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Kumaraswamy-Half-Cauchy Distribution: Characterizations and Related Results

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Abstract

We present various characterizations of a recently introduced distribution (Ghosh 2014), called Kumaraswamy-Half- Cauchy distribution based on: (*i*) a simple relation between two truncated moments; (*ii*) truncated moment of certain function of the 1^{st} order statistic; (*iii*) truncated moment of certain function of the random variable; (*iv*) hazard function; (*v*) distribution of the 1^{st} order statistic; (*vi*) via record values. We also provide some remarks on bivariate Gumbel copula distribution whose marginal distributions are Kumaraswamy- Half-Cauchy distributions.

Keywords: Kumaraswamy-Half-Cauchy, distribution

1. Introduction

In designing a stochastic model for a particular modeling problem, an investigator will be vitally interested to know if their model fits the requirements of a specific underlying probability distribution. To this end, the investigator will rely on the characterizations of the selected distribution. Generally speaking, the problem of characterizing a distribution is an important problem in various fields and has recently attracted the attention of many researchers. Consequently, various characterization results have been reported in the literature. These characterizations have been established in many different directions. The present work deals with the characterizations of a newly introduced univariate continuous distribution, called Kumaraswamy- Half- Cauchy (KwHC) distribution. These characterizations are based on: (*i*) a simple relation between two truncated moments; (*ii*) truncated moment of certain function of the 1^{st} order statistic; (*iii*) truncated moment of certain function of the random variable; (*iv*) hazard function; (*v*) distribution of the 1^{st} order statistic; (*vi*) via record values. We also make some remarks on bivariate Gumbel copula distribution whose marginal distributions are (KwHC) distributions.

The (KwHC) distribution introduced by Ghosh (2014), has probability density function (pdf) and cumulative distribution function (cdf) given, respectively, by

$$f(x) = f(x; a, b, \delta) = \frac{ab2^a}{\delta\pi^a}\left(1 + \left(\frac{x}{\delta}\right)^2\right)^{-1}\left(\arctan\left(\frac{x}{\delta}\right)\right)^{a-1} \times \left[1 - \left(\frac{2}{\pi}\arctan\left(\frac{x}{\delta}\right)\right)^a\right]^{b-1}, \tag{1.1}$$

for $x > 0$, and

$$F(x) = F(x; a, b, \delta) = 1 - \left[1 - \left(\frac{2}{\pi}\arctan\left(\frac{x}{\delta}\right)\right)^a\right]^b, \quad x \geq 0$$

where a, b, δ are all positive parameters. For a detailed treatment of this distribution, we refer the interested reader to Ghosh (2014).

The paper is organized as follows. We present our characterization results in section 2 via 6 subsections 2.1-2.6. Section 3 deals with certain remarks concerning bivariate Gumel copula distribution.

2. Characterization Results

We divide this section to 6 subsections each of which deals with one of the characterizations mentioned in the Abstract as well as in the Introduction.

2.1 Characterizations Based on Two Truncated Moments

In this subsection we present characterizations of (KwCH) distribution in terms of a simple relationship between two truncated moments. We like to mention here the works of Glänzel (1987, 1990), Glänzel et al. (1984), Glänzel and Hamedani (2001) and Hamedani (2002, 2006, 2010) in this direction. Our characterization results presented here will employ an interesting result due to Glänzel (1987) (Theorem 2.1.1 below). The advantage of the characterizations given here is that, cdf F need not have a closed form and are given in terms of an integral whose integrand depends on the solution of a first order differential equation, which can serve as a bridge between probability and differential equation.

Theorem 2.1.1. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a given probability space and let $H = [a, b]$ be an interval for some $a < b$ $(a = -\infty\ ,\ b = \infty$ might as well be allowed$)$. Let $X : \Omega \to H$ be a continuous random variable with the distribution function F and let g and h be two real functions defined on H such that

$$\mathbf{E}\left[g(X) \mid X \geq x\right] = \mathbf{E}\left[h(X) \mid X \geq x\right] \eta(x), \qquad x \in H,$$

is defined with some real function η. Assume that $g, h \in C^1(H)$, $\eta \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $h\eta = g$ has no real solution in the interior of H. Then F is uniquely determined by the functions g, h and η, particularly

$$F(x) = \int_a^x C \left| \frac{\eta'(u)}{\eta(u)\, h(u) - g(u)} \right| \exp(-s(u))\ du,$$

where the function s is a solution of the differential equation $s' = \frac{\eta' h}{\eta h - g}$ and C is a constant, chosen to make $\int_H dF = 1$.

Again, following our previous work, we like to mention that this kind of characterization based on the ratio of truncated moments is stable in the sense of weak convergence, in particular, let us assume that there is a sequence $\{X_n\}$ of random variables with distribution functions $\{F_n\}$ such that the functions g_n, h_n and η_n $(n \in \mathbb{N})$ satisfy the conditions of Theorem 2.1.1 and let $g_n \to g$, $h_n \to h$ for some continuously differentiable real functions g and h. Let, finally, X be a random variable with distribution F. Under the condition that $g_n(X)$ and $h_n(X)$ are uniformly integrable and the family $\{F_n\}$ is relatively compact, the sequence X_n converges to X in distribution if and only if η_n converges to η, where

$$\eta(x) = \frac{E\left[g(X) \mid X \geq x\right]}{E\left[h(X) \mid X \geq x\right]}.$$

This stability theorem makes sure that the convergence of distribution functions is reflected by corresponding convergence of the functions g, h and η, respectively. It guarantees, for instance, the 'convergence' of characterization of the Wald distribution to that of the Lévy-Smirnov distribution if $\alpha \to \infty$, as was pointed out in (2001).

A further consequence of the stability property of Theorem 2.1.1 is the application of this theorem to special tasks in statistical practice such as the estimation of the parameters of discrete distributions. For such purpose, the functions g, h and, specially, η should be as simple as possible. Since the function triplet is not uniquely determined it is often possible to choose η as a linear function. Therefore, it is worth analyzing some special cases which helps to find new characterizations reflecting the relationship between individual continuous univariate distributions and appropriate in other areas of statistics.

Clearly, Theorem 2.1.1 can be stated in terms of two functions g and η by taking $h(x) \equiv 1$, which will reduce the condition given in Theorem 2.1.1 to $E\left[g(X) \mid X \geq x\right] = \eta(x)$. However, adding an extra function will give a lot more flexibility, as far as its application is concerned.

Proposition 2.1.2. Let $X : \Omega \to (0,\infty)$ be a continuous random variable and let $h(x) = \left[1 - \left(\frac{2}{\pi}\arctan\left(\frac{x}{\delta}\right)\right)^a\right]^{1-b}$ and $g(x) = \left(\arctan\left(\frac{x}{\delta}\right)\right)^a h(x)$ for $x \in (0,\infty)$. The pdf of X is (1.1) if and only if the function η defined in Theorem 2.1.1 has the form

$$\eta(x) = \frac{\pi^a}{2^{a+1}}\left\{1 + \left(\frac{2}{\pi}\arctan\left(\frac{x}{\delta}\right)\right)^a\right\}, \quad x > 0.$$

Proof. Let X have density (1.1) , then

$$(1 - F(x))\ \mathbf{E}\left[h(X) \mid X \geq x\right] = b\left\{1 - \left(\frac{2}{\pi}\arctan\left(\frac{x}{\delta}\right)\right)^a\right\}, \quad x > 0,$$

and

$$(1 - F(x))\ \mathbf{E}\left[g(X) \mid X \geq x\right] = \frac{b\pi^a}{2^{a+1}}\left\{1 - \left(\frac{2}{\pi}\arctan\left(\frac{x}{\delta}\right)\right)^{2a}\right\}, \quad x > 0,$$

and finally

$$\eta(x)h(x) - g(x) = \frac{h(x)}{2}\left\{\left(\frac{\pi}{2}\right)^a - \left(\arctan\left(\frac{x}{\delta}\right)\right)^a\right\} > 0 \quad for\ \ x > 0.$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x)\ h(x)}{\eta(x)\ h(x) - g(x)} = \frac{\frac{a}{\delta}\left(1 + \left(\frac{x}{\delta}\right)^2\right)^{-1}\left(\arctan\left(\frac{x}{\delta}\right)\right)^{a-1}}{\left\{\left(\frac{\pi}{2}\right)^a - \left(\arctan\left(\frac{x}{\delta}\right)\right)^a\right\}}, \quad x > 0,$$

and hence

$$s(x) = -\ln\left\{1 - \left(\frac{2}{\pi}\arctan\left(\frac{x}{\delta}\right)\right)^a\right\}, \quad x > 0.$$

Now, in view of Theorem 2.1.1, X has density (1.1) .

Corollary 2.1.3. Let $X : \Omega \to (0,\infty)$ be a continuous random variable and let $h(x)$ be as in Theorem 2.1.1. The pdf of X is (1.1) if and only if there exist functions g and η defined in Theorem 2.1.1 satisfying the differential equation

$$\frac{\eta'(x)h(x)}{\eta(x)h(x) - g(x)} = \frac{a\left(1 + \left(\frac{x}{\delta}\right)^2\right)^{-1}\left(\arctan\left(\frac{x}{\delta}\right)\right)^{a-1}}{\delta\left\{\left(\frac{\pi}{2}\right)^a - \left(\arctan\left(\frac{x}{\delta}\right)\right)^a\right\}}, \quad x > 0.$$

Remarks 2.1.4. (a) The general solution of the differential equation in Corollary 2.1.3 is

$$\eta(x) = \left[1 - \left(\frac{2}{\pi}\arctan\left(\frac{x}{\delta}\right)\right)^a\right]^{-1} \times \left[\begin{array}{c} -\int \frac{a2^a}{\delta\pi^a}\left(1 + \left(\frac{x}{\delta}\right)^2\right)^{-1}\left(\arctan\left(\frac{x}{\delta}\right)\right)^{a-1} \times \\ \left[1 - \left(\frac{2}{\pi}\arctan\left(\frac{x}{\delta}\right)\right)^a\right]^{b-1} g(x)\,dx + D \end{array}\right],$$

for $x > 0$, where D is a constant. One set of appropriate functions is given in Proposition 2.1.2 with $D = \frac{\pi^a}{2^{a+1}}$.

(b) Clearly there are other triplets of functions (h, g, η) satisfying the conditions of Theorem 2.1.1. We presented one such triplet in Proposition 2.1.2.

2.2 Characterizations Based on Truncated Moment of Certain Functions of the 1^{th} Order Statistic

Let $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ be n order statistics from a continuous cdf F. We present here a characterization result base on some functions of the 1^{st} order statistic. Our characterization will be a consequence of the following proposition, which is an extended version of the one appeared in our previous work (Hamedani, 2010).

Proposition 2.2.1. Let $X : \Omega \to (\alpha,\beta)$ be a continuous random variable with cdf F . Let $\psi(x)$ and $q(x)$ be two differentiable functions on (α,β) such that $\lim_{x\to\beta} \psi(x)[1-F(x)]^n = 0$ and $\int_\alpha^\beta \frac{q'(t)}{[q(t)-\psi(t)]}dt = \infty$. Then

$$E[\psi(X_{1:n}) \mid X_{1:n} > t] = q(t) \ , \ \ t > \alpha \ ,$$

implies

$$F(x) = 1 - \exp\left\{-\int_a^x \frac{q'(t)}{n[\psi(t) - q(t)]}dt\right\}, \quad \alpha \leq x < \beta.$$

Remarks 2.2.2. (a) Taking, e.g., $\psi(x) = 2\left[1 - \left(\frac{2}{\pi}\arctan\left(\frac{x}{\delta}\right)\right)^a\right]^{nb}$, $q(x) = \frac{1}{2}\psi(x)$ and $(\alpha,\beta) = (0,\infty)$, Proposition 2.2.1 provides a characterization of (KwHC) distribution. (b) Clearly there are other suitable functions $\psi(x)$ and $q(x)$. (c) We like to point out that Proposition 2.2.1 holds true (with of course appropriate modifications) if we replace $X_{1:n}$ with the base random variable X .

2.3 Characterization Based on Single Truncated Moment of Certain Function of the Random Variable

In this subsection we employ a single function ψ_1 of X and characterize the distribution of X in terms of the truncated moment of $\psi_1(X)$. The following proposition have already appeared in our previous work (Hamedani, Technical Report, 2013), so we will just state it here which can be used to characterize (KwCH) distribution.

Proposition 2.3.1. Let $X : \Omega \to (\alpha,\beta)$ be a continuous random variable with cdf F . Let $\psi_1(x)$ be a differentiable function on (α,β) with $\lim_{x\to\alpha} \psi_1(x) = 1$. Then for $\xi \neq 1$,

$$E[\psi_1(X) | X > x] = \xi\psi_1(x), \qquad x \in (\alpha,\beta),$$

if and only if

$$\psi_1(x) = (1 - F(x))^{\frac{1}{\xi}-1}, \qquad x \in (\alpha,\beta).$$

Remark 2.3.2. For $\xi = \frac{b}{b+1}$, $\psi_1(x) = 1 - \left(\frac{2}{\pi}\arctan\left(\frac{x}{\delta}\right)\right)^a$ and $(\alpha,\beta) = (0,\infty)$, we have a characterization of (KwHC) distribution.

2.4 Characterization Based on the Hazard Function

The following definition is stated here for the sake of completeness.

Definition 2.4.1. Let F be an absolutely continuous distribution with the corresponding pdf f . The hazard function corresponding to F is denoted by h_F and is defined by

$$h_F(y) = \frac{f(y)}{1 - F(y)}, \quad y \in Supp\, F, \tag{2.4.1}$$

where $Supp\, F$ is the support of F .

It is obvious that the hazard function of a twice differentiable distribution function satisfies the first order differential equation

$$\frac{h_F'(y)}{h_F(y)} - h_F(y) = q(y) \ ,$$

where $q(y)$ is an appropriate integrable function. Although this differential equation has an obvious form since

$$\frac{f'(y)}{f(y)} = \frac{h'_F(y)}{h_F(y)} - h_F(y) , \tag{2.4.2}$$

for many univariate continuous distributions (2.4.2) seems to be the only differential equation in terms of the hazard function. The goal of the characterization based on hazard function is to establish a differential equation in terms of hazard function, which has as simple form as possible and is not of the trivial form (2.4.2) . For some general families of distributions this may not be possible.

Proposition 2.4.2. Let $X : \Omega \to (0, \infty)$ be a continuous random variable. The random variable X has pdf (1.1) (WLOG, for $\delta = 1$ and $a > 1$) if and only if its hazard function $h_F(x)$ satisfies the differential equation

$$h'_F(x) - (a-1)\left(\left(1+x^2\right)\arctan x\right)^{-1} h_F(x) = \tfrac{ab2^a}{\pi^a}\left(1+x^2\right)^{-2}(\arctan x)^{a-1}\times$$

$$\left[1-\left(\frac{2}{\pi}\arctan x\right)^a\right]^{-2}\left\{\frac{2}{\pi}\left(\frac{2}{\pi}\arctan x\right)^{a-1}(a+2x\arctan x) - 2x\right\},\ x>0, \tag{2.4.3}$$

with initial condition $h_F(0) = 0$.

Proof: If X has pdf (1.1) , then clearly (2.4.3) holds. Now, if (2.4.3) holds, then

$$\frac{d}{dx}\left\{(\arctan x)^{1-a} h_F(x)\right\} = \frac{ab2^a}{\pi^a}\frac{d}{dx}\left\{\left(1+x^2\right)^{-1}\left[1-\left(\frac{2}{\pi}\arctan x\right)^a\right]^{-1}\right\},$$

from which we have

$$h_F(x) = \frac{ab2^a}{\pi^a}\left(1+x^2\right)^{-1}(\arctan x)^{a-1}\left[1-\left(\frac{2}{\pi}\arctan x\right)^a\right]^{-1}.$$

Integrating both sides of the last equation from 0 to x , we arrive at

$$-\ln(1-F(x)) = -b\ln\left\{\left[1-\left(\frac{2}{\pi}\arctan x\right)^a\right]\right\},$$

from which, in view of the initial condition, we obtain

$$1-F(x) = \left[1-\left(\frac{2}{\pi}\arctan x\right)^a\right]^b, \quad x \ge 0.$$

Remark 2.4.3. For $a = 1$, (2.4.3) will have the following simple form

$$h'_F(x) + 2x\left(1+x^2\right)^{-1} h_F(x) = \frac{4b}{\pi^2}\left(1+x^2\right)^{-2}\left[1-\frac{2}{\pi}\arctan x\right]^{-2},$$

with initial condition $h_F(0) = \frac{2b}{\pi}$. It is easy to show that the solution of the above equation is

$$F(x) = 1-\left[1-\frac{2}{\pi}\arctan x\right]^b, \quad x \ge 0.$$

2.5 Characterization Based on the Distribution of the **1st** *Order Statistic*

We consider the following characterization based on the 1^{st} order statistic.

Proposition 2.5.1. Suppose $X_1, X_2, \cdots, X_n$ are *i.i.d.* (independent and identically distributed) random variables. The common distribution of $X_i's$ is (KwHC) distribution if only if $X_{1:n}$ has a (KwHC) distribution.

Proof. Follows immediately from the fact that

$$\begin{aligned} P(X_{1:n} > x) &= (P(X_1 > x))^n \\ &= (1 - F(x))^n \\ &= \left(1 - \left(\frac{2}{\pi}\arctan\left(\frac{x}{\delta}\right)\right)^a\right)^{nb} \qquad x \geq 0. \end{aligned}$$

2.6 Characterization via Record Values

We will consider a direct application of the following theorem of Athar et al. (2014) as it has appeared in their paper. In what follows, $X_{U(r)}$ is the rth record values of a random sample $X_1, X_2, ..., X_n$. Characterization of distributions through conditional expectation of record values have been considered among others by Nagaraja (1988), Franco and Ruiz (1997), Dembinska and Wesolowski (2000), Khan and Alzaid (2004) and Wu (2004). The following theorem is a particular type of generalization of the results mentioned in Khan et al. (2010). For a detailed discussion on the related topic the reader is suggested to see Arnold et al. (1998) and the references therein.

Theorem 2.6.1. Let X be a continuous random variable with cdf $F(x)$ and pdf $f(x)$ on the support $(-\infty, \infty)$. Then for two consecutive values of r and s and $1 \leq r < s \leq n$, $E\left[(h(X_{U(s)} - X_{U(r)}))^p \mid X_{U(r)} = x\right] = a^* \sum_{j=0}^{p} \binom{p}{j} (h(x))^{p-j} (b/a)^j$ if and only if $F(x) = 1 - (ah(x) + b)^c$, $a \neq 0$, where $a^* = \sum_{i=0}^{p} (-1)^{i+p} \binom{p}{i} \left(\frac{c}{c+i}\right)^{s-r}$ and $h(x)$ is a continuous differentiable function of x.

Remark 2.6.2. Taking $a = -1$, $b = 1$, $c = 1$, $h(x) = \left(\frac{2}{\pi}\arctan\left(\frac{x}{\delta}\right)\right)^a$ for $x \in (0, \infty)$, we have a characterization of (KwHC) distribution.

3. Related Remarks

In this section, we consider the general structure of a bivariate Kumaraswamy-G model (Nadarajah et al. 2011). Nadarajah et al. (2011) define a bivariate Kumaraswamy-G distribution as follows. Let $G(x_1, x_2)$ be an arbitrary absolutely continuous bivariate cdf with corresponding pdf $g(x_1, x_2)$. For positive parameters a and b, define

$$F(x_1, x_2) = 1 - (1 - G^a(x_1, x_2))^b. \tag{3.1}$$

In order for (3.1) to be a valid absolutely continuous bivariate cdf, it must satisfy $F(\infty, \infty) = 1$, $F(-\infty, x_2) = 0$, $F(x_1, -\infty)) = 0$ and $\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}F(x_1, x_2) \geq 0$ for all x_1, x_2.

Next, consider a bivariate Gumbel copula of the form $G(x_1, x_2) = x_1 x_2 \exp(-c \log x_1 \log x_2)$, $0 < x_1 < 1, 0 < x_2 < 1$. Note that, we will have Kumaraswamy marginals since $F_{X_1}(x_1) = 1 - (1 - G^a(x_1, 1))^b = 1 - (1 - x_1^a)^b$ and $F_{X_2}(x_2) = 1 - (1 - G^a(1, x_2))^b = 1 - (1 - x_2^a)^b$, using the fact that G is a copula. If we make the following transformation $X_1 = \frac{2}{\pi}\arctan\left(\frac{Y_1}{\delta}\right)$ and $X_2 = \frac{2}{\pi}\arctan\left(\frac{Y_2}{\delta}\right)$, for some $\delta > 0$ then, the resulting marginals will be (KwHC), since

$$F_{Y_1}(y_1) = 1 - (1 - G^a(y_1, 1))^b = 1 - \left(1 - \left(\frac{2}{\pi}\arctan\left(\frac{y_1}{\delta}\right)\right)^a\right)^b, \quad y_1 \geq 0,$$

and

$$F_{Y_2}(y_2) = 1 - (1 - G^a(y_2, 1))^b = 1 - \left(1 - \left(\frac{2}{\pi}\arctan\left(\frac{y_2}{\delta}\right)\right)^a\right)^b, \quad y_2 \geq 0.$$

So, we can state the following proposition: if the bivariate distribution (or bivariate Gumbel copula) is of the form mentioned above, then the marginal distributions will be (KwHC) distributions with appropriate parameters.

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