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A Note on $\mathbb{L}_2$-structure of Continuous-time Bilinear Processes with Time-varying Coefficients

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Abstract

This paper is concerned with the investigation of $\mathbb{L}_2$-structure issue of time-varying coefficients continuous-time bilinear processes (*COBL*) driven by a Brownian motion (*BM*). Such processes are very useful for modeling irregular spacing non linear and non Gaussian datasets and may be proposed to model for instance some financial returns representing high amplitude oscillations and thus make it a serious candidate for describe processes with time-varying degree of persistence and other complex systems. Our attention is focused however on the probabilistic structure of *COBL* processes, so, we establish necessary and sufficient conditions for the existence of regular solutions in term of their transfer function. Explicite formulas for the mean and covariance functions are given. As a consequence, we observe that the second order structure is similar to a *CARMA* processes with some uncorrelated noise. Therefore, it is necessary to look into higher-order cumulant in order to distinguish between *COBL* and *CARMA* processes.

Keywords: nonlinear SDE, continuous-time bilinear processes, CARMA, transfer function

1. Introduction

Discrete-time series such as the well-known *ARMA* models, provide an effective and tractable tool to describe many datasets assumed to be equally spaced. However, in many practical situations, the data generating the process, are often observed irregularly spaced. This phenomenon happens for instance in physics, engineering problems, economy and so on. Therefore, the resort to continuous-time (which can be interpreted as a solution of some stochastic differential equations (*SDE*)) models is unavoidable. These models are often assumed to be linear and may be Gaussian which continue to gain a growing interest of researchers (see for instance Brockwell (2001) and the references therein). However, recent studies have been shown that the linearity and/or Gaussianity assumptions is very unrealistic. So, various nonlinear models have been widely proposed in economics, sociology and in industrial panel data in order to describe these features. Indeed, a continuous-time *GARCH* (*COGARCH*) process, was recently introduced and studied by Kluppelberg et al. (2004) and by Brockwell et al. (2006). A general class of non-linear continuous-time Autoregressive and threshold (*CTAR*) and threshold *ARMA* processes are constructed and briefly discussed by Brockwell (2001). A continuous-time bilinear models (*COBL*) was introduced firstly by Mohler (1973) in control theory and has popularized in time series analysis by Lebreton and Musiela (1984), Subba Rao and Terdik (2003) and by Iglói and Terdik (1999).

The main purpose of this paper is, on hand, to generalize the time-invariant *COBL* model proposed by Iglói and Terdik (1999) to time-varying one and on other hand, to extend the results by Lebreton and Musiela (1984) in time domain to frequency domain. So, in the next section, we present a powerful frame for studying the nonlinear *SDE* in terms of their transfer function. This approach allows us to distinguish between linear and nonlinear and between regular and singular solutions. Section 3, describes the *COBL* equation with respect to its evolutionary (time-dependent) transfer function, so in Section 4 we use this representation to give sufficient and necessary conditions ensuring the existence of second-order regular solutions. In section 5, an exact expression for the covariance function and the spectral density are given and it is shown that the second-order structure is the same as a *CARMA* process with time-varying coefficients. This result, we then conduct to investigate the third-order cumulants and showing that the bispectrum is zero if the process is linear. In Section 6, we conclude and discuss possible extensions.

2. Wiener's Chaos Representation

Let $(\epsilon(t))_{t\geq 0}$ be a real Brownian motion defined on some filtered space $(\Omega, \mathcal{A}, (\mathcal{A}_t)_{t\geq 0}, P)$ with associated spectral representation $\epsilon(t) = \int_{\mathbb{R}} \frac{e^{it\lambda}-1}{i\lambda} dZ(\lambda)$, where $dZ(\lambda)$ is an orthogonal complex-valued stochastic measure on $\mathbb{R}$ with zero mean, $E\left\{|dZ(\lambda)|^2\right\} = dF(\lambda) = \frac{d\lambda}{2\pi}$ and uniquely determined by $Z([a,b[) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-i\lambda a}-e^{-i\lambda b}}{i\lambda} d\epsilon(\lambda)$, for all $-\infty < a < b < +\infty$. So the process $(Z([0,t[))_{t>0}$ is also a Brownian motion. Consider the Hilbert space $\mathcal{H} = \mathbb{L}_2(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, F)$ of the complex squared integrable functions f satisfying $f(\lambda) = \overline{f(-\lambda)}$ for any $\lambda \in \mathbb{R}$ where $\overline{f(\lambda)}$ denotes the complex conjugate of $f(\lambda)$. For any $n \geq 1$, we associated three real Hilbert spaces based on $\mathcal{H}$. The first is $\mathcal{H}_n = \mathcal{H}^{\otimes n}$ the n-fold tensor product of $\mathcal{H}$ endowed by the inner product $< f, f >= \int_{\mathbb{R}^n} f(\lambda_{(n)})\overline{f(\lambda_{(n)})} dF(\lambda_{(n)})$ where $\lambda_{(n)} = (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{R}^n$, with $\lambda_{(0)} = 0$, $dF(\lambda_{(n)}) = \prod_{i=1}^{n} dF(\lambda_i)$ and $f(\lambda_{(n)}) = \overline{f(-\lambda_{(n)})}$ such that $\|f\|^2 = \int_{\mathbb{R}^n} \left|f(\lambda_{(n)})\right|^2 dF(\lambda_{(n)}) < \infty$. The second one is $\widehat{\mathcal{H}}_n = \mathcal{H}^{\oplus n} \subset \mathcal{H}_n$ the $n-$fold symmetrized tensor product of $\mathcal{H}$ defined by $f \in \widehat{\mathcal{H}}_n$ if and only if f is invariant under permutation of their arguments i.e., $f(\lambda_{(n)}) = Sym\{f(\lambda_{(n)})\}$ where $Sym\{f(\lambda_{(n)})\} = \widetilde{f}(\lambda_{(n)}) = \frac{1}{n!}\sum_{p\in\mathcal{P}_n} f(\lambda_{(p(n))})$, $\mathcal{P}_n$ denotes the group of all permutations of the set $\{1, 2, ..., n\}$, and we endowed $\widehat{\mathcal{H}}_n$ with the inner product $< f, f >_{\oplus} = n! < f, f >$ for $f, f \in \widehat{\mathcal{H}}_n$. The last space is called Fock-space over $\mathcal{H}$ denoted by $\mathfrak{I}(\mathcal{H})$ defined by $\mathfrak{I}(\mathcal{H}) = \bigoplus_{r=0}^{\infty} \frac{1}{r!}\widehat{\mathcal{H}}_r$ with $\widehat{\mathcal{H}}_0 = \mathcal{H}_0 = \mathbb{R}$ ($\bigoplus$ denotes the direct orthogonal sum) whose elements $\underline{\mathbf{f}} = (f(\lambda_{(r)}), r \geq 0)$ fulfilled the condition

$$\left\|\underline{\mathbf{f}}\right\|^2 = \sum_{r\geq 0} \frac{1}{r!} \int_{\mathbb{R}^r} \left|f(\lambda_{(r)})\right|^2 dF(\lambda_{(r)}) < \infty. \tag{2.1}$$

Let $\mathfrak{I} = \mathfrak{I}(\epsilon) := \sigma(\epsilon(t), t \geq 0)$ (resp $\mathfrak{I}_{\leq t}(\epsilon) := \sigma(\epsilon(s), s \leq t)$) be the σ-algebra generated by $(\epsilon(t))_{t\geq 0}$ (resp. generated by $\epsilon(s)$ up to time t) and let $\mathbb{L}_2(\mathfrak{I}) = \mathbb{L}_2(\mathbb{R}, \mathfrak{I}, P)$ be the real Hilbert space of nonlinear $\mathbb{L}_2-$functional of $(\epsilon(t))_{t\geq 0}$. It is well known (see Major (1981) and Bibi (2006)) that $\mathbb{L}_2(\mathfrak{I})$ is isometrically isomorphic to Fock-space $\mathfrak{I}(\mathcal{H})$, so, for any random process $(X(t))_{t\in\mathbb{R}}$ (not necessary stationary) of $\mathbb{L}_2(\mathfrak{I})$ admits the so-called Wiener-Itô orthogonal representation

$$X(t) = f_t(0) + \sum_{r\geq 1} \frac{1}{r!} \int_{\mathbb{R}^r} e^{it\underline{\lambda}_{(r)}} f_t(\lambda_{(r)}) dZ(\lambda_{(r)}) \tag{2.2}$$

where $\underline{\lambda}_{(r)} = \sum_{i=1}^{r} \lambda_i$ and the integrals are the multiple Wiener-Itô stochastic integrals with respect to the stochastic measure $dZ(\lambda)$, $f_t(0) = E\{X(t)\}$, $dZ(\lambda_{(r)}) = \prod_{i=1}^{r} dZ(\lambda_i)$ and $f(\lambda_{(r)}) \in \widehat{\mathcal{H}}_r$ are referred as the $r-th$ evolutionary transfer function of $(X(t))_{t\in\mathbb{R}}$, uniquely determined and fulfill the condition $\sum_{r\geq 0} \frac{1}{r!} \int_{\mathbb{R}^r} \left|f_t(\lambda_{(r)})\right|^2 dF(\lambda_{(r)}) < \infty$. As a property of the representation (2.2) is that for any $f_t \in \mathcal{H}_n$ and $f_s \in \mathcal{H}_m$, we have

$$E\left\{\int_{\mathbb{R}^n} f_t(\lambda_{(n)}) dZ(\lambda_{(n)}) \int_{\mathbb{R}^m} f_s(\lambda_{(m)}) dZ(\lambda_{(m)})\right\} = \delta_n^m n! \int_{\mathbb{R}^n} \widetilde{f}_t(\lambda_{(n)}) \overline{\widetilde{f}_s(\lambda_{(n)})} dF(\lambda_{(n)}) \tag{2.3}$$

where δ_n^m is the delta function. This means that the spaces $\widehat{\mathcal{H}}_n$ are orthogonal. Two interesting properties related to the multiple Wiener-Itô stochastic integrals which is important for future use are the diagram and Itô formulas summarized in the following lemma due to Dobrushin (1979).

Lemma 2.1 *Let $(\varphi_i)_{1\leq i\leq k}$ be an orthonormal system in $\mathcal{H}$, $\left(n_j\right)_{1\leq j\leq k}$ is a sequence of positive integers such that $n = n_1 + ... + n_k$ and let h_j be the $j-th$ Hermite polynomial with highest coefficient 1, i.e., $h_j(x) = (-1)^j e^{\frac{x^2}{2}} \frac{d^j}{dx^j} e^{-\frac{x^2}{2}}$. Then,*

1. *The Itô's formula states that*

$$\prod_{j=1}^{k} h_{n_j}\left(\int_{\mathbb{R}} \varphi_j(\lambda) dZ(\lambda)\right) = \int_{\mathbb{R}^n} \prod_{j=1}^{k} \prod_{i=1}^{n_j} \varphi_j(\lambda_{\underline{n}_{(j-1)}+i}) dZ(\lambda_{(n)}) = \int_{\mathbb{R}^n} Sym\left\{\prod_{j=1}^{n} \varphi_j(\lambda_j)\right\} dZ(\lambda_{(n)})$$

2. *The diagram formulas state that for any $f \in \mathcal{H}_1$ and $g \in \mathcal{H}_n$ we have*

$$\int_{\mathbb{R}} f(\lambda)dZ(\lambda) \int_{\mathbb{R}^n} g\left(\lambda_{(n)}\right) dZ(\lambda_{(n)}) = \int_{\mathbb{R}^{n+1}} g\left(\lambda_{(n)}\right) f\left(\lambda_{n+1}\right) dZ(\lambda_{(n+1)}) + \sum_{k=1}^{n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} g\left(\lambda_{(n)}\right) \overline{f\left(\lambda_k\right)} dF\left(\lambda_k\right) dZ(\lambda_{(n\backslash k)})$$

where $dZ(\lambda_{(n\backslash k)}) = dZ\left(\lambda_1\right)...dZ\left(\lambda_{k-1}\right).dZ\left(\lambda_{k+1}\right)...dZ\left(\lambda_n\right)$.

Remark 2.2 For any $n \geq 0$, let $\mathcal{K}_n$ be the closed vector subspace of $\mathbb{L}_2(\mathfrak{I})$ spanned by $\{h_n(X), X \in \mathfrak{I} : Var(X) = 1\}$ with $\mathcal{K}_0 = \{C^t\}$. Then, it can be shown that the subspaces $\mathcal{K}_n$ are mutually orthogonal and $\mathbb{L}_2(\mathfrak{I}) = \bigoplus_{n=0}^{\infty} \mathcal{K}_n$ (see Peccati and Taqqu (2011)). Noting that the condition $Var(X) = 1$ in the definition of the subspaces $(\mathcal{K}_n)_{n\geq 0}$ is necessary. Indeed, $h_2(X) = X^2 - 1$ is orthogonal to $h_0(X) = C^t$, if and only if $E\{X^2 - 1\} = 0$.

Noting here that the representation (2.2) is referred as regular solution if $\mathfrak{I}_{\leq t}(X) \subset \mathfrak{I}_{\leq t}(\epsilon)$ i.e., the process $X(t)$ depends only on the past of the noise $\epsilon(t)$. So, a stochastic process $(X(t))_{t\in\mathbb{R}}$ having a representation (2.2) has a regular (or also causal) second-order solution if for all $t \in \mathbb{R}$ and every $r \in \mathbb{N}$, the evolutionary transfer functions $f_t(\lambda_{(r)})$ satisfies Szego's condition, i.e., $\forall t \in \mathbb{R}$: $\int_{\mathbb{R}^r} \frac{1}{1 + \left\|\lambda_{(r)}\right\|^2} \log\left(\left|f_t(\lambda_{(r)})\right|\right) dF(\lambda_{(r)}) > -\infty$. (see Ibragimov and Rozanov (1978) for further discussions).

In the following section, a class of non linear diffusion processes admitting the representation (2.2) will be investigated.

3. Evolutionary Transfer Functions of Time-varying COBL Processes

The representation (2.2) can be describe general nonlinear stochastic differential equation (SDE) with a great accuracy and can be enlarged to include the processes $(X(t))_{t\in\mathbb{R}}$ solving the following SDE

$$dX(t) = f\left(X^{(s)}(t), \epsilon^{(r)}(t), 0 \leq s \leq p, 0 \leq r \leq q\right) dt + d\epsilon(t) \tag{3.1}$$

in which the superscript $^{(k)}$ denotes the $k-$fold differentiation with respect to t and some measurable function f. The derivative $\epsilon^{(j)}(t)$, $j > 0$ do not exist in usual sense, hence it can be interpret in the Itô sense. The main objective here is to derive the evolutionary transfer functions system associated with some subclass of (3.1) and we establish necessary and sufficient conditions ensuring the existence of second$-$order regular solutions. More precisely, we shall restrict ourself to the diffusion processes $(X(t))_{t\in\mathbb{R}}$ generated by the following SDE,

$$dX(t) = (\alpha(t)X(t) + \mu(t))\, dt + (\gamma(t)X(t) + \beta(t))\, d\epsilon(t), t \geq 0, X(0) = X_0 \tag{3.2}$$

in which $\alpha(t)$, $\mu(t)$, $\gamma(t)$ and $\beta(t)$ are measurable deterministic functions to the conditions $\forall t \geq 0$, $\alpha(t) \neq 0$, $\gamma(t)\mu(t) \neq \alpha(t)\beta(t)$ and the following conditions: $\forall T > 0$, $\int_0^T |\alpha(t)|\, dt < \infty$ and $\int_0^T |\mu(t)|\, dt < \infty$, $\int_0^T |\gamma(t)|^2\, dt < \infty$, $\int_0^T |\beta(t)|^2\ dt < \infty$.The initial state $X(0)$ is a random variable, defined on $(\Omega, \mathcal{A}, P)$, independent of ϵ such that $E\{X(0)\} = m(0)$ and $Var\{X(0)\} = K(0)$. Equation (3.2) is called continuous$-$time bilinear ($COBL(1, 1)$) (resp. linear) SDE whenever $\gamma(t) \neq 0$ (resp. $\gamma(t) = 0$) for all $t > 0$, in other words, when the solution is not Gaussian or it is.

SDE given by (3.2) encompasses many commonly used models in the literature. Some specific examples among others are:

1. $COGARCH(1, 1)$: This classes of processes is defined as a SDE by $dX(t) = \sigma(t)\, dB_1(t)$ with $d\sigma^2(t) = \theta\left(\gamma - \sigma^2(t)\right) dt + \rho\sigma^2(t)\, dB_2(t), t > 0$ where B_1 and B_2 are independent Brownian motions. So, the stochastic volatility equation may be regarded as particular case of (3.2) by assuming constant the functions $\alpha(t)$, $\mu(t)$, $\gamma(t)$ and $\beta(t)$ with $\beta(t) = 0$ for all $t \geq 0$. (see Kluppelberg et al. (2004) and the reference therein).

2. $CAR(1)$: This classes of SDE may be obtained by assuming $\gamma(t) = 0$ for all $t \geq 0$. (see Brockwell (2001) and the reference therein)

3. Gaussian Ornstein-Uhlenbeck (GOU) process: The GOU process is defined as $dX(t) = \gamma(\mu - X(t))\,dt + \beta d\epsilon(t), t \geq 0$. So it can be obtained from (3.2) by assuming constant the functions $\alpha(t)$, $\mu(t)$, $\gamma(t)$ and $\beta(t)$ with $\gamma(t) = 0$ for all $t \geq 0$. (see Brockwell (2001) and the reference therein).

The solution of equation (3.2) may be obtained according to :

3.1 The Time Domain Approach

The existence and uniqueness of the Itô solution process $(X(t))_{t\geq 0}$ of equation (3.2) in time domain is ensured by the general results on stochastic differential equations and under the above assumptions (see Lebreton and Musiela (1984))

$$X(t) = \varphi(t)\left\{X(0) + \int_0^t \varphi^{-1}(s)\left(\mu(s) - \gamma(s)\beta(s)\right)ds + \int_0^t \varphi^{-1}(s)\beta(s)\,d\epsilon(s)\right\}$$

where $\varphi(t) = \exp\left\{\int_0^t \left(\alpha(s) - \frac{1}{2}\gamma^2(s)\right)ds + \int_0^t \gamma(s)\,d\epsilon(s)\right\}$, $t \geq 0$, which reduce to that given by Iglói and Terdik (1999) in constant coefficients case and provide a solution for non-stationary Gaussian Ornstein-Uhlenbeck process corresponding to the case when $\gamma(s) = 0$ for all s. In this case we obtain

$$X(t) = \psi(t)\left\{X(0) + \int_0^t \varphi^{-1}(s)\mu(s)ds + \int_0^t \varphi^{-1}(s)\beta(s)\,d\epsilon(s)\right\} \tag{3.3}$$

where $(\psi(t))_{t\geq 0}$ is the mean function of $(\varphi(t))_{t\geq 0}$ i.e., $\psi(t) = \exp\left\{\int_0^t \alpha(s)ds\right\}$, $t \geq 0$ and the stochastic integral on the right-hand side of (3.3) has an expected value zero and by (2.3) we obtain

$$E\left\{\int_0^t f(s)\,d\epsilon(s)\int_0^t g(s')d\epsilon(s')\right\} = \left\{\int_0^t f(s)\,g(s)d(s)\right\}$$

for any squared integrable functions f and g with respect to Lebesgue measure on $[0, t]$.

3.2 The Frequency Domain Approach

In frequency domain, we have

Theorem 3.1

Assume that the process $(X(t))_{t\geq 0}$ generated by the SDE (3.2) has a regular second-order solution. Then the evolutionary symmetrized transfer functions $f_t(\lambda_{(r)}), (t, r) \in \mathbb{R}\times\mathbb{N}$ of this solution are given by symmetrization of the following differential equations of order 1*,*

$$f_t^{(1)}(\lambda_{(r)}) = \begin{cases} \alpha(t)f_t(0) + \mu(t), & \text{if } r = 0 \\ \left(\alpha(t) - i\underline{\lambda}_{(r)}\right)f_t(\lambda_{(r)}) + r\left(\gamma(t)f_t(\lambda_{(r-1)}) + \delta_{\{r=1\}}\beta(t)\right), & \text{if } r \geq 1 \end{cases} \tag{3.4}$$

Proof. Suppose that the system (3.2) has a regular second-order solution of the form (2.2) with evolutionary symmetrized transfer functions $f_t(\lambda_{(r)}), (t, r) \in \mathbb{R}\times\mathbb{N}$, then by putting the solution (2.2) into (3.2) and using the diagram formula we obtain
$X(t)d\epsilon(t)$

$$\begin{aligned}
&= \left\{f_t(0) + \sum_{r=1}^{\infty}\frac{1}{r!}\int_{\mathbb{R}^r} f_t(\lambda_{(r)})e^{it\underline{\lambda}_{(r)}}dZ(\lambda_{(r)})\right\}\int_{\mathbb{R}} e^{it\lambda}dZ(\lambda) \\
&= \int_{\mathbb{R}} f_t(0)e^{it\lambda}dZ(\lambda) \\
&+ \sum_{r=1}^{\infty}\frac{1}{r!}\left\{\int_{\mathbb{R}^{r+1}} f_t(\lambda_{(r)})e^{it\underline{\lambda}_{(r+1)}}dZ(\lambda_{(r+1)}) + r\int_{\mathbb{R}^{r-1}} e^{it\underline{\lambda}_{(r-1)}}\left(\int_{\mathbb{R}} f_t(\lambda_{(r)})dF(\lambda_r)\right)dZ(\lambda_{(r-1)})\right\},
\end{aligned}$$

since a regular solution does not depends on $\epsilon(s), s > t$, and depends on $\epsilon(t)$ linearly, then the last term in above expression is equal to 0, and hence, the recursion (3.4) follows by identifying the $r-th$ order evolutionary transfer functions.□

Remark 3.2 The existence and uniqueness of the evolutionary symmetrized transfer functions $f_t(\lambda_{(r)}), (t,r) \in \mathbb{R}\times\mathbb{N}$ of this solution is ensured by general results on linear ordinary differential equations (see, Kelly (2010), ch. 1) so,

$$f_t(\lambda_{(r)}) = \begin{cases} \varphi_t(0)\left(f_0(0) + \int_0^t \varphi_s^{-1}(0)\mu(s)ds\right) \text{ if } r = 0 \\ \varphi_t\left(\underline{\lambda}_{(r)}\right)\left(f_0(\lambda_{(r)}) + r\int_0^t \varphi_s^{-1}\left(\underline{\lambda}_{(r)}\right)(\gamma(s)f_s(\lambda_{(r-1)}) + \delta_{\{r=1\}}\beta(s))\,ds\right) \text{ if } r \geq 1 \end{cases} \tag{3.5}$$

where $\varphi_t\left(\underline{\lambda}_{(r)}\right) = \exp\left\{\int_0^t \left(\alpha(s) - i\underline{\lambda}_{(r)}\right)ds\right\}$.

Remark 3.3 When $\alpha(t), \mu(t), \gamma(t)$ and $\beta(t)$ are constant, then the recursion (3.4) reduces to

$$f(0) = -\frac{\mu}{\alpha}, \text{ and } f(\lambda_{(r)}) = r\left(i\underline{\lambda}_{(r)} - \alpha\right)^{-1}(\gamma f(\lambda_{(r-1)}) + \delta_{\{r=1\}}\beta) \text{ if } r \geq 1$$

or equivalently $f(\lambda_{(r)}) = r!\gamma^{r-1}\left(\beta - \frac{\mu}{\alpha}\gamma\right)\prod_{j=1}^{r}\left(i\underline{\lambda}_{(j)} - \alpha\right)^{-1}$ and the symmetrized version can be rewritten as $Sym\{f(\lambda_{(r)})\}$ $= (\mu\gamma - \alpha\beta)\gamma^{r-1}\int_0^{+\infty}\exp\{\alpha\lambda\}\prod_{j=1}^{r}\frac{1-\exp\{-i\lambda\lambda_j\}}{i\lambda_j}d\lambda.$

Remark 3.4 Under the coditions specified in Remark 3.3 and when $\gamma = 0$, it is not difficult to see that $(X(t))_{t\geq 0}$ has a unique strictly stationary solution given by $X(t) = f(0) + \int_{\mathbb{R}} g(t-u)d\epsilon(u)$ where $\int_{\mathbb{R}} g(u)e^{-i\lambda u}du = \frac{\beta}{i\lambda-\alpha}, \lambda \in \mathbb{R}$.

4. Condition for the Existence of Regular Solutions

In Theorem 3.1 a first-order ordinary differential equation is derived for evolutionary transfer function of SDE (3.2). In order that these transfer function define a second-order regular solutions, we need to check that its belong to $\Im(\mathcal{H})$ and satisfies the condition (2.1). For this purpose, let $\mathbb{L}_\infty$ (resp. $\mathbb{L}_\infty \otimes \mathbb{L}_\infty$) be the Banach space of infinite dimensional vectors of essentially bounded complex functions on $\mathbb{C}$ (resp. on $\mathbb{C}^2$) and denote by $\underline{\mathbf{f}}(\lambda_{(r)}) = (f_t(\lambda_{(r)}), t \in \mathbb{R}) \in \mathbb{L}_\infty$ for any $\lambda_{(r)} \in \mathbb{R}^r$. Define the following bounded operators $\underline{\alpha} = diag\{\alpha(t)\}$, $\underline{\gamma} = diag\{\gamma(t)\}$, $\underline{\mu} = diag\{\mu(t)\}$ and $\underline{\beta} = diag\{\beta(t)\}$ where "$diag$" denotes the diagonal operator induced by a function in $\mathbb{L}_\infty$, i.e., $(\underline{\alpha}v)_t = \alpha(t)v(t)$ for all $v = (v(t), t \in \mathbb{R}) \in \mathbb{L}_\infty$. Moreover, let D (resp. I) be the differentiation (with respect to t) (resp. identity) operator on $\mathbb{L}_\infty$, i.e., $(Dv)_t = v^{(1)}(t), t \in \mathbb{R}$ (resp $(Iv)_t = v(t)$) for all $v \in \mathbb{L}_\infty$ with derivable components and set $P(z) = Iz - \Phi$ with $\Phi = \underline{\alpha} - D$ (see Dunford and Schwartz, 1963 for further details). With this notation, the functions (3.4) may be rewritten as $\underline{\mathbf{f}}^{(1)}(0) = \underline{\alpha}\underline{\mathbf{f}}(0) + \underline{\mu}\mathbf{1}$, $\underline{\mathbf{f}}^{(1)}(\lambda_{(r)}) = \left(\underline{\alpha} - i\underline{\lambda}_{(r)}I\right)\underline{\mathbf{f}}(\lambda_{(r)}) + r\left(\underline{\gamma}\underline{\mathbf{f}}(\lambda_{(r-1)}) + \delta_{\{r=1\}}\underline{\beta}\mathbf{1}\right)$, or equivalently

$$\underline{\mathbf{f}}(0) = P^{-1}(0)\underline{\mu}\mathbf{1}, \underline{\mathbf{f}}(\lambda) = P^{-1}(i\lambda)\left(\underline{\gamma}\underline{\mathbf{f}}(0) + \underline{\beta}\mathbf{1}\right), \underline{\mathbf{f}}(\lambda_{(r)}) = rP^{-1}\left(i\underline{\lambda}_{(r)}\right)\underline{\gamma}\underline{\mathbf{f}}(\lambda_{(r-1)}), r \geq 2. \tag{4.1}$$

However, it is easily follows from (2.2) that the necessary and sufficient condition for the existence of second-order regular solution of SDE (3.2) is that the components of $\sum_{r\geq 0}\frac{1}{r!}\int_{\mathbb{R}^r}\widetilde{\underline{\mathbf{f}}}(\lambda_{(r)}) \otimes \overline{\widetilde{\underline{\mathbf{f}}}(\lambda_{(r)})}dF(\lambda_{(r)})$ be finite where $\widetilde{\underline{\mathbf{f}}}(\lambda_{(r)}) = Sym\{\underline{\mathbf{f}}(\lambda_{(r)})\}$. Thus we have

Proposition 4.1 *The SDE (3.2) has a regular solution if and only if the following two conditions hold true.*

C1. *The spectrum of the operator Φ lie in $\overline{C} = \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$.*

C2. *The spectrum of the operator $\left(I \otimes \Phi + \Phi \otimes I + \underline{\gamma}^{\otimes 2}\right)^{-1}\left(\underline{\gamma}^{\otimes 2} - \Phi \otimes I - I \otimes \Phi\right)$ lie in $\overline{C}$.*

Moreover, the solution process is unique, bounded up to the second order moments and its infinite dimensional evolutionary transfer functions $\underline{\mathbf{f}}(\lambda_{(r)})$ satisfy the recursion (4.1) is such that $\left\{r!\underline{\mathbf{f}}_t(\lambda_{(r)}), r \in \mathbb{N}\right\} \in \Im(\mathcal{H})$.

To show the Proposition 4.1, we need the following lemmas

Lemma 4.2 *Under the condition **C1** of Proposition 4.1 the functions* $\underline{\mathbf{f}}_t(\lambda_{(r)})$ *and* $\widetilde{\underline{\mathbf{f}}}_t(\lambda_{(r)})$ *are well defined.*

Proof. First we define the tensor product of two bounded linear operators A and B on $\mathbb{L}_\infty$ by $((A \otimes B)(u \otimes v))_{t,s} = (Au)_t \,.\, (Av)_s$ for all $u, v \in \mathbb{L}_\infty$ and note that under **C1**, the operator $P(z)$ has an analytic inverse. For any $r \in \mathbb{N}$, let $\left\|\underline{\mathbf{f}}(r)\right\|$ be the norm of the vector $\underline{\mathbf{f}}(\lambda_{(r)})$, then $\left\|\underline{\mathbf{f}}(0)\right\| \leq \left\|P^{-1}\right\|\left\|\underline{\mu}\right\|, \left\|\underline{\mathbf{f}}(1)\right\| \leq \left\|P^{-1}\right\|\left\|\underline{\mathbf{f}}\right\|$ where $\underline{\mathbf{f}} = \underline{\gamma}\underline{\mathbf{f}}(0) + \underline{\beta}\mathbf{1}$ and $\left\|\underline{\mathbf{f}}(r)\right\| \leq r!\left\|P^{-1}\underline{\gamma}\right\|^{r-1}\left\|\underline{\mathbf{f}}\right\|$ for $r \geq 2$. This mean that $(\underline{\mathbf{f}}(\lambda_{(r)}), r \geq 0)$ is well defined and $\underline{\mathbf{f}}(\lambda_{(r)}) \in \mathbb{L}_\infty$ for all $r \in \mathbb{N}$. Since $\left\|Sym\underline{\mathbf{f}}(r)\right\| \leq \left\|\underline{\mathbf{f}}(r)\right\|$, then $\left(\widetilde{\underline{\mathbf{f}}}(\lambda_{(r)}), r \geq 0\right)$ is also well defined. Now we prove that the coordinates $f_t(\lambda_{(r)})$ of the vector $\underline{\mathbf{f}}(\lambda_{(r)})$ belong to $\Im(\mathcal{H})$ for all $(t, n) \in \mathbb{R} \times \mathbb{N}$ or more generally, the integral $\upsilon_{t,s}(r) = \frac{1}{r!}\int_{\mathbb{R}^r} \widetilde{f_t}(\lambda_{(r)})\,\overline{\widetilde{f_s}(\lambda_{(r)})}dF(\lambda_{(r)})$ (which is the $r-th$ term of $Cov(X_t, X_s)$) exist and finite. Indeed, $\upsilon_{t,s}(r) \leq \frac{1}{r!}\int_{\mathbb{R}^r}\left|\widetilde{f_t}(\lambda_{(r)})\right|\left|\overline{\widetilde{f_s}(\lambda_{(r)})}\right|dF(\lambda_{(r)}) \leq \left(\sup_{\lambda_{(r)}\in\mathbb{R}^r}\left\|\underline{\mathbf{f}}(\lambda_{(r)})\right\|\right)^2$. Since $\upsilon_{t,t}(r) = \frac{1}{r!}\left\|\widetilde{\underline{\mathbf{f}}}(\lambda_{(r)})\right\|^2$, then the functions $f_t(\lambda_{(r)}) \in \Im(\mathcal{H})$ for all $(t, r) \in \mathbb{R} \times \mathbb{N}$.□

In the following lemma an explicit formula for the quantities $\upsilon_{t,s}(r)$ is given.

Lemma 4.3 *Let* $V(r) = (v_{t,s}(r), t, s \in \mathbb{R}) = \frac{1}{r!}\int_{\mathbb{R}^r}\widetilde{\underline{\mathbf{f}}}(\lambda_{(r)}) \otimes \overline{\widetilde{\underline{\mathbf{f}}}(\lambda_{(r)})}dF(\lambda_{(r)}$, *then under the condition* **C1** *of Proposition 4.1 we have* $V(r) = \frac{1}{r!}\int_{\mathbb{R}^r}\widetilde{\underline{\mathbf{f}}}(\lambda_{(r)}) \otimes \overline{\widetilde{\underline{\mathbf{f}}}(\lambda_{(r)})}dF(\lambda_{(r)} = G^{r-1}V(1)$, *where* $G = -(\Phi \otimes I + I \otimes \Phi)^{-1}\underline{\gamma}^{\otimes 2}$ *and* $V(1) = -(\Phi \otimes I + I \otimes \Phi)^{-1}\underline{\mathbf{f}}^{\otimes 2}$.

Proof. First we note that $V(r) \in \mathbb{L}_\infty \otimes \mathbb{L}_\infty$ for all $r \geq 0$. On the other hand from (4.1) we obtain for $r = 1$,

$$V(1) = \int_{\mathbb{R}} \underline{\mathbf{f}}(\lambda) \otimes \overline{\underline{\mathbf{f}}(\lambda)}\frac{d\lambda}{2\pi} = \int_{\mathbb{R}} P^{-1}(i\lambda) \otimes P^{-1}(-i\lambda)\frac{d\lambda}{2\pi}\underline{\mathbf{f}}^{\otimes 2},$$

so by the residue theorem $V(1)$ reduces to $-(\Phi \otimes I + I \otimes \Phi)^{-1}\underline{\mathbf{f}}^{\otimes 2}$. For $r = 2$, we obtain

$$\begin{aligned} V(2) &= \frac{1}{2!}\int_{\mathbb{R}^2}\widetilde{\underline{\mathbf{f}}}(\lambda_{(2)}) \otimes \overline{\widetilde{\underline{\mathbf{f}}}(\lambda_{(2)})}\frac{d\lambda_1 d\lambda_2}{(2\pi)^2} \\ &= \frac{1}{2!}\int_{\mathbb{R}^2} P^{-1}(i\underline{\lambda}_{(2)})\underline{\gamma} \otimes P^{-1}(-i\underline{\lambda}_{(2)})\underline{\gamma}\{P^{-1}(i\lambda_1) \otimes P^{-1}(-i\lambda_1) + P^{-1}(i\lambda_1) \otimes P^{-1}(-i\lambda_2) \\ &+ P^{-1}(i\lambda_2) \otimes P^{-1}(-i\lambda_1) + P^{-1}(i\lambda_2) \otimes P^{-1}(-i\lambda_2)\}\frac{d\lambda_1 d\lambda_2}{(2\pi)^2}\underline{\mathbf{f}}^{\otimes 2} \\ &= \frac{1}{2!}\int_{\mathbb{R}^2} P^{-1}(i\underline{\lambda}_{(2)})\underline{\gamma} \otimes P^{-1}(-i\underline{\lambda}_{(2)})\underline{\gamma}\left\{P^{-1}(i\lambda_1) \otimes P^{-1}(-i\lambda_1) + P^{-1}(i\lambda_2) \otimes P^{-1}(-i\lambda_2)\right\}\frac{d\lambda_1 d\lambda_2}{(2\pi)^2}\underline{\mathbf{f}}^{\otimes 2} \\ &= GV(1). \end{aligned}$$

Now, assume that the Lemma 4.3 is valid up to $r-1$. Then using the identity $\widetilde{\underline{\mathbf{f}}}(\lambda_{(r-1)}) = \frac{1}{r}\sum_{k=1}^{r}\widetilde{\underline{\mathbf{f}}}(\lambda_{(r\backslash k)})$, we obtain after some tedious computations

$$
\begin{aligned}
&\frac{1}{r!}\int_{\mathbb{R}^r} \underline{\widetilde{\mathbf{f}}}(\lambda_{(r)}) \otimes \overline{\underline{\widetilde{\mathbf{f}}}(\lambda_{(r)})} \frac{d\lambda_{(r)}}{(2\pi)^r} \\
&= \frac{1}{r!}\int_{\mathbb{R}^r} P^{-1}(i\underline{\lambda}_{(r)})\underline{\gamma} \otimes P^{-1}(-i\underline{\lambda}_{(r)})\underline{\gamma} r\underline{\widetilde{\mathbf{f}}}(\lambda_{(r-1)}) \otimes \overline{r\underline{\widetilde{\mathbf{f}}}(\lambda_{(r-1)})} \frac{d\lambda_{(r)}}{(2\pi)^r} \\
&= \frac{1}{r!}\sum_{k,l=1}^{r}\int_{\mathbb{R}^r} P^{-1}(i\underline{\lambda}_{(r)})\underline{\gamma} \otimes P^{-1}(-i\underline{\lambda}_{(r)})\underline{\gamma}\underline{\widetilde{\mathbf{f}}}(\lambda_{(r\backslash l)}) \otimes \overline{\underline{\widetilde{\mathbf{f}}}(\lambda_{(r\backslash k)})} \frac{d\lambda_{(r)}}{(2\pi)^r} \\
&= \frac{1}{r!}\sum_{k=1}^{r}\int_{\mathbb{R}^r} P^{-1}(i\underline{\lambda}_{(r)})\underline{\gamma} \otimes P^{-1}(-i\underline{\lambda}_{(r)})\underline{\gamma}\underline{\widetilde{\mathbf{f}}}(\lambda_{(r\backslash k)}) \otimes \overline{\underline{\widetilde{\mathbf{f}}}(\lambda_{(r\backslash k)})} \frac{d\lambda_{(r)}}{(2\pi)^r} \\
&+ \frac{1}{r!}\sum_{k\neq l}^{r}\int_{\mathbb{R}^r} P^{-1}(i\underline{\lambda}_{(r)})\underline{\gamma} \otimes P^{-1}(-i\underline{\lambda}_{(r)})\underline{\gamma}\underline{\widetilde{\mathbf{f}}}(\lambda_{(r\backslash k)}) \otimes \overline{\underline{\widetilde{\mathbf{f}}}(\lambda_{(r\backslash l)})} \frac{d\lambda_{(r)}}{(2\pi)^r} \\
&= \frac{r}{r!}\int_{\mathbb{R}^r} P^{-1}(i\underline{\lambda}_{(r)})\underline{\gamma} \otimes P^{-1}(-i\underline{\lambda}_{(r)})\underline{\gamma}\underline{\widetilde{\mathbf{f}}}(\lambda_{(r-1)}) \otimes \overline{\underline{\widetilde{\mathbf{f}}}(\lambda_{(r-1)})} \frac{d\lambda_{(r)}}{(2\pi)^r} \\
&+ \frac{r(r-1)}{r!}\int_{\mathbb{R}^r} P^{-1}(i\underline{\lambda}_{(r)})\underline{\gamma} \otimes P^{-1}(-i\underline{\lambda}_{(r)})\underline{\gamma}\underline{\widetilde{\mathbf{f}}}(\lambda_{(r-1)}) \otimes \overline{\underline{\widetilde{\mathbf{f}}}(\lambda_{(r\backslash r-1)})} \frac{d\lambda_{(r)}}{(2\pi)^r}.
\end{aligned}
$$

Now, since the last term is zero (See Terdik (1990)), the Lemma 4.3 follows.□

Proof. of Proposition 4.1. The proof follows essentially from the fact that if the spectrum of any operator A lies in $\overline{C} = \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$, then the spectrum of $(I - A)^{-1}(I + A)$ lies in $\{|z| \in \mathbb{C} : z < 1\}$. Hence, for any $r \geq 1, V(r)$ converges to zero at an exponential rate as $r \to \infty$, so $\left(r!\underline{\mathbf{f}}_t(\lambda_{(r)}), r \geq 0\right) \in \mathfrak{I}(\mathcal{H})$ for all $t \in \mathbb{R}$ and their components constitute however a regular second-order solution for SDE (3.2).□

Remark 4.4 The assumption **C1** ensure also that the *CAR* part has a regular solution, however, **C2** is the infinite dimensional generalization of the condition that we found in literature of differential equations systems (see Kelly (2010) chap. 6).

Remark 4.5 It is worth noting that the condition **C2** in Proposition 4.1, may be replaced by

C0. The spectrum of the operator

$$\Psi = \int_{\mathbb{R}} (P(i\lambda) \otimes P(-i\lambda))^{-1} \underline{\gamma}^{\otimes 2} dF(\lambda)$$

lies in $\overline{C} = \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$.

Though the conditions **C1** and **C2** (or equivalently **C0**) of Proposition 4.1 could be used as a sufficient condition for the existence of regular solution of SDE (3.2), they are of little use in practice because they are based on the properties of infinite dimentional operators. Hence, some simple sufficient conditions can be given. Indeed, define $P_t(z) = z - \alpha(t)$ and $R_t(z) = \gamma(t)z$, it can be seen that $P(z)\mathbf{1} = (P_t(z))_{t\in\mathbb{R}}$ and $R(z)\mathbf{1} = (R_t(z))_{t\in\mathbb{R}}$. So,

Proposition 4.6 *A sufficient condition for the SDE (3.2) to have a second-order regular solution is that the following two conditions hold true*

C′1. $P_t(z) \neq 0$ for all $t \in \mathbb{R}$ and $z \in \mathbb{C}$.

C′2. $\Upsilon = \int_{\mathbb{R}} \sup_t |R_t(1)/P_t(i\lambda)|^2 d\lambda < 1$

Proof. The condition **C′1** implie the analycity of $P^{-1}(z)$, so for all $u, v \in \mathbb{L}_\infty$, the equation $P^*(z)u = v$ has a solution. On the other hand, the condition **C′2** ensure the convergence of the series $V_\infty = \sum_{r\geq 1} G^{r-1}$ (in the operator norm) or equivalently by remark 4.5, $\Psi(\mathbf{1}\otimes\mathbf{1}) \leq \Upsilon$. Indeed, for any $\upsilon \in \ell_1$ and $V \in \ell_\infty \otimes \ell_\infty$ let $\langle \upsilon\otimes\overline{\upsilon}, V\rangle = \int_{\mathbb{R}}\int_{\mathbb{R}} \upsilon_s \overline{\upsilon}_t V_{s,t} dt ds$, $G(z) = P^{-1}(z)R(z)$ and set $k(z) = \sup_t |R_t(1)/P_t(z)|^2$. Then we have

$$\begin{aligned}\langle \upsilon\otimes\overline{\upsilon}, (G(z)\otimes G(\overline{z}))(\mathbf{1}\otimes\mathbf{1})\rangle &= |\langle \upsilon, G(z)\mathbf{1}\rangle|^2 = \left|\left\langle P^*(z)u, P^{-1}(z)R(1)\mathbf{1}\right\rangle\right|^2 = |\langle u, R(1)\mathbf{1}\rangle|^2 \\ &= \left|\int_{\mathbb{R}} u_t R_t(1)dt\right|^2 \leq \left|\int_{\mathbb{R}} u_t P_t(z)dt\right|^2 k(z) = \langle u, P(z)\mathbf{1}\rangle^2 k(z) \\ &= \langle \upsilon, 1\rangle^2 k(z) = \langle \upsilon\otimes\overline{\upsilon}, \mathbf{1}\otimes\mathbf{1}\rangle k(z).\end{aligned}$$

The result follows by integrating this inequality along $\mathbb{R}$ for $z = i\lambda$.□

Example 4.1 Consider the simplest $COBL(1,1)$ defined by

$$dX(t) = \alpha(t)X(t)dt + \gamma(t)X(t)d\epsilon(t), t \geq 0, X(0) = X_0 \tag{4.2}$$

Let $a = \sup_{t\in\mathbb{R}} |\alpha(t)|$ and $\gamma = \sup_{t\in\mathbb{R}} |R_t(1)|^2$. Then by the residue theorem we have

$$\int_{\mathbb{R}} \sup_t \frac{|R_t(1)|^2}{|i\lambda - \alpha(t)|^2} dF(\lambda) \leq \frac{\gamma}{2\pi}\int_{\mathbb{R}} \frac{1}{\lambda^2 - a^2} d\lambda = \frac{\gamma}{2a}.$$

So the sufficient condition for (4.2) to has a regular solution is that $\gamma < 2a$.

5. Applications

In this section, the general results of previous sections are particularized for the computation of the first and second-order moments totally describes the statistical properties of a Gaussian processes.

5.1 Second order structure of $COBL(1,1)$ processes

Proposition 5.1 *Under the conditions of Proposition 4.6 we have*

$$E\{X(t)\} = \varphi_t(0)\left(f_0(0) + \int_0^t \varphi_s^{-1}(0)\mu(s)ds\right) \text{ and } Cov(X(t), X(s)) = \varphi_t(0)\varphi_s^{-1}(0)K(s), t \geq s \geq 0$$

where $K(t) = \psi_t\left(K(0) + \int_0^t \psi_s^{-1}(\gamma(s)f_s(0) + \beta(s))^2 ds\right)$ *with* $\psi_t = \exp\left\{\int_0^t \left(2\alpha(s) + \gamma^2(s)\right)ds\right\}$

Proof. From the representation (2.2) we have $E\{X(t)\} = f_t(0)$, so the expression of $E\{X(t)\}$ follows from the first equation of the recursion (3.5). On the other hand, for any $t \geq 0$, let $X(t) - f_t(0) = \varphi_t(0)Y(t)$ where

$$Y(t) = \sum_{r\geq 1}\int_{\mathbb{R}^r}\left(f_0(\lambda_{(r)}) + r\int_0^t \varphi_s^{-1}\left(\underline{\lambda}_{(r)}\right)(\gamma(s)f_s(\lambda_{(r-1)}) + \delta_{\{r=1\}}\beta(s))ds\right)dZ(\lambda_{(r)}),$$

since $K(t) = Cov(X(t), X(t)) = \varphi_t^2(0)Cov(Y(t), Y(t))$, then we have

$$dK(t) = 2\alpha(t)\varphi_t^2(0)Cov(Y(t), Y(t))dt + \varphi_t^2(0)dCov(Y(t), Y(t))$$

where

$$dCov(Y(t), Y(t)) = \varphi_t^{-2}(\gamma(t)f_t(0) + \beta(t))^2 dt + \varphi_t^{-2}\gamma^2(t)dt$$

which implies the differential equation $dK(t) = \left(2\alpha(t) + \gamma^2(t)\right)K(t)dt + (\gamma(t)f_t(0) + \beta(t))^2 dt$ and its solution given by $K(t) = \psi_t\left(K(0) + \int_0^t \psi_s^{-1}(\gamma(s)f_s(0) + \beta(s))^2 ds\right)$ with $\psi_t = \exp\left\{\int_0^t \left(2\alpha(s) + \gamma^2(s)\right)ds\right\}$.□

Remark 5.2 In stationary case, i.e., when the functions $\alpha(.)$, $\mu(.)$, $\gamma(.)$ and $\beta(.)$ are constants, we have

$E\{X(t)\} = -\dfrac{\mu}{\alpha}$, $K(0) = Var\{X(t)\} = \dfrac{(\alpha\beta - \mu\gamma)^2}{\alpha^2\left|2\alpha + \gamma^2\right|}$ and $Cov(X(t), X(t+h)) = K(0)e^{\alpha|h|}, h \in \mathbb{R}$.

The result of the Proposition 5.1 shows that the covariance function of $COBL(1,1)$ processes defined by (3.2) has the same form as that of a $CAR(1)$. So we have the following result due to Lebreton and Musiela (1984).

Proposition 5.3 *There exists a wide-sense Wiener process* $(\epsilon^*(t), t \geq 0)$ *uncorrelated with* $X(0)$ *such that* $(X(t), t \geq 0)$ *admits the* $CAR(1)$ *representation , i.e.,*

$$dX(t) = (\alpha(t)X(t) + \mu(t))\,dt + \left\{\gamma^2(t)K(t) + (\gamma(t)m(t) + \beta(t))^2\right\}^{1/2} d\epsilon^*(t), t \geq 0, X(0) = X_0.$$

Remark 5.4 To draw the conclusion that the second-order properties of $COBL(1,1)$ cannot be distinguished from an $CAR(1)$ process. This makes it necessary for us to look into third-order cumulant in order to distinguish the nonlinear random processes.

5.2 Third-order structure of $COBL(1,1)$ *processes*

For the sake of convenience and simplicity, we shall assume constant the coefficients $\alpha(t), \mu(t), \gamma(t), \beta(t)$ and $\mu(t) = 0$ in Equation (3.2). Moreover, we assume the process solution is given by the Wiener-It$\hat{o}$ representation in the form $X(t) = \sum_{r=1}^{+\infty} \int_{\mathbb{R}^r} g\left(\lambda_{(r)}\right) e^{it\underline{\lambda}_{(r)}} dZ\left(\lambda_{(r)}\right)$. Using the above representation, we can obtain the following approximation

$$X(t) = \int_{\mathbb{R}} g\left(\lambda_1\right) e^{it\lambda_1} dZ\left(\lambda_1\right) + \int_{\mathbb{R}^2} g\left(\lambda_{(2)}\right) e^{it\underline{\lambda}_{(2)}} dZ\left(\lambda_{(2)}\right) + \xi(t) = X^{(1)}(t) + X^{(2)}(t) + \xi(t),$$

where $\xi(t)$ is a second-order stationary process which it is orthogonal to the first two terms. The transfer functions $g(\lambda_1)$, $g\left(\lambda_{(2)}\right)$ are given by $g(\lambda_1) = (i\lambda_1 - \alpha)^{-1}\beta$, $g(\lambda_1, \lambda_2) = \gamma\left(i(\lambda_1 + \lambda_2) - \alpha\right)^{-1}\left(i\lambda_1 - \alpha\right)^{-1}\beta$. It can be shown that

$$\begin{aligned}
C_X(s,u) &= E\left\{X(t)X(t+s)X(t+u)\right\} \\
&= \left[E\left\{X^{(1)}(t)X^{(1)}(t+s)X^{(2)}(t+u)\right\} + E\left\{X^{(1)}(t)X^{(2)}(t+s)X^{(1)}(t+u)\right\}\right] \\
&\quad + \left[E\left\{X^{(2)}(t)X^{(1)}(t+s)X^{(1)}(t+u)\right\}\right] + O(1).
\end{aligned}$$

We calculate $E\left\{X^{(1)}(t)X^{(1)}(t+s)X^{(2)}(t+u)\right\}$, and the other terms can be obtained by symmetry. First we observe that

$$\begin{aligned}
&E\left\{X^{(1)}(t)X^{(1)}(t+s)X^{(2)}(t+u)\right\} \\
&= E\left[\int_{\mathbb{R}^2} g(\lambda_1)g(\lambda_2)e^{it\lambda_1 + i(t+s)\lambda_2} dZ(\lambda_{(2)}) \times \int_{\mathbb{R}^2} g(\lambda_3,\lambda_4)e^{i(t+u)(\lambda_3+\lambda_4)} dZ(\lambda_3,\lambda_4)\right] \\
&= 2!\int_{\mathbb{R}^2} Sym\left\{g(\lambda_1)g(\lambda_2)e^{it\lambda_1 + i(t+s)\lambda_2}\right\} \overline{Sym\left\{g(\lambda_1,\lambda_2)e^{i(t+u)(\lambda_1+\lambda_2)}\right\}} dF\left(\lambda_{(2)}\right) \\
&= 2\int_{\mathbb{R}^2} \gamma g(\lambda_1)g(\lambda_2)g(-\lambda_1-\lambda_2)\frac{1}{\beta} Sym\left\{g(-\lambda_1)\right\} Sym\left\{e^{is\lambda_1}\right\} e^{-iu(\lambda_1+\lambda_2)} \frac{d\lambda_1\lambda_2}{(2\pi)^2}.
\end{aligned}$$

We calculate $E\left\{X^{(1)}(t)X^{(2)}(t+s)X^{(1)}(t+u)\right\}$,we get

$$\begin{aligned}
&E\left\{X^{(1)}(t)X^{(2)}(t+s)X^{(1)}(t+u)\right\} \\
&= 2\int_{\mathbb{R}^2} \gamma g(\lambda_1)g(\lambda_2)g(-\lambda_1-\lambda_2)\frac{1}{\beta} Sym\left\{g(-\lambda_1)\right\} Sym\left\{e^{iu\lambda_1}\right\} e^{-s(\lambda_1+\lambda_2)} \frac{d\lambda_1\lambda_2}{(2\pi)^2}.
\end{aligned}$$

It remains to compute $E\left\{X^{(2)}(t)X^{(1)}(t+s)X^{(1)}(t+u)\right\}$, we have

$$
\begin{aligned}
& E\left\{X^{(2)}(t)X^{(1)}(t+s)X^{(1)}(t+u)\right\} \\
&= E\left[\int_{\mathbb{R}^2} g(\lambda_1,\lambda_2)e^{it(\lambda_1+\lambda_2)}Z(d\lambda_{(2)}) \times \int_{\mathbb{R}^2} g(\lambda_3)g(\lambda_4)e^{i(t+s)\lambda_3+i(t+u)\lambda_4}dZ(\lambda_3,d\lambda_4)\right] \\
&= 2!\int_{\mathbb{R}^2} Sym\left\{g(\lambda_1,\lambda_2)e^{it(\lambda_1+\lambda_2)}\right\}\overline{Sym\left\{g(\lambda_1)g(\lambda_2)e^{i(t+s)\lambda_1+i(t+u)\lambda_2}\right\}}dF\left(\lambda_{(2)}\right) \\
&= 2\int_{\mathbb{R}^2} \gamma g(\lambda_1+\lambda_2)\frac{1}{\beta}Sym\left\{g(\lambda_1)\right\}g(-\lambda_1)g(-\lambda_2)Sym\left\{e^{-is\lambda_1-iu\lambda_2}\right\}\frac{d\lambda_1\lambda_2}{(2\pi)^2}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
C_X(s,u) = E\left\{X(t)X(t+s)X(t+u)\right\} = 2\frac{\gamma}{\beta}\int_{\mathbb{R}^2} & g(\lambda_1)g(\lambda_2)g(-\lambda_1-\lambda_2)Sym\left\{g(-\lambda_1)\right\} \\
& \times Sym\left[e^{i(s-u)\lambda_1-u\lambda_2} + e^{e^{i(u-s)\lambda_1-s\lambda_2}} + e^{i(s\lambda_1+u\lambda_2)}\right]\frac{d\lambda_1\lambda_2}{(2\pi)^2}.
\end{aligned}
$$

By taking Fourier transforms (omitting the terms of $O(1)$), the bispectral density function $f(\lambda_1,\lambda_2)$ can be shown to be

$$
f(\lambda_1,\lambda_2) = 2\frac{\gamma}{\beta}\frac{1}{(2\pi)^2}\left\{S(\lambda_1\lambda_2) + S(\lambda_2,-\lambda_1-\lambda_2) + S(\lambda_1,-\lambda_1-\lambda_2)\right\}
$$

where $S(\lambda_1,\lambda_2) = g(\lambda_1)g(\lambda_2)g(-\lambda_1-\lambda_2)Sym\{g(-\lambda_1)\}$. It is clear that the bispectrum is zero for all frequencies λ_1 and λ_1 if and only if the process is linear($\gamma = 0$) (and Gaussian).

6. Conclusion

In this paper, we have extended some results of Terdik (1990) on time-invariant bilinear SDE to time-varying one. So, we have analyzed the probabilistic structure of general nonlinear continuous-time processes via Wiener's chaos. In particular, necessary and sufficient conditions for the existence of regular second-order solutions are given for a $COBL(1,1)$ driven by a standard Brownian motion with explicit expression in terms of higher-order evolutionary transfer functions. The main advantage of the frequency approach is that beside its adaptation with nonlinear effects, it preserves the mathematically tractable $CARMA$ structure. In particular, it was seen in Section 4, that the spectrum (second-order properties) does not generally provide sufficient information about the structure of the process. Hence, it is necessary to look at third-order cummulant in order to discriminate the $COBL(1,1)$ from a $CAR(1)$ process which seem to be too difficult and tedious. However, specific tools, for instance wavelet methods as an alternative to Fourier methods should be adapted to analysis the general bilinear SDE with time-varying coefficients. We leave this important issue for future researches.

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