International Journal of Statistics and Probability; Vol. 4, No. 4; 2015
ISSN 1927-7032 E-ISSN 1927-7040
Published by Canadian Center of Science and Education

The Transmuted Marshall-Olkin Fréchet Distribution: Properties and Applications

Ahmed Z. Afify[1], G. G. Hamedani[2], Indranil Ghosh[3], & M. E. Mead[4]

[1] Department of Statistics, Mathematics and Insurance, Benha University, Egypt

[2] Department of Mathematics, Statistics and Computer Science, Marquette University, Milwaukee, USA

[3] Department of Mathematics and Statistics, University of North Carolina Wilmington, USA

[4] Department of Statistics and Insurance, Faculty of Commerce, Zagazig University, Egypt

Correspondence: Indranil Ghosh, Department of Mathematics & Statistics, University of North Carolina Wilmington, Wilmington, USA. Tel: 1-910-962-7644. E-mail: ghoshi@uncw.edu

Received: September 12, 2015 Accepted: September 28, 2015 Online Published: October 26, 2015

doi:10.5539/ijsp.v4n4p132 URL: http://dx.doi.org/10.5539/ijsp.v4n4p132

Abstract

This paper introduces a new four-parameter lifetime model, which extends the Marshall-Olkin Fréchet distribution introduced by Krishna et al. (2013), called the transmuted Marshall-Olkin Fréchet distribution. Various structural properties including ordinary and incomplete moments, quantile and generating function, Rényi and q-entropies and order statistics are derived. The maximum likelihood method is used to estimate the model parameters. We illustrate the superiority of the proposed distribution over other existing distributions in the literature in modeling two real life data sets.

Keywords: Transmuted family, generating Function, Rényi Entropy, order Statistics, maximum Likelihood estimation, Marshall-Olkin Fréchet distribution.

1. Introduction

Recently, there has been an increased interest in developing generalized continuous univariate distributions which have been extensively used for analyzing and modeling data in many applied areas such as lifetime analysis, engineering, economics, insurance and environmental sciences. However, these applied areas clearly require extended forms of these probability distributions when the parent models do not provide adequate fits. So, several families of distributions have been proposed by extending common families of continuous distributions. These generalized distributions provide more flexibility by adding one or more parameters to the baseline model. One example is the Marshal-Olkin-G (MO-G in short) family proposed by Marshal and Olkin (1997) by adding one parameter to the reliability function (rf) $\overline{G}(x) = 1 - G(x)$, where $G(x)$ is the baseline cumulative distribution function (cdf). Using the MO-G family, Krishna et al. (2013) defined and studied the Marshall-Olkin Fréchet (MOFr) distribution extending the Fréchet distribution.

The cdf of the MOFr is given (for $x > 0$) by

$$G(x,\alpha,\beta,\sigma) = \frac{e^{-\left(\frac{\sigma}{x}\right)^{\beta}}}{\alpha + (1-\alpha)\, e^{-\left(\frac{\sigma}{x}\right)^{\beta}}}, \tag{1}$$

where $\sigma > 0$ is a scale parameter and α and β are positive shape parameters.

The corresponding probability density function (pdf) is given by

$$g(x,\alpha,\beta,\sigma) = \frac{\alpha\beta\sigma^{\beta} x^{-(\beta+1)} e^{-\left(\frac{\sigma}{x}\right)^{\beta}}}{\left(\alpha + (1-\alpha)\, e^{-\left(\frac{\sigma}{x}\right)^{\beta}}\right)^{2}}. \tag{2}$$

The Fréchet distribution is one of the important distributions in extreme value theory and has applications in life testing, floods, rainfall, wind speeds, sea waves and track race records. Further details were explored by Kotz and Nadarajah (2000). Many authors constructed generalizations of the Fréchet distribution. For example, Nadarajah and Kotz (2003) studied the exponentiated Fréchet (EFr), Nadarajah and Gupta (2004) and Barreto-Souza et al.

(2011) independently introduced the beta Fréchet (BFr), Mahmoud and Mandouh (2013) proposed the transmuted Fréchet (TFr), Silva et al. (2013) proposed the gamma extended Fréchet (GEFr), Elbatal et al. (2014) studied the transmuted exponentiated Fréchet (TEFr), Mead and Abd-Eltawab (2014) introduced the Kumaraswamy Fréchet (Kw-Fr) and Afify et al. (2015) proposed the Weibull Fréchet (WFr) distributions.

In this paper, we define and study a new model by adding one parameter in equation (1) to provide more flexibility to the generated model. In fact, based on the transmuted-G (T-G) family pioneered by Shaw and Buckley (2007), we construct a new distribution called the *transmuted Marshall-Olkin Fréchet* (henceforth in short TMOFr) distribution and provide a comprehensive description of some of its mathematical properties. We hope that the new model will attract wider applications in reliability, engineering and other areas of research.

Recently, many authors used the T-G family to propose new generalizations of some well- known distributions. For example, Aryal and Tsokos (2009) defined the transmuted generalized extreme value, Aryal and Tsokos (2011) proposed the transmuted Weibull, Khan and King (2013) introduced the transmuted modified Weibull, Afify et al. (2014) defined the transmuted complementary Weibull geometric and Afify et al. (2015) proposed the transmuted Weibull Lomax distributions. For a detailed study on the general properties of the transmuted family of distributions, the interested reader is referred to Bourguignon, Ghosh and Cordeiro (2015).

Consider a baseline cdf $G(x)$ and pdf $g(x)$. Then, the cdf and pdf of the T-G family of distributions are ,respectively, defined by

$$F(x;\lambda) = G(x)\left[1+\lambda-\lambda G(x)\right] \tag{3}$$

and

$$f(x;\lambda) = g(x)\left[1+\lambda-2\lambda G(x)\right], \tag{4}$$

where $|\lambda| \leq 1$.

Note that if $\lambda = 0$, equation (4) gives the baseline distribution. Further details can be found in Shaw and Buckley (2007).

The rest of the paper is outlined as follows. In Section 2, we define the TMOFr distribution and give some plots for its pdf and hazard rate function (hrf). We derive useful mixture representations for the pdf and cdf in Section 3. We provide in Section 4 some mathematical properties of the TMOFr distribution including, ordinary and incomplete moments, moments of the residual life, reversed residual life, quantile and generating functions and Rényi and q-entropies. In Section 5, the order statistics and their moments are determined. Certain characterizations are presented in Section 6. The maximum likelihood estimates (MLEs) of the model parameters are obtained in Section 7. In Section 8, the TMOFr distribution is applied to two real data sets to illustrate its potentiality. Finally, in Section 9, we provide some concluding remarks.

2. The TMOFr Model

By inserting (1) into (3), we obtain the cdf of TMOFr (for $x > 0$)

$$F(x) = \frac{e^{-\left(\frac{\sigma}{x}\right)^{\beta}}}{\alpha+(1-\alpha)\,e^{-\left(\frac{\sigma}{x}\right)^{\beta}}}\left[1+\lambda-\frac{\lambda e^{-\left(\frac{\sigma}{x}\right)^{\beta}}}{\alpha+(1-\alpha)\,e^{-\left(\frac{\sigma}{x}\right)^{\beta}}}\right], \tag{5}$$

whereas its pdf can be expressed, from (1), (2) and (4) as

$$f(x) = \frac{\alpha\beta\sigma^{\beta}x^{-(\beta+1)}e^{-\left(\frac{\sigma}{x}\right)^{\beta}}}{\left[\alpha+(1-\alpha)\,e^{-\left(\frac{\sigma}{x}\right)^{\beta}}\right]^{2}}\left[1+\lambda-\frac{2\lambda e^{-\left(\frac{\sigma}{x}\right)^{\beta}}}{\alpha+(1-\alpha)\,e^{-\left(\frac{\sigma}{x}\right)^{\beta}}}\right], \tag{6}$$

where $\sigma > 0$ is a scale parameter, α and β are positive shape parameters and $|\lambda| \leq 1$.

A physical interpretation of the cdf of TMOFr is possible if we take a system consisting of two independent components functioning independently at a given time. So, if the two components are connected in parallel, the overall system will have the TMOFr cdf with $\lambda = -1$.

The rf, hrf, reversed hazard rate function (rhrf) and Cumulative hazard rate function (chrf) are, respectively, given by

$$R(x) = \frac{\alpha^{2}+\left(\alpha-\alpha\lambda-2\alpha^{2}\right)e^{-\left(\frac{\sigma}{x}\right)^{\beta}}+\left(\alpha^{2}+\alpha\lambda-\alpha\right)e^{-2\left(\frac{\sigma}{x}\right)^{\beta}}}{\left[\alpha+(1-\alpha)\,e^{-\left(\frac{\sigma}{x}\right)^{\beta}}\right]^{2}},$$

$$h(x) = \frac{\alpha\beta\sigma^{\beta}x^{-(\beta+1)}e^{-\left(\frac{\sigma}{x}\right)^{\beta}}}{\left[\alpha + (1-\alpha)\,e^{-\left(\frac{\sigma}{x}\right)^{\beta}}\right]}\left\{\alpha\,(1+\lambda) - \left[\lambda\,(\alpha+1)+\alpha-1\right]e^{-\left(\frac{\sigma}{x}\right)^{\beta}}\right\}$$
$$\times\left\{\alpha^2 + \left[\alpha\,(1-\lambda-2\alpha) + \left(\alpha^2+\alpha\lambda-\alpha\right)e^{-\left(\frac{\sigma}{x}\right)^{\beta}}\right]e^{-\left(\frac{\sigma}{x}\right)^{\beta}}\right\}^{-1},$$

$$r(x) = \frac{\alpha\beta\sigma^{\beta}x^{-(\beta+1)}\left\{\alpha\,(1+\lambda) - \left[\lambda\,(\alpha+1)+\alpha-1\right]e^{-\left(\frac{\sigma}{x}\right)^{\beta}}\right\}}{\left[\alpha\,(1+\lambda) - (\alpha\lambda+\alpha-1)\,e^{-\left(\frac{\sigma}{x}\right)^{\beta}}\right]\left[\alpha + (1-\alpha)\,e^{-\left(\frac{\sigma}{x}\right)^{\beta}}\right]}$$

and

$$H(x) = \ln\left\{\frac{\left[\alpha + (1-\alpha)\,e^{-\left(\frac{\sigma}{x}\right)^{\beta}}\right]^2}{\alpha^2 + (\alpha-\alpha\lambda-2\alpha^2)\,e^{-\left(\frac{\sigma}{x}\right)^{\beta}} + (\alpha^2+\alpha\lambda-\alpha)\,e^{-2\left(\frac{\sigma}{x}\right)^{\beta}}}\right\}.$$

Some of the plots of the pdf and hrf of TMOFr for different values of the parameters α, β, σ and λ are displayed in Figures 1 and 2.

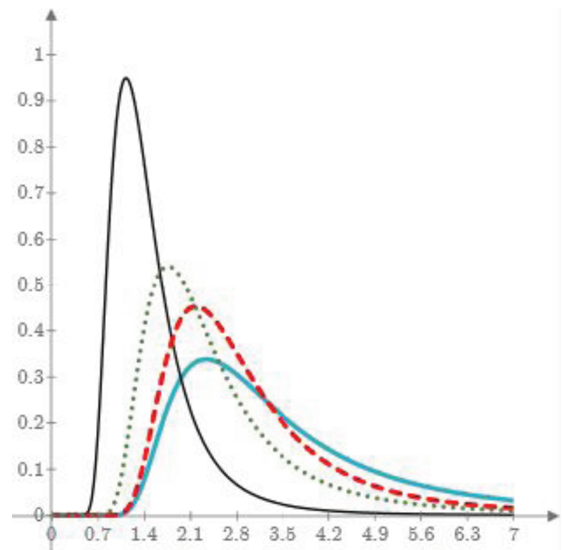

Figure 1. The pdf of TMOFr: (a) For $\alpha = 0.5 : \beta = \lambda = 0.5$ and $\sigma = 2$ (thick line), $\beta = 0.9, \sigma = 0.6$ and $\lambda = -0.2$(black line), $\beta = 0.4, \sigma = 5$ and $\lambda = 1$(dashed line) and $\beta = 0.3, \sigma = 1.5$ and $\lambda = 0.5$.

The TMOFr distribution shows flexible properties as it contains some well known distributions as special cases such as MOFr, transmuted Fréchet (TFr), transmuted inverse exponential (TIE), transmuted inverse Rayleigh (TIR), inverse exponential (IE) and inverse Rayleigh (IR) distributions among others. The flexibility of the TMOFr is explained in Table 1 where it has eleven sub-models when their parameters are carefully chosen.

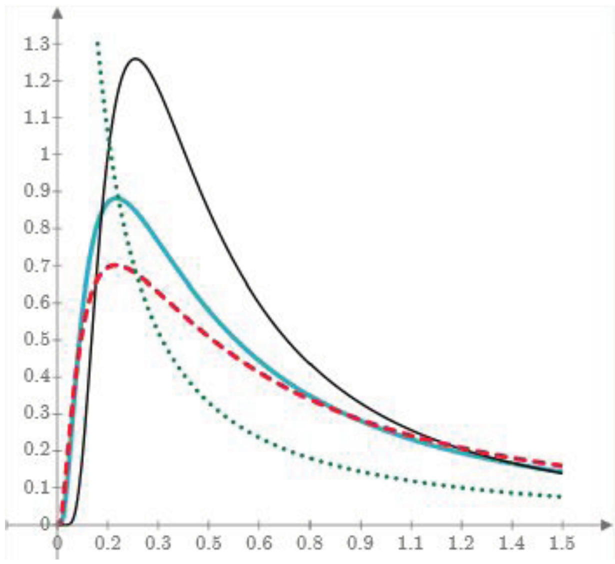

Figure 2. The pdf of TMOFr: (dotted line) (b) For $\alpha = 1.5$ and $\beta = 2.5$: $\sigma = 2.5$ and $\lambda = 0.1$ (thick line), $\sigma = 1.3$ and $\lambda = 0.9$(black line), $\sigma = 2.5$ and $\lambda = 0.7$ (dashed line) and $\sigma = 2$ and $\lambda = 0.6$ (dotted line).

Table 1. Sub-models of the TMOFr

α	β	σ	λ	Reduced Model	Author
α	1	σ	λ	TMOIE	New
α	2	σ	λ	TMOIR	New
α	β	σ	0	MOFr	Krishna et al. (2013)
α	1	σ	0	MOIE	–
α	2	σ	0	MOIR	–
1	β	σ	λ	TFr	Mahmoud and Mandouh (2013)
1	1	σ	λ	TIE	Oguntunde and Adejumo (2015)
1	2	σ	λ	TIR	Ahmad et al. (2014)
1	β	σ	0	Fr	Fréchet (1924)
1	1	σ	0	IE	Keller and Kamath (1982)
1	2	σ	0	IR	Trayer (1964)

3. Mixture Representation

The pdf in (6) can be expressed as

$$f(x)\, \frac{(1+\lambda)\,\alpha\beta\sigma^{\beta}x^{-(\beta+1)}e^{-\left(\frac{\sigma}{x}\right)^{\beta}}}{\left[\alpha + (1-\alpha)\,e^{-\left(\frac{\sigma}{x}\right)^{\beta}}\right]^{2}} - \frac{2\lambda\alpha\beta\sigma^{\beta}x^{-(\beta+1)}e^{-2\left(\frac{\sigma}{x}\right)^{\beta}}}{\left[\alpha + (1-\alpha)\,e^{-\left(\frac{\sigma}{x}\right)^{\beta}}\right]^{3}}.$$

Expansions for the density of TMOFr can be derived using the series expansion

$$(1-z)^{-k} = \sum_{j=0}^{\infty} \frac{\Gamma(k+j)}{j!\,\Gamma(k)} z^{j}, \;\; |z| < 1, \;\; k > 0.$$

Applying the above series expansion, the pdf of the TMOFr can be expressed in the mixture form

$$f(x) = \sum_{k=0}^{\infty} \left[\upsilon_k h_{\beta,\sigma(k+1)^{1/\beta}}(x) - \omega_k h_{\beta,\sigma(k+2)^{1/\beta}}(x) \right], \tag{7}$$

where $\upsilon_k = \frac{(1+\lambda)}{\alpha}\left(1-\frac{1}{\alpha}\right)^k$, $\omega_k = \frac{\nu(k+1)}{\alpha^2}\left(1-\frac{1}{\alpha}\right)^k$ and $h_{\beta,\sigma(\delta)^{1/\beta}}(x)$ is the Fréchet (Fr) density with shape parameter β and scale parameter $\sigma(\delta)^{1/\beta}$.

Since the density function TMOFr is expressed as a mixture of Fr densities, one may obtain some of its mathematical properties directly from the properties of the Fr distribution.

By integrating (7), we obtain

$$F(x) = \sum_{k=0}^{\infty} \left[\upsilon_k H_{\beta,\sigma(k+1)^{1/\beta}}(x) - \omega_k H_{\beta,\sigma(k+2)^{1/\beta}}(x) \right],$$

where $H_{\beta,\sigma(\delta)^{1/\beta}}(x)$ is the cdf of Fr distribution with shape parameter β and scale parameter $\sigma(\delta)^{1/\beta}$.

4. Mathematical Properties

Employing established algebraic expansions to determine some structural quantities of the TMOFr distribution can be more efficient than computing those directly by numerical integration of its density function.

4.1 Moments

Henceforth, let Z be a random variable having the Fr distribution with scale $\sigma > 0$ and shape $\beta > 0$. Then, the pdf of Z is given by

$$g(z;\beta,\sigma) = \beta\sigma^{\beta} z^{-(\beta+1)} e^{-\left(\frac{\sigma}{z}\right)^{\beta}},\ z > 0.$$

For $r < \beta$, the rth ordinary and incomplete moments of Z are given by

$$\mu'_{r,Z} = \sigma^r\,\Gamma\left(1-\frac{r}{\beta}\right) \quad \text{and} \quad \varphi_{r,Z}(t) = \sigma^r\,\gamma\left(1-\frac{r}{\beta},(\sigma/t)^{\beta}\right),$$

respectively, where $\gamma(s,t) = \int_0^t x^{s-1}\,e^{-x}dx$ is the lower incomplete gamma function.

Then, the rth moment of X, say μ'_r, can be expressed as

$$\mu'_r = E(X^r) = \sum_{k=0}^{\infty} \sigma^r \left[\upsilon_k\ (k+1)^{r/\beta} - \omega_k\ (k+2)^{r/\beta}\right]\Gamma\left(1-\frac{r}{\beta}\right). \tag{8}$$

Using the relation between the central and non-central moments, we obtain the nth central moment of X, say μ_n, as follows

$$\mu_n = \sum_{r=0}^{n}\sum_{k=0}^{\infty} \sigma^r\left(-\mu'_1\right)^{n-r}\ \left[\upsilon_k\ (k+1)^{r/\beta} - \omega_k\ (k+2)^{r/\beta}\right]\binom{n}{r}\Gamma\left(1-\frac{r}{\beta}\right).$$

The skewness and kurtosis measures can be determined from the central moments using established results.

4.2 Incomplete Moments

The main application of the first incomplete moment refers to the Bonferroni and Lorenz curves. These curves are very useful in economics, reliability, demography, insurance and medicine. The answers to many important questions in economics require more than just knowing the mean of the distribution, but its shape as well. This is obvious not only in the study of econometrics but in other areas as well. The sth incomplete moments, say $\varphi_s(t)$, is given by

$$\varphi_s(t) = \int_0^t x^s f(x)\,dx,$$

Using equation (7), we can write

$$\varphi_s(t) = \sum_{k=0}^{\infty}\left[\upsilon_k \int_0^t x^s h_{\beta,\sigma(k+1)^{1/\beta}}(x) - \omega_k \int_0^t x^s h_{\beta,\sigma(k+2)^{1/\beta}}(x)\right],$$

and then using the lower incomplete gamma function, we obtain (for $s < \alpha$)

$$\varphi_s(t) = \sum_{k=0}^{\infty} \upsilon_k \sigma^s (k+1)^{s/\beta} \gamma\left(1 - \frac{s}{\beta}, (k+1)\left(\frac{\sigma}{t}\right)^{\beta}\right) - \sum_{k=0}^{\infty} \omega_k \sigma^s (k+2)^{s/\beta} \gamma\left(1 - \frac{s}{\beta}, (k+2)\left(\frac{\sigma}{t}\right)^{\beta}\right),$$

where $\gamma(a, z)$ is, the lower incomplete gamma function, defined in subsection 4.1.

The first incomplete moment of the TMOFr distribution can be obtained by setting $s = 1$ in the last equation.

Another application of the first incomplete moment is related to mean residual life and mean waiting time given by $m_1(t) = [1 - \varphi_1(t)]/R(t) - t$ and $M_1(t) = t - [\varphi_1(t)/F(t)]$, respectively.

The amount of scattering in a population is evidently measured, to some extent, by the totality of deviations from the mean and median. The mean deviations about the mean $[\delta_\mu(X) = E(|X - \mu_1^{'}|)]$ and about the median $[\delta_\mu(X) = E(|X - M|)]$ of X can be, used as measures of spread in a population, expressed by $\delta_\mu(X) = \int_0^\infty |X - \mu_1^{'}| f(x)\, dx = 2\mu_1^{'} F(\mu_1^{'}) - 2\varphi_1(\mu_1^{'})$ and $\delta_M(X) = \int_0^\infty |X - M| f(x)\, dx = \mu_1^{'} - 2\varphi_1(M)$, respectively, where $\mu_1^{'} = E(X)$ comes from (8), $F(\mu_1^{'})$ is simply calculated, $\varphi_1(\mu_1^{'})$ is the first incomplete moments and M is the median of X.

4.3 Residual Life Function

Several functions are defined related to the residual life. The failure rate function, mean residual life function and the left censored mean function, also called vitality function. It is well known that these three functions uniquely determine $F(x)$, see Gupta (1975), Kotz and Shanbhag (1980) and Zoroa et al. (1990).

Moreover, the nth moment of the residual life, say $m_n(t) = E[(X - t)^n \mid X > t]$, $n = 1, 2, \ldots$, uniquely determine $F(x)$ (see Navarro et al., (1998). The nth moment of the residual life of X is given by

$$m_n(t) = \frac{1}{R(t)} \int_t^\infty (x - t)^n f(x) dx.$$

Then, we can write (for $r < \beta$)

$$\begin{aligned} m_n(t) = {} & \frac{1}{R(t)} \sum_{r=0}^{n} \sum_{n=0}^{\infty} (-t)^{n-r} \upsilon_k \sigma^r (k+1)^{\frac{r}{\beta}} \binom{n}{r} \left\{1 - \gamma\left(1 - \frac{r}{\beta}, (k+1)\left(\frac{\sigma}{t}\right)^{\beta}\right)\right\} \\ & - \frac{1}{R(t)} \sum_{r=0}^{n} \sum_{n=0}^{\infty} (-t)^{n-r} \omega_k \sigma^r (k+2)^{\frac{r}{\beta}} \binom{n}{r} \left\{1 - \gamma\left(1 - \frac{r}{\beta}, (k+2)\left(\frac{\sigma}{t}\right)^{\beta}\right)\right\}. \end{aligned}$$

Another interesting function is the mean residual life (MRL) function or the life expectation at age x defined by $m_1(x) = E[(X - x) \mid X > x]$, which represents the expected additional life length for a unit which is alive at age x. The MRL of the TMOFr distribution can be obtained by setting $n = 1$ in the last equation. Guess and Proschan (1988) derived an extensive coverage of possible applications of the MRL applications in survival analysis, biomedical sciences, life insurance, maintenance and product quality control, economics, social studies and demography (see, Lai and Xie, 2006).

4.4 Reversed Residual Life Function

The nth moment of the reversed residual life, say $M_n(t) = E[(t - X)^n \mid X \le t]$ for $t > 0$, $n = 1, 2, \ldots$ uniquely determines $F(x)$ (Navarro et al., 1998). We obtain

$$M_n(t) = \int_0^t (t - x)^n dF(x).$$

Therefore, the nth moment of the reversed residual life of X given that $r < \beta$ becomes

$$\begin{aligned} M_n(t) = {} & \frac{1}{F(t)} \sum_{r=0}^{n} \sum_{n=0}^{\infty} (-1)^r t^{n-r} \upsilon_k \sigma^r (k+1)^{\frac{r}{\beta}} \binom{n}{r} \gamma\left(1 - \frac{r}{\beta}, (k+1)\left(\frac{\sigma}{t}\right)^{\beta}\right) \\ & - \frac{1}{F(t)} \sum_{r=0}^{n} \sum_{n=0}^{\infty} (-1)^r t^{n-r} \omega_k \sigma^r (k+2)^{\frac{r}{\beta}} \binom{n}{r} \gamma\left(1 - \frac{r}{\beta}, (k+2)\left(\frac{\sigma}{t}\right)^{\beta}\right). \end{aligned}$$

The mean inactivity time (MIT) or mean waiting time (MWT), also called the mean reversed residual life function, is defined by $M_1(t) = E[(t - X) \mid X \leq t]$, and it represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0, x)$. The MRRL of X can be obtained by setting $n = 1$ in the above equation. The properties of the mean inactivity time have been considered by many authors, see e.g., Kayid and Ahmad (2004) and Ahmad et al. (2005).

4.5 Quantile and Generating Functions

The quantile function (qf) of X is the real solution of the equation $F(x_q) = q$. Then by inverting (5), we obtain

$$x_q = \sigma\left[\ln\left(\frac{b + \alpha\sqrt{1 + \lambda(\lambda - 4q + 2)}}{2\alpha^2 q}\right)\right]^{-1/\beta}, 0 < q < 1,$$

where $b = 2\alpha q(\alpha - 1) + \alpha(\lambda + 1)$.

Simulating a TMOFr random variable is straightforward. If U is a uniform variate on the unit interval $(0, 1)$, then the random variable $X = x_q$ follows (6).

First, we provide the moment generating function (mgf) of the Fr model as discussed by Afify et al. (2015). We can write the mgf of Z as

$$M(t;\beta,\sigma) = \beta\sigma^{\beta}\int_0^{\infty} e^{t/y}\, y^{(\beta-1)}\, e^{-(\sigma y)^{\beta}}\, dy.$$

By expanding the first exponential and determining the integral, we obtain

$$M(t;\beta,\sigma) = \sum_{m=0}^{\infty}\frac{\sigma^m t^m}{m!}\,\Gamma\left(\frac{\beta - m}{\beta}\right).$$

Consider the Wright generalized hypergeometric function (Kilbas et al., 2006) defined by

$${}_p\Psi_q\left[\begin{array}{c}(\gamma_1, A_1), \ldots, (\gamma_p, A_p)\\(\beta_1, B_1), \ldots, (\beta_q, B_q)\end{array};x\right] = \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma\left(\gamma_j + A_j n\right)}{\prod_{j=1}^{q}\Gamma\left(\beta_j + B_j n\right)}\,\frac{x^n}{n!}.$$

Then, we can write $M(t;\beta,\sigma)$ as

$$M(t;\beta,\sigma) = {}_1\Psi_0\left[\begin{array}{c}\left(1, -\beta^{-1}\right)\\-\end{array};\sigma t\right].$$

Combining the last expression and (7), the mgf of X can be expressed as

$$M_X(t) \;=\; \sum_{k=0}^{\infty}\upsilon_k\, {}_1\Psi_0\left[\begin{array}{c}\left(1, -\beta^{-1}\right)\\-\end{array};\sigma(k+1)^{1/\beta}\, t\right] - \sum_{k=0}^{\infty}\omega_k\, {}_1\Psi_0\left[\left(1, -\beta^{-1}\right) - ;\sigma(k+2)^{1/\beta}\, t\right].$$

4.6 Rényi and q-Entropies

The Rényi entropy of a random variable X represents a measure of variation of the uncertainty. The Rényi entropy is defined by

$$I_\gamma(X) = \frac{1}{(\gamma - 1)}\log\int_{-\infty}^{\infty} f(x)^{\gamma}\, dx,\ \gamma > 0 \text{ and } \gamma \neq 1.$$

Then, using (6), we can write

$$f(x)^{\gamma} = \frac{\left(\alpha\beta\sigma^{\beta}\right)^{\gamma} x^{-\gamma(\beta+1)} e^{-\gamma\left(\frac{\sigma}{x}\right)^{\beta}}}{\left[\alpha + (1-\alpha)\, e^{-\left(\frac{\sigma}{x}\right)^{\beta}}\right]^{2\gamma}}\underbrace{\left[1 + \lambda - 2\lambda\frac{e^{-\left(\frac{\sigma}{x}\right)^{\beta}}}{\alpha + (1-\alpha)\, e^{-\left(\frac{\sigma}{x}\right)^{\beta}}}\right]^{\gamma}}_{A}.$$

Applying the generalized binomial expansion to the quantity A, we obtain

$$f(x)^{\gamma} = \left[\alpha\beta\sigma^{\beta}(1+\lambda)\right]^{\gamma} x^{-\gamma(\beta+1)}\sum_{j=0}^{\infty}\frac{(-1)^j\,\Gamma(\gamma+1)\, d^j}{j!\,\Gamma(\gamma - j + 1)} \times e^{-(j+\gamma)\left(\frac{\sigma}{x}\right)^{\beta}}\underbrace{\left[\alpha + (1-\alpha)\, e^{-\left(\frac{\sigma}{x}\right)^{\beta}}\right]^{-(2\gamma+j)}}_{B},$$

where $d = 2\lambda/(1+\lambda)$.

Applying the series expansion defined in Section 3 to B, we can write

$$f(x)^{\gamma} = \left(\beta\sigma^{\beta}\right)^{\gamma} \sum_{j,k=0}^{\infty} b_{j,k} x^{-\gamma(\beta+1)} e^{-(k+j+\gamma)\left(\frac{\sigma}{x}\right)^{\beta}},$$

where

$$b_{j,k} = \frac{(-1)^{j}\,\Gamma(2\gamma+j+k)\,\Gamma(\gamma+1)\,d^{j}\,(1+\lambda)^{\gamma}}{j!k!\Gamma(2\gamma+j)\,\Gamma(\gamma-j+1)\,\alpha^{\gamma+j}}\left(1-\frac{1}{\alpha}\right)^{k}.$$

Then the Rényi entropy of X is given by

$$I_{\gamma}(X) = \frac{1}{(\gamma-1)}\log\left[\left(\beta\sigma^{\beta}\right)^{\gamma}\sum_{j,k=0}^{\infty} b_{j,k}\int_{-\infty}^{\infty} x^{-\gamma(\beta+1)} e^{-(k+j+\gamma)\left(\frac{\sigma}{x}\right)^{\beta}} dx\right],$$

and by making the substitution $u = (\sigma/x)^{\beta}$, for $\gamma(1+\beta) > 1$, we have that

$$I_{\gamma}(X) = \frac{1}{(\gamma-1)}\log\left[\left(\frac{\beta}{\sigma}\right)^{\gamma-1}\sum_{j,k=0}^{\infty} b_{j,k}\,(k+j+\gamma)^{-s}\,\Gamma(s)\right],$$

where $s = [\gamma(1+\beta)-1]/\beta$.

The q-entropy (for $q > 0$ and $q \neq 1$), say $H_q(X)$, is given by

$$H_q(X) = \frac{1}{(q-1)}\log\left\{1-\int_{-\infty}^{\infty} f(x)^{q}\,dx\right\},$$

and then

$$H_q(X) = \frac{1}{(q-1)}\log\left\{1-\left[\left(\frac{\beta}{\sigma}\right)^{q-1}\sum_{j,k=0}^{\infty} b^{\star}_{j,k}\,(k+j+q)^{-s^{\star}}\,\Gamma\left(s^{\star}\right)\right]\right\},$$

where $s^{\star} = [q(1+\beta)-1]/\beta$ and

$$b^{\star}_{j,k} = \frac{(-1)^{j}\,\Gamma(2q+j+k)\,\Gamma(q+1)\,d^{j}\,(1+\lambda)^{q}}{j!k!\Gamma(2q+j)\,\Gamma(q-j+1)\,\alpha^{q+j}}\left(1-\frac{1}{\alpha}\right)^{k}.$$

5. Order Statistics

The order statistics and their moments have great importance in many statistical problems and they have many applications in reliability analysis and life testing. Let $X_1, \ldots, X_n$ be a random sample of size n from the TMOFr(α, β, σ, λ) with cdf (5) and pdf (6), respectively. Let $X_{1:n}, \ldots, X_{n:n}$ be the corresponding order statistics. Then, the pdf of rth order statistic, say $X_{r:n}$, $1 \leq r \leq n$, denoted by $f_{r:n}(x)$, can be expressed as

$$f_{r:n}(x) = C_{r:n}\frac{\alpha\beta\sigma^{\beta}x^{-(\beta+1)}e^{-\left(\frac{\sigma}{x}\right)^{\beta}}\left[\ell_1 e^{-\left(\frac{\sigma}{x}\right)^{\beta}} - \ell_3 e^{-2\left(\frac{\sigma}{x}\right)^{\beta}}\right]^{r-1}}{\left[\alpha+(1-\alpha)e^{-\left(\frac{\sigma}{x}\right)^{\beta}}\right]^{3}\left[\alpha+(1-\alpha)e^{-\left(\frac{\sigma}{x}\right)^{\beta}}\right]^{2(r-1)}} \times \frac{\left[\ell_1 - \ell_2 e^{-\left(\frac{\sigma}{x}\right)^{\beta}}\right]\left[\alpha^2 + \ell_4 e^{-\left(\frac{\sigma}{x}\right)^{\beta}} + \ell_5 e^{-2\left(\frac{\sigma}{x}\right)^{\beta}}\right]^{n-r}}{\left[\alpha+(1-\alpha)e^{-\left(\frac{\sigma}{x}\right)^{\beta}}\right]^{2(n-r)}},$$

where $C_{r:n} = \frac{n!}{(r-1)!(n-r)!}$, $\ell_1 = \alpha(1+\lambda)$, $\ell_2 = \alpha\lambda+\alpha+\lambda-1$, $\ell_3 = \alpha\lambda+\alpha-1$, $\ell_4 = \alpha-\alpha\lambda-2\alpha^2$ and $\ell_5 = \alpha^2+\alpha\lambda-\alpha$.

The pdf $f_{r:n}(x)$, can also be expressed as

$$f_{r:n}(x) = \frac{f(x)}{\mathrm{B}(r, n-r+1)}\sum_{s=0}^{n-1}(-1)^{s}\binom{n-1}{s}F^{s+r-1}(x). \tag{9}$$

Further, we can write

$$F^{s+r-1}(x) = \sum_{i=0}^{\infty}(-\lambda)^{i}\binom{s+r-1}{i}\frac{(1+\lambda)^{s+r-i-1}\,e^{-(s+r+i-1)\left(\frac{\sigma}{x}\right)^{\beta}}}{\left[\alpha+(1-\alpha)e^{-\left(\frac{\sigma}{x}\right)^{\beta}}\right]^{s+r+i-1}}.$$

Using equation (6) and the last equation and, after some simplification, we can write

$$f(x)F^{s+r-1}(x) = (1+\lambda)\sum_{i=0}^{\infty}\frac{d_i\alpha\beta\sigma^{\beta}x^{-(\beta+1)}e^{-(s+r+i)\left(\frac{\sigma}{x}\right)^{\beta}}}{\left[\alpha+(1-\alpha)\,e^{-\left(\frac{\sigma}{x}\right)^{\beta}}\right]^{s+r+i+1}} - 2\lambda\sum_{i=0}^{\infty}\frac{d_i\alpha\beta\sigma^{\beta}x^{-(\beta+1)}e^{-(s+r+i+1)\left(\frac{\sigma}{x}\right)^{\beta}}}{\left[\alpha+(1-\alpha)\,e^{-\left(\frac{\sigma}{x}\right)^{\beta}}\right]^{s+r+i+2}}, \tag{10}$$

where $d_i = (-\lambda)^i \binom{s+r-1}{i}(1+\lambda)^{s+r-i-1}$.

By inserting (10) in equation (9) and, after some simplification, we obtain

$$f_{r:n}(x) = \sum_{s=0}^{n-1}\sum_{i,j=0}^{\infty} a_{i,j}\, h_{s+r+i+j}(x) - \sum_{s=0}^{n-1}\sum_{i,j=0}^{\infty} b_{i,j}\, h_{s+r+i+j+1}(x), \tag{11}$$

where

$$a_{i,j} = \frac{(-1)^s\,\Gamma(s+r+i+j+1)\,(1+\lambda)\,d_i\left(1-\frac{1}{\alpha}\right)^j}{j!\,\mathrm{B}(r,n-r+1)\,\Gamma(s+r+i+1)\,(s+r+i+j)\,\alpha^{s+r+i}}\binom{n-1}{s},$$

$$b_{i,j} = \frac{(-1)^s\,\Gamma(s+r+i+j+2)\,2\lambda d_i\left(1-\frac{1}{\alpha}\right)^j}{j!\,\mathrm{B}(r,n-r+1)\,\Gamma(s+r+i+2)\,(s+r+i+j+1)\,\alpha^{s+r+i+1}}\binom{n-1}{s}$$

and $h_\eta(x)$ denotes to the Fr density with shape parameter β and scale parameter $\sigma\,(\eta)^{1/\beta}$.

Equation (11) reveals that the pdf of the TMOFr order statistics is a mixture of Fr densities. So, some of their mathematical properties can also be obtained from those of the Fr distribution. For example, the qth moment of $X_{r:n}$ can be expressed as

$$E\left(X_{r:n}^q\right) = \sum_{s=0}^{n-1}\sum_{i,j=0}^{\infty} a_{i,j}\, E\left(Y_{s+r+i+j}^q\right) - \sum_{s=0}^{n-1}\sum_{i,j=0}^{\infty} b_{i,j}\, E\left(Y_{s+r+i+j+1}^q\right), \tag{12}$$

where $Y_{s+r+i+j}^q \sim \mathrm{Fr}(\sigma\,(s+r+i+j)^{1/\beta},\beta)$.

The L-moments are analogous to the ordinary moments but can be estimated by linear combinations of order statistics. Based upon the moments in equation (12), we can derive explicit expressions for the L-moments of X as infinite weighted linear combinations of the means of suitable TMOFr distribution. They are linear functions of expected order statistics defined by

$$\lambda_r = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}E\left(X_{r-k:r}\right),\ r \geq 1.$$

The first four L-moments are given by

$$\lambda_1 = E\left(X_{1:1}\right),\quad \lambda_2 = \frac{1}{2}E\left(X_{2:2}-X_{1:2}\right),\ \lambda_3 = \frac{1}{3}E\left(X_{3:3}-2X_{2:3}+X_{1:3}\right)$$

and

$$\lambda_4 = \frac{1}{4}E\left(X_{4:4}-3X_{3:4}+3X_{2:4}-X_{1:4}\right).$$

6. Characterizations

The problem of characterizing a distribution is an important problem in various fields which has recently attracted the attention of many researchers. These characterizations have been established in many different directions. This section deals with various characterizations of TMOFr distribution. These characterizations are based on: (*i*) a simple relationship between two truncated moments; (*ii*) the hazard function; (*iii*) a single function of the random variable. It should be mentioned that for characterization (*i*) the cdf need no have a closed form. We believe, due to the nature of the cdf of TMOFr, there may not be other possible characterizations than the ones presented in this section.

6.1 Characterizations Based on Two Truncated Moments

In this subsection we present characterizations of TMOFr distribution in terms of a simple relationship between two truncated moments. Our first characterization result borrows from a theorem due to Glanzel (1987) see Theorem A of Appendix A. We refer the interested reader to Glanzel (1987) for a proof of Theorem A. Note that the result holds also when the interval H is not closed. Moreover, as mentioned above, it could be also applied when the cdf F does not have a closed form. As shown in Afify et al. (2015), this characterization is stable in the sense of weak convergence.

Proposition 6.1. Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $h(x) = \left[1 + \lambda - \frac{\lambda e^{-(\frac{\sigma}{x})\beta}}{\alpha + (1-\alpha) e^{-(\frac{\sigma}{x})\beta}}\right]^{1-b}$ and $g(x) = h(x)\left[\alpha + (1-\alpha) e^{-(\frac{\sigma}{x})\beta}\right]^{-1}$ for $x > 0$. The random variable X belongs to TMOFr family (6) if and only if the function η defined in Theorem A has the form

$$\eta(x) = \frac{1}{2}\left\{1 + \left[\alpha + (1-\alpha) e^{-(\frac{\sigma}{x})\beta}\right]^{-1}\right\}, \quad x > 0. \tag{13}$$

Proof. Let X be a random variable with density (6), then

$$(1 - F(x)) E[h(x) \mid X \geq x] = \frac{1}{1-\alpha}\left\{\left[\alpha + (1-\alpha) e^{-(\frac{\sigma}{x})\beta}\right]^{-1} - 1\right\}, \ x > 0,$$

and

$$(1 - F(x)) E[g(x) \mid X \geq x] = \frac{1}{2(1-\alpha)}\left\{\left[\alpha + (1-\alpha) e^{-(\frac{\sigma}{x})\beta}\right]^{-2} - 1\right\}, \ x > 0,$$

and finally

$$\eta(x) h(x) - g(x) = \frac{1}{2} h(x)\left\{1 - \left[\alpha + (1-\alpha) e^{-(\frac{\sigma}{x})\beta}\right]^{-1}\right\} > 0 \text{ for } x > 0.$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x) h(x)}{\eta(x) h(x) - g(x)} = \frac{-(1-\alpha)\beta\sigma^{\beta} e^{-(\frac{\sigma}{x})^{\beta}}\left[\alpha + (1-\alpha) e^{-(\frac{\sigma}{x})\beta}\right]^{-2}}{x^{\beta+1}\left\{1 - \left[\alpha + (1-\alpha) e^{-(\frac{\sigma}{x})\beta}\right]^{-1}\right\}}, \ x > 0,$$

and hence

$$s(x) = -\ln\left\{\left\{1 - \left[\alpha + (1-\alpha) e^{-(\frac{\sigma}{x})\beta}\right]^{-1}\right\}\right\}, \ x > 0.$$

Now, in view of Theorem A, X has density (6).

Corollary 6.1. Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $h(x)$ be as in Proposition 6.1. The pdf of X is (6) if and only if there exist functions g and η defined in Theorem A satisfying the differential equation

$$\frac{\eta'(x) h(x)}{\eta(x) h(x) - g(x)} = \frac{-(1-\alpha)\beta\sigma^{\beta} e^{-(\frac{\sigma}{x})^{\beta}}\left[\alpha + (1-\alpha) e^{-(\frac{\sigma}{x})\beta}\right]^{-2}}{x^{\beta+1}\left\{1 - \left[\alpha + (1-\alpha) e^{-(\frac{\sigma}{x})\beta}\right]^{-1}\right\}}, \ x > 0. \tag{14}$$

The general solution of the differential equation in Corollary 6.1 is

$$\eta(x) = \left\{1 - \left[\alpha + (1-\alpha) e^{-(\frac{\sigma}{x})\beta}\right]^{-1}\right\}^{-1}\left\{\int \frac{(1-\alpha)\beta\sigma^{\beta} x^{-(\beta+1)} e^{-(\frac{\sigma}{x})^{\beta}}}{h(x)\left[\alpha + (1-\alpha) e^{-(\frac{\sigma}{x})\beta}\right]^{2}} g(x)\, dx + D\right\}$$

where D is a constant. Note that a set of functions satisfying the differential equation (14) is given in Proposition 6.1 with $D = \frac{1}{2}$. However, it should be also noted that there are other triplets (h, g, η) satisfying the conditions of Theorem A.

6.2 Characterization Based on Hazard Function

It is known that the hazard function, h_F, of a twice differentiable distribution function, F, satisfies the first order differential equation

$$\frac{f'(x)}{f(x)} = \frac{h'_F(x)}{h_F(x)} - h_F(x). \tag{15}$$

For many univariate continuous distributions, this is the only characterization available in terms of the hazard function. The following characterization establishes a non-trivial characterization for TMOFr distribution in terms of the hazard function when $\alpha = 1$, which is not of the trivial form given in (15). We assume, without loss of generality, that $\sigma = 1$ in the following Proposition.

Proposition 6.2. Let $X : \Omega \to (0, \infty)$ be a continuous random variable. Then for $\alpha = 1$, the pdf of X is (6) if and only if its hazard function $h_F(x)$ satisfies the differential equation

$$h'_F(x) + (\beta + 1)\, x^{-1} h_F(x) = \beta x^{-(\beta+1)} \frac{d}{dx} \left\{ \frac{e^{-(\frac{\sigma}{x})^\beta} \left[1 + \lambda - 2\lambda e^{-(\frac{\sigma}{x})^\beta}\right]}{1 + \left[\lambda e^{-(\frac{\sigma}{x})^\beta} - (1+\lambda)\right] e^{-(\frac{\sigma}{x})^\beta}} \right\}, \tag{16}$$

with the boundary condition $\lim_{x\to\infty} h_F(x) = 0$.

Proof. If X has pdf (6), then clearly (16) holds. Now, if (16) holds, then

$$\frac{d}{dx} \left\{ x^{\beta+1} h_F(x) \right\} = \frac{d}{dx} \left\{ \frac{\beta e^{-(\frac{\sigma}{x})^\beta} \left[1 + \lambda - 2\lambda e^{-(\frac{\sigma}{x})^\beta}\right]}{1 + \left[\lambda e^{-(\frac{\sigma}{x})^\beta} - (1+\lambda)\right] e^{-(\frac{\sigma}{x})^\beta}} \right\},$$

or, equivalently,

$$h_F(x) = \frac{\beta x^{-(\beta+1)} e^{-(\frac{\sigma}{x})^\beta} \left[1 + \lambda - 2\lambda e^{-(\frac{\sigma}{x})^\beta}\right]}{1 + \left[\lambda e^{-(\frac{\sigma}{x})^\beta} - (1+\lambda)\right] e^{-(\frac{\sigma}{x})^\beta}},$$

which is the hazard function of the TMOFr distribution.

6.3 Characterizations Based on a Single Function of the Random Variable

In this subsection we present a characterization result in terms of a function of the random variable X.

Proposition 6.3. Let $X : \Omega \to (a, b)$ be a continuous random variable with cdf F and corresponding pdf f. Let $\psi(x)$ be a differentiable function greater than 1 on (a, b) such that $\lim_{x\to a^+} \psi(x) = 1$ and $\lim_{x\to b^-} \psi(x) = 1 + c$. Then, for $0 < c < 1$,

$$E\left[\psi(X) \mid X \le x\right] = c + (1 - c)\, \psi(x), \tag{17}$$

if and only if

$$F(x) = \left(\frac{\psi(x) - 1}{c} \right)^{\frac{1-c}{c}}. \tag{18}$$

Proof. If (17) holds, then

$$\int_a^x \psi(u) f(u)\, du = \{c + (1 - c)\, \psi(x)\} F(x).$$

Differentiating both sides of the above equation with respect to x and rearranging the terms, we arrive at

$$\frac{f(x)}{F(x)} = \frac{1-c}{c}\left(\frac{\psi'(x)}{\psi(x)-1}\right). \tag{19}$$

Integrating both sides of (19) from x to b and using the condition $\lim_{x\to b^-}\psi(x) = 1+c$, we arrive at (18).

Conversely, if (18) holds, then $\psi(x) = 1 + c(F(x))^{\frac{c}{1-c}}$ and

$$\begin{aligned} E[\psi(X) \mid X \le x] &= \frac{\int_a^x \left\{1 + c(F(u))^{\frac{c}{1-c}}\right\} f(u)\,du}{F(x)} \\ &= \frac{F(x) + c(1-c)(F(x))^{\frac{1}{1-c}}}{F(x)} \\ &= 1 + c(1-c)(F(x))^{\frac{c}{1-c}} \\ &= 1 + (1-c)(\psi(x)-1) \\ &= c + (1-c)\psi(x), \end{aligned}$$

which is (17).

Remark 6.1. Taking, e.g., $(a,b) = (0,\infty)$ and

$$\psi(x) = 1 + c\left\{\frac{e^{-(\frac{\sigma}{x})\beta}}{\alpha + (1-\alpha)e^{-(\frac{\sigma}{x})\beta}}\left[1 + \lambda - \frac{\lambda e^{-(\frac{\sigma}{x})\beta}}{\alpha + (1-\alpha)e^{-(\frac{\sigma}{x})\beta}}\right]\right\}^{\frac{c}{1-c}}.$$

Proposition 6.3 gives a characterization of TMOFr distribution.

7. Maximum Likelihood Estimation

Several approaches for parameter estimation were proposed in the literature but the maximum likelihood method is the most commonly employed. The MLEs enjoy desirable properties and can be used when constructing confidence intervals and regions and also in test statistics. The normal approximation for these estimators in large sample distribution theory is easily handled either analytically or numerically. In this section, we consider the estimation of the parameters of the TMOFr$(\alpha,\beta,\sigma,\lambda,x)$ distribution by maximum likelihood. Consider the random sample $x_1,\ldots,x_n$ of size n from this distribution. The log-likelihood function for the parameter vector $\phi = (\alpha,\beta,\sigma,\lambda)^\intercal$, say (ϕ), is given by

$$\begin{aligned} (\phi) &= n(\log\alpha + \ln\beta - \log\sigma) - \sum_{i=1}^n \left(\frac{\sigma}{x_i}\right)^\beta + (\beta+1)\sum_{i=1}^n \log\left(\frac{\sigma}{x_i}\right) \\ &\quad -3\sum_{i=1}^n \log\left[\alpha + (1-\alpha)e^{-\left(\frac{\sigma}{x_i}\right)^\beta}\right] + \sum_{i=1}^n \log\left[\alpha(\lambda+1) - pe^{-\left(\frac{\sigma}{x_i}\right)^\beta}\right], \end{aligned}$$

where $p = \lambda(\alpha+1) + \alpha - 1$. The above equation can be maximized either directly by using the `MATH-CAD` program, `R` (`optim` function), `SAS` (`PROC NLMIXED`) or by solving the nonlinear equations obtained by differentiating the log-likelihood. Therefore, the corresponding score function, say $\mathbf{U}(\phi) = \frac{\partial(\phi)}{\partial\phi}$, is given by $\mathbf{U}(\phi) = \left(\frac{\partial(\phi)}{\partial\alpha}, \frac{\partial(\phi)}{\partial\beta}, \frac{\partial(\phi)}{\partial\sigma}, \frac{\partial(\phi)}{\partial\lambda}\right)^\intercal$. Then,

$$\frac{\partial(\phi)}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^n \frac{3 - 3e^{-\left(\frac{\sigma}{x_i}\right)^\beta}}{\alpha + (1-\alpha)e^{-\left(\frac{\sigma}{x_i}\right)^\beta}} + \sum_{i=1}^n \frac{1 + \lambda - (1+\lambda)e^{-\left(\frac{\sigma}{x_i}\right)^\beta}}{\alpha(\lambda+1) - pe^{-\left(\frac{\sigma}{x_i}\right)^\beta}},$$

$$\frac{\partial(\phi)}{\partial\beta} = \sum_{i=1}^n \left[1 - \left(\frac{\sigma}{x_i}\right)^\beta\right]\log\left(\frac{\sigma}{x_i}\right) + \sum_{i=1}^n \frac{p\left(\frac{\sigma}{x_i}\right)^\beta e^{-\left(\frac{\sigma}{x_i}\right)^\beta}}{\alpha(\lambda+1) - pe^{-\left(\frac{\sigma}{x_i}\right)^\beta}}\log\left(\frac{\sigma}{x_i}\right) + \frac{n}{\beta} + \sum_{i=1}^n \frac{3(1-\alpha)\left(\frac{\sigma}{x_i}\right)^\beta e^{-\left(\frac{\sigma}{x_i}\right)^\beta}}{\alpha + (1-\alpha)e^{-\left(\frac{\sigma}{x_i}\right)^\beta}}\log\left(\frac{\sigma}{x_i}\right),$$

$$\frac{\partial(\phi)}{\partial\sigma}=\frac{1}{\sigma}\left[n-\beta\sum_{i=1}^{n}\left(\frac{\sigma}{x_i}\right)^{\beta}\right]-\frac{\beta}{\sigma}\sum_{i=1}^{n}\frac{p\left(\frac{\sigma}{x_i}\right)^{\beta}e^{-\left(\frac{\sigma}{x_i}\right)^{\beta}}}{\alpha(\lambda+1)-pe^{-\left(\frac{\sigma}{x_i}\right)^{\beta}}}+\frac{\beta}{\sigma}\sum_{i=1}^{n}\frac{3(1-\alpha)e^{-\left(\frac{\sigma}{x_i}\right)^{\beta}}}{\alpha+(1-\alpha)e^{-\left(\frac{\sigma}{x_i}\right)^{\beta}}}\left(\frac{\sigma}{x_i}\right)^{\beta}$$

and

$$\frac{\partial(\phi)}{\partial\lambda}=\sum_{i=1}^{n}\frac{\alpha-(1+\alpha)e^{-\left(\frac{\sigma}{x_i}\right)^{\beta}}}{\alpha(\lambda+1)-pe^{-\left(\frac{\sigma}{x_i}\right)^{\beta}}}.$$

We can obtain the estimates of the unknown parameters by setting the score vector to zero, $\mathbf{U}(\widehat{\phi})=0$. Solving these equations simultaneously gives the MLEs $\widehat{\alpha},\widehat{\beta},\widehat{\sigma}$ and $\widehat{\lambda}$. If they can not be solved analytically and statistical software can be used to solve them numerically by means of iterative techniques such as the Newton-Raphson algorithm. For the TMOFr distribution all the second order derivatives exist.

For interval estimation of the model parameters, we require the 4×4 observed information matrix $J(\phi)=\{J_{rs}\}$ for $r,s=\alpha,\beta,\sigma,\lambda$. Under standard regularity conditions, the multivariate normal $N_4(0,J(\widehat{\phi})^{-1})$ distribution can be used to construct approximate confidence intervals for the model parameters. Here, $J(\widehat{\phi})$ is the total observed information matrix evaluated at $\widehat{\phi}$. Therefore, approximate $100(1-\varphi)\%$ confidence intervals for α,β,σ and λ can be determined as:

$\widehat{\alpha}\pm z_{\frac{\varphi}{2}}\sqrt{\widehat{J_{\alpha\alpha}}}$, $\widehat{\beta}\pm z_{\frac{\varphi}{2}}\sqrt{\widehat{J_{\beta\beta}}}$, $\widehat{\sigma}\pm z_{\frac{\varphi}{2}}\sqrt{\widehat{J_{\sigma\sigma}}}$ and $\widehat{\lambda}\pm z_{\frac{\varphi}{2}}\sqrt{\widehat{J_{\lambda\lambda}}}$, where $z_{\frac{\varphi}{2}}$ is the upper φth percentile of the standard normal distribution.

8. Applications

In this section, We provide two applications to two real data sets to prove the flexibility of the TMOFr model. We compare the fit of the TMOFr with competitve models namely: MOFr, BFr, GEFr, TFr and Fr distributions. The pdfs of these distributions are, respectively, given by (for $x>0$):

MOFr: $f(x)=\alpha\beta\sigma^{\beta}x^{-(\beta+1)}e^{-\left(\frac{\sigma}{x}\right)^{\beta}}\left[\alpha+(1-\alpha)e^{-\left(\frac{\sigma}{x}\right)^{\beta}}\right]^{-2}$;

BFr: $f(x)=\frac{\beta\sigma^{\beta}}{B(a,b)}x^{-(\beta+1)}e^{-a\left(\frac{\sigma}{x}\right)^{\beta}}\left[1-e^{-\left(\frac{\sigma}{x}\right)^{\beta}}\right]^{b-1}$;

GEFr: $f(x)=\frac{a\beta\sigma^{\beta}}{\Gamma(b)}x^{-(\beta+1)}e^{-\left(\frac{\sigma}{x}\right)^{\beta}}\left[1-e^{-\left(\frac{\sigma}{x}\right)^{\beta}}\right]^{a-1}\left\{-\log\left[1-e^{-\left(\frac{\sigma}{x}\right)^{\beta}}\right]^{a}\right\}^{b-1}$;

TFr: $f(x)=\beta\sigma^{\beta}x^{-(\beta+1)}e^{-\left(\frac{\sigma}{x}\right)^{\beta}}\left[1+\lambda-2\lambda e^{-\left(\frac{\sigma}{x}\right)^{\beta}}\right]$;

Fr: $f(x)=\beta\sigma^{\beta}x^{-(\beta+1)}e^{-\left(\frac{\sigma}{x}\right)^{\beta}}$.

The parameters of the above densities are all positive real numbers except for the TFr distribution for which $|\lambda|\leq1$.

The first data set refers to breaking stress of carbon fibres (in Gba) (Nichols and Padgett, 2006) and consists of 100 observations: 3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2.00, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.80, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65. The second data set is obtained from Smith and Naylor (1987). The data are the strengths of 1.5 cm glass fibres, measured at the National Physical Laboratory, England. Unfortunately, the units of measurement are not given in the paper. The data set consisting of 63 observations are: 0.55, 0.93,1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2 , 0.74, 1.04, 1.27, 1.39, 1.49, 1.53,1.59, 1. 61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.5, 1.54, 1.6, 1.62, 1.66, 1.69,1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.5, 1.55, 1.61, 1.62, 1.66, 1.7, 1.77, 1.84, 0.84, 1.24, 1.3, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.7, 1.78, 1.89.

In order to compare the distributions, we consider the measures of goodness-of-fit including the Akaike information criterion (AIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion ($HQIC$) and consistent

Akaike information criterion ($CAIC$). These measures of goodness-of-fit evaluated at the MLEs$\widehat{\ }$, where$\widehat{\ }$is the maximized log-likelihood.

We also consider the Cramér–von Mises (W^*) and Anderson–Darling (A^*) statistics. The statistics W^* and A^* are described in details in Chen and Balakrishnan (1995). In general, the smaller the values of these statistics, the better the fit to the data.

In Table 2, we list the numerical values of the statistics W^*, A^*, AIC, BIC, $HQIC$ and $CAIC$ using the two real data sets (DS), whereas the MLEs and their corresponding standard errors and the statistics of the model parameters are shown in Table 3. These numerical results are obtained using the MATH-CAD program.

Table 2. The statistics W^*, A^*, AIC, BIC, $HQIC$ and $CAIC$ for the two data sets

DS	Models	W^*	A^*	AIC	BIC	$HQIC$	$CAIC$
	TMOFr	0.2376	1.26771	309.973	320.393	314.19	310.394
	BFr	0.25137	1.39536	311.133	321.553	315.35	311.554
	GEFr	0.25872	1.43853	311.96	332.381	316.178	312.381
I	MOFr	0.59267	3.38252	351.328	359.143	354.491	351.578
	TFr	0.55598	3.17823	350.475	358.29	353.638	350.725
	Fr	0.54849	3.13643	348.308	353.519	350.417	348.432
	TMOFr	0.56541	3.10166	56.46	65.032	59.831	57.149
	MOFr	0.59607	3.2897	57.08	63.509	59.609	57.487
II	BFr	0.76879	4.20206	68.63	77.202	72.002	69.32
	GEFr	0.78121	4.27204	69.557	78.13	72.929	70.247
	TFr	1.17022	6.45074	100.078	106.507	102.606	100.484
	Fr	1.16252	6.40749	97.707	101.993	99.392	97.907

The figures in Table 2 indicate that the TMOFr model has the smallest values of the statistics W^*, A^*, AIC, BIC, $HQIC$ and $CAIC$ except BIC and $HQIC$ for the second data set. Hence, it can be chosen as the best model among all fitted models. Based on these criteria in Table 2, we conclude that the TMOFr distribution provides a better fit than the other models.

9. Concluding Remarks

In this paper, we propose a new four-parameter model, called the *transmuted Marshall-Olkin Fréchet* (TMOFr) distribution, which extends the Marshall-Olkin Fréchet (MOFr) distribution introduced by Krishna et al. (2013). In fact, the TMOFr distribution is motivated by the wide use of the Fréchet distribution in practice and also in view of the fact that the generalization provides more flexibility to analyze real life data. We provide some of its mathematical properties. The density function of TMOFr can be expressed as a mixture of Fréchet densities. We derive explicit expressions for the ordinary and incomplete moments, residual life and reversed residual life functions, quantile and generating functions, Rényi and q-entropies. We obtain the density function of order statistics and their moments. We discuss the maximum likelihood estimation of the model parameters. Two applications illustrate that the proposed distribution provides consistently better fit than other non-nested models.

Appendix A

Theorem A. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a given probability space and let $H = [d, e]$ be an interval for some $d < e$ ($d = -\infty$, $e = \infty$ might as well be allowed). Let $X : \Omega \to H$ be a continuous random variable with the distribution function F and let g and h be two real functions defined on H such that

$$\mathbf{E}\left[g(X) \mid X \geq x\right] = \mathbf{E}\left[h(X) \mid X \geq x\right]\eta(x), \quad x \in H,$$

is defined with some real function η. Assume that $g, h \in C^1(H)$, $\eta \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $h\eta = g$ has no real solution in the interior of H. Then F is uniquely determined by the functions g, h and η, particularly

$$F(x) = \int_a^x C\left|\frac{\eta'(u)}{\eta(u)h(u) - g(u)}\right|\exp(-s(u))\,du,$$

where the function s is a solution of the differential equation $s' = \frac{\eta' h}{\eta h - g}$ and C is the normalization constant, such that $\int_H dF = 1$.

Table 3. MLEs and their standard errors (in parentheses) for the two data sets

DS	Models	E			stimates
	TMOFr($\beta, \sigma, \alpha, \lambda$)	3.3313	0.6496	101.923	0.2936
		(0.206)	(0.068)	(47.625)	(0.27)
	BFr(β, σ, a, b)	0.4046	1.6097	22.0143	29.7617
		(0.108)	(2.498)	(21.432)	(17.479)
	GEFr(β, σ, a, b)	0.4776	1.3692	27.6452	17.4581
I		(0.133)	(2.017)	(14.136)	(14.818)
	MOFr(β, σ, α)	1.5796	2.3066	0.5988	
		(0.16)	(0.489)	(0.3091)	
	TFr(β, σ, λ)	1.7435	1.9315	0.0819	
		(0.076)	(0.097)	(0.198)	
	Fr(β, σ)	1.7766	1.8705		
		(0.113)	(0.112)		
	TMOFr($\beta, \sigma, \alpha, \lambda$)	6.8744	0.65	376.268	0.1499
		(0.596)	(0.049)	(246.832)	(0.302)
	MOFr(β, σ, α)	6.4655	0.6812	161.6114	
		(0.559)	(0.045)	(91.499)	
	BFr(β, σ, a, b)	0.6466	2.0518	15.0756	36.9397
II		(0.163)	(0.986)	(12.057)	(22.649)
	GEFr(β, σ, a, b)	0.7421	1.6625	32.112	13.2688
		(0.197)	(0.952)	(17.397)	(9.967)
	TFr(β, σ, λ)	2.7898	1.3068	0.1298	
		(0.165)	(0.034)	(0.208)	
	Fr(β, σ)	2.8876	1.2643		
		(0.234)	(0.059)		

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