) = \{\sin[u_1(y_t - \theta)], \cos[u_2(y_t-\theta)] - \exp(-|cu_2|^\alpha), t = 1, \ldots, n\}$$

then the optimal estimating function for $(\theta, c)'$ is given by

$$\mathbf{g}_n^*(\theta, c) = \begin{pmatrix} -\frac{2u_1 e^{-|cu_1|^\alpha}}{1-e^{-|2cu_1|^\alpha}} \sum_{t=1}^n \sin[u_1(y_t-\theta)] \\ \frac{2\alpha c^{\alpha-1}|u_2|^\alpha e^{-|cu_2|^\alpha}}{1+e^{|2cu_2|^\alpha} - 2e^{-2|cu_2|^\alpha})} \sum_{t=1}^n \{\cos[u_2(y_t-\theta)] - \exp(-|cu_2|^\alpha)\} \end{pmatrix}$$

and the corresponding information matrix is

$$\mathbf{I}_{\mathbf{g}_n^*} = \begin{pmatrix} \frac{2nu_1^2\exp(-2|cu_1|^\alpha)}{1-\exp(-|2cu_1|^\alpha)} & 0 \\ 0 & \frac{2n\alpha^2 c^{2(\alpha-1)} u_2^{2\alpha} e^{-2|cu_2|^\alpha}}{(1+e^{-|2cu_2|^\alpha} - 2e^{-2|cu_2|^\alpha})} \end{pmatrix}.$$

Note. We can choose the optimal values of u_1 and u_2 which maximize the associated information. For example, suppose $y_t, t = 1, \cdots, n$ are independent Cauchy random variables having probability density function

$$f(y; \theta) = \frac{1}{\pi}\left[\frac{1}{(y-\theta)^2+1}\right].$$

Based on the optimal estimating function $\sum_{t=1}^n \sin[u_1(y_t - \theta)]$ for θ, the optimal value of u_1 can be obtained by maximizing $\frac{nu_1^2}{e^{2u_1}-1}$. Moreover, when $y_t, t = 1, \cdots, n$ are independent Cauchy random variables having probability density function $f(x; c) = \frac{1}{\pi}\left[\frac{c}{y^2+c^2}\right]$, then for any real u_2, the optimal estimate of e^{-u_2c} is given by $n^{-1}\sum_{t=1}^n \cos u_2 y_t$, and the optimal value of u_2 is obtained by maximizing $\frac{2nu_2^2 e^{-2cu_2}}{1-e^{-2cu_2}}$.

2.1 Estimating Function for Discrete Circular Models

One difficulty in estimating the parameter(s) of wrapped Poisson (WP) or wrapped zero inflated Poisson (WZIP) is that neither the likelihood of WP/WZIP random variable nor the score function is available in closed form, which leads one to use either trigonometric method of moment estimation (TMME) or an estimating function approach.

In a recent paper, Thavaneswaran et al. (2013) studied parameter estimation for nonlinear time series with stable errors using combinations of bounded sine and cosine estimating functions. However, there is still a gap in the literature in terms of effective use of estimating functions for inference for discrete circular distributions. In this section, we study joint estimation of circular mean and mean resultant length parameters via estimating functions for a class of circular distributions.

Circular distributions often involve circular parameters, i.e., parameters taking values on the unit circle. Because we cannot directly take expectations of circular variables, it is not immediately obvious how to define unbiasedness of estimates of circular parameters. Mardia and Jupp (2000) defined a statistic t taking values on the unit circle as an unbiased estimator of ω if the mean direction of t is ω, defined as $E[\sin(t-\omega)] = 0$ and provided a lower bound on the variability of unbiased estimators as

$$Var(\sin(t-\omega)) \geq \frac{\rho_\omega^2(t)}{I_\omega}$$

where $\rho_\omega(t)$ is the mean resultant length of t and I_ω denotes the Fisher information

$$I_\omega = E\left[\left(\frac{\partial l}{\partial \omega}\right)\right] = E\left[-\frac{\partial^2 l}{\partial \omega^2}\right],$$

which is assumed to be positive.

Theorem 3. *If $\phi_1, \cdots, \phi_n$ are independent and identically distributed symmetric circular random variables with circular mean θ and the pth trigonometric moments given by $\bar{\alpha}_p = E[\cos p(\phi_t - \theta)]$ and $\bar{\beta}_p = E[\sin p(\phi_t - \theta)] = 0$ where $\bar{\alpha}_p$ depends on the concentration parameter c, then based on the elementary estimating functions*

$$\mathbf{h}_t(\theta, c) = \left\{\sin[p(\phi_t-\theta)], \cos[p(\phi_t-\theta)] - \bar{\alpha}_p(c)\right\}'$$

the optimal estimating function for $(\theta, c)'$ is given by

$$\mathbf{g}^*(\theta, c) = \sum_{t=1}^{n} \begin{pmatrix} \frac{2\bar{\alpha}_p(c)}{1-\bar{\alpha}_{2p}(c)} \sin[p(\phi_t-\theta)] \\ -\frac{\frac{2\partial\bar{\alpha}_p(c)}{\partial c}}{1+\bar{\alpha}_{2p}(c)-2\bar{\alpha}_p^2(c)} \left\{\cos[p(\phi_t-\theta)] - \bar{\alpha}_p(c)\right\} \end{pmatrix},$$

and the information matrix is given by

$$\mathbf{I}_{\mathbf{g}^*} = \begin{pmatrix} \frac{2n\bar{\alpha}_p^2(c)}{1-\bar{\alpha}_{2p}(c)} & 0 \\ 0 & \frac{2n\left(\frac{\partial\bar{\alpha}_p(c)}{\partial c}\right)^2}{1+\bar{\alpha}_{2p}(c)-2\bar{\alpha}_p^2(c)} \end{pmatrix}.$$

Corollary 2. *Suppose $\phi_1, \cdots, \phi_n$ are i.i.d. WP(λ) random angles in $[0, 2\pi)$ with circular mean $\theta = \lambda \sin\frac{2\pi}{m}$ and centered trigonometric moments given by $\bar{\alpha}_p(\lambda) = e^{-\lambda(1-\cos\frac{2\pi p}{m})}\cos\left(\lambda\sin\frac{2\pi p}{m}\right)$, then the corresponding optimal estimating function and the information matrix are respectively*

$$\mathbf{g}^*(\lambda) = \sum_{t=1}^{n} -\frac{2\bar{\alpha}_1(\lambda)\sin\frac{2\pi}{m}}{1-\bar{\alpha}_2(\lambda)} \sin(\phi_t - \lambda\sin\frac{2\pi}{m})$$

and

$$\mathbf{I}_{\mathbf{g}^*} = \left(\frac{2n\bar{\alpha}_1^2(\lambda)\sin^2\frac{2\pi}{m}}{1-\bar{\alpha}_2(\lambda)}\right).$$

The following table gives the values of the information for $m = 12$ and λ from 1 to 6.

Table 1. Poisson Simulation

λ $(m = 12)$	1.0	2.0	3.0	4.0	5.0	6.0
I_{g^*}	4401.00	2401.87	1728.55	1357.06	1128.65	957.07

Corollary 3. *Suppose $\phi_1, \cdots, \phi_n$ are independent and identically distributed WZIP random variables with the circular mean*

$$\theta(\lambda, w) = atan^{-1}\left[\frac{(1-w)e^{-\lambda(1-\cos\frac{2\pi}{m})}\sin\left(\lambda\sin\frac{2\pi}{m}\right)}{w + (1-w)e^{-\lambda(1-\cos\frac{2\pi}{m})}\cos\left(\lambda\sin\frac{2\pi}{m}\right)}\right]$$

and the trigonometric moments $\bar{\alpha}_1 = E[\cos(\phi_t - \theta)]$ and $\bar{\alpha}_2 = E[\cos 2(\phi_t - \theta)]$,

then based on the martingale differences

$$\mathbf{h}_t(\theta, c) = (\sin[(\phi_t - \theta)], \cos[(\phi_t - \theta)] - \bar{\alpha}_1(c))'$$

the optimal estimating function is given by

$$\mathbf{g}^*(\lambda, w) = \sum_{t=1}^{n}\begin{pmatrix} -\frac{2\bar{\alpha}_1(\lambda,w)}{1-\bar{\alpha}_2(\lambda,w)}\frac{\partial\theta(\lambda,w)}{\partial\lambda}\sin(\phi_t - \theta) \\ -\frac{2\bar{\alpha}_1(\lambda,w)}{1-\bar{\alpha}_2(\lambda,w)}\frac{\partial\theta(\lambda,w)}{\partial w}\sin(\phi_t - \theta)\end{pmatrix}$$

and the corresponding information matrix is given by

$$\mathbf{I}_{\mathbf{g}^*} = \frac{2n\bar{\alpha}_1^2(\lambda, w)}{1 - \bar{\alpha}_2(\lambda, w)}\begin{pmatrix} \left(\frac{\partial\theta(\lambda,w)}{\partial\lambda}\right)^2 & \left(\frac{\partial\theta(\lambda,w)}{\partial\lambda}\right)\left(\frac{\partial\theta(\lambda,w)}{\partial w}\right) \\ \left(\frac{\partial\theta(\lambda,w)}{\partial w}\right)\left(\frac{\partial\theta(\lambda,w)}{\partial\lambda}\right) & \left(\frac{\partial\theta(\lambda,w)}{w}\right)^2\end{pmatrix}.$$

2.2 Inference on the Mean Direction

We derive the asymptotic normal distribution of the tangent of the mean direction in the following theorem.

Theorem 4. *(a) Suppose $\phi_1, \cdots, \phi_n$ are symmetric i.i.d. circular random angles in $[0, 2\pi)$, i.e. with trigonometric moments $E\cos(\phi_1 - \mu) = \bar{\alpha}_1, E\cos 2(\phi_1 - \mu) = \bar{\alpha}_2$ and $E\sin(\phi_1 - \mu) = \bar{\beta}_1 = 0$. Let $\hat{\mu}$ be the estimate of μ obtained by solving $g^* = \sum_{i=1}^{n}\sin(\phi_i - \mu) = 0$. Then the limiting distribution of the tangent of the mean direction is given by*

$$\sqrt{n}(\tan\hat{\mu} - \tan\mu) \xrightarrow{\mathcal{D}} N(0, \sigma^2)$$

where the asymptotic variance σ^2 is

$$\sigma^2 = \frac{1 - \bar{\alpha}_2}{2\bar{\alpha}_1^2\cos^4\mu}.$$

(b) Suppose $\phi_1, \cdots, \phi_n$ are i.i.d. WP(λ) random angles in $[0, 2\pi)$ with circular mean $\mu = \lambda\sin\frac{2\pi}{m}$ and centered trigonometric moments $\bar{\alpha}_p = e^{-\lambda(1-\cos\frac{2\pi p}{m})}\cos\left(\lambda\sin\frac{2\pi p}{m}\right)$. Then the limiting distribution of the tangent of the mean direction is given by

$$\sqrt{n}(\tan\hat{\mu} - \tan\mu) \xrightarrow{\mathcal{D}} N(0, \sigma^2)$$

where the asymptotic variance σ^2 is $\sigma^2 = \frac{1-\bar{\alpha}_2}{2\bar{\alpha}_1^2\cos^4\mu}$.

(c) Suppose $\phi_1, \cdots, \phi_n$ are i.i.d. WZIP(λ, w) random angles in $[0, 2\pi)$ with circular mean

$$\mu = atan^{-1}\left[\frac{(1-w)e^{-\lambda(1-\cos\frac{2\pi}{m})}\sin\left(\lambda\sin\frac{2\pi}{m}\right)}{w + (1-w)e^{-\lambda(1-\cos\frac{2\pi}{m})}\cos\left(\lambda\sin\frac{2\pi}{m}\right)}\right],$$

and centered trigonometric moments $\bar{\alpha}_p = w\cos p\mu + (1-w)e^{-\lambda(1-\cos\frac{2\pi p}{m})}\cos\left(\lambda\sin\frac{2\pi p}{m} - p\mu\right)$. *Then the limiting distribution of the tangent of the mean direction is given by*

$$\sqrt{n}(\tan\hat{\mu} - \tan\mu) \xrightarrow{\mathcal{D}} N(0, \sigma^2),$$

where the asymptotic variance σ^2 *is*

$$\sigma^2 = \frac{1 - \bar{\alpha}_2}{2\bar{\alpha}_1^2\cos^4\mu}.$$

The proof of this theorem is similar to Gatto and Jammalamadaka (2003).

We have studied the sampling distribution of the estimators of the circular mean μ, mean resultant length ρ and variance V_0 (= 1- ρ) of the wrapped zero inflated Poisson distribution. For this purpose, we have simulated 100 samples of size 1000 from wrapped zero inflated Poisson distribution with $\lambda = 4$ and $w = 0.4$. The theoretical values of μ, ρ and V_0 are given by $\mu = 0.8989205$, $\rho = 0.4078955$ and $V_0 = 0.5921045$. We produced the histograms of the above estimates as well as the normal Q-Q plots of each. The plots are given in Figure 1. It is clear that plots show a good fit of the normal distribution.

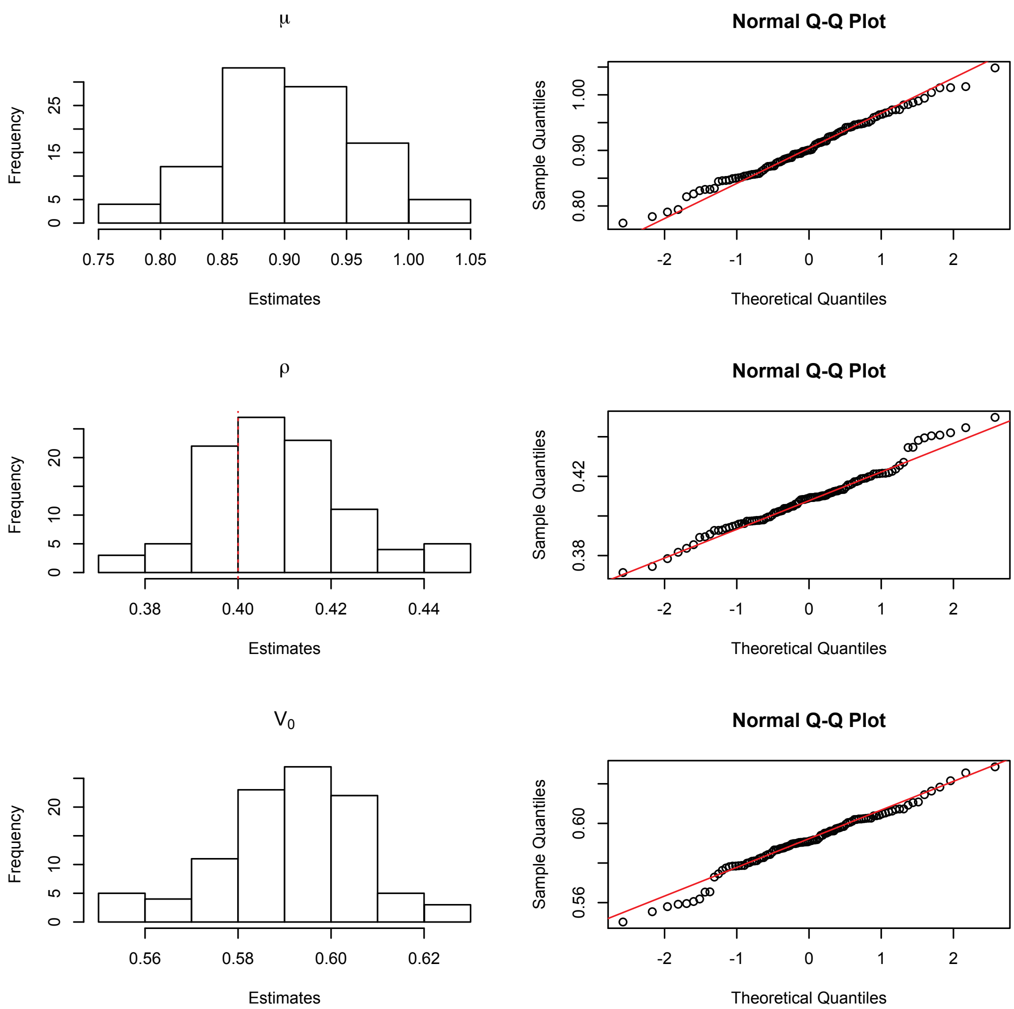

Figure 1. Sampling distributions of the estimators of circular mean μ, mean resultant length ρ and circular variance V_0.

We have also studied the sampling distribution of the estimators of w and λ of the wrapped zero inflated Poisson distribution. The plots are given in Figure 2. It also shows a good fit of the normal distribution.

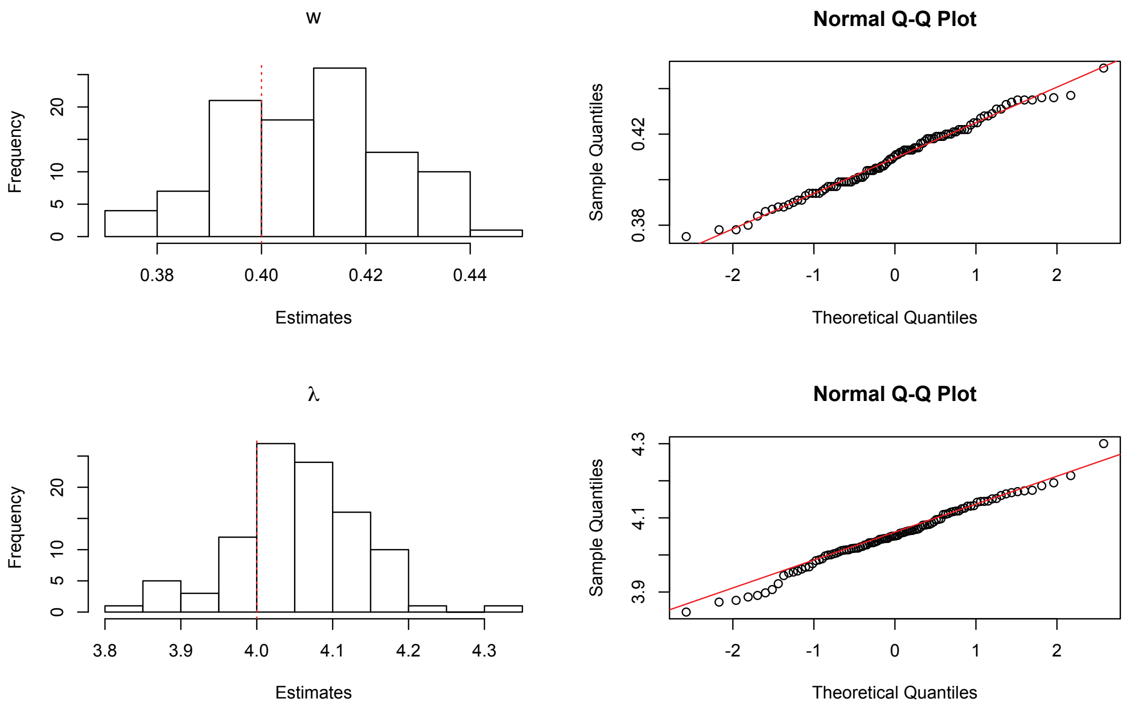

Figure 2. Sampling distributions of the estimators of w and λ.

3. Concluding Remarks

In the present work we have tried to address an important problem of estimation of wrapped zero-inflated Poisson distribution using estimating functions. We have studied estimating functions based on characteristic function and indicated how we can maximize the D-optimality criterion, that is, by maximizing the determinant of the Godambe's information matrix. We have studied the joint estimation of circular mean and mean resultant length parameters via estimating functions for a class of circular distributions. In particular, we have studied this for wrapped Poisson and wrapped zero-inflated Poisson. Further, we have derived the asymptotic normal distribution of the tangent of the mean direction for both the wrapped Poisson and wrapped zero-inflated Poisson distributions. Our simulation studies (based on the histograms of the estimates and the normal Q-Q plots) show that the estimates are normally distributed.

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