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Characterizations of Probability Distributions with Completely Monotone Hazard Functions

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Abstract

In this article, we characterize the classes of absolutely continuous distributions concentrated on $(0, \infty)$ and discrete distributions concentrated on $\{0, 1, 2, ...\}$, with (non-vanishing survivor functions having) completely monotone hazard functions; in the latter case, we refer to the hazard functions also as the hazard sequences. These provide us with characterizations of the certain specialized versions of mixtures of exponential and geometric distributions with mixing distributions, satisfying some further criteria, which by the Goldie-Steutel theorem and a result of Kaluza are seen to be specialized versions of infinitely divisible distributions. We shed light on the implications of our findings, giving some pertinent examples and remarks.

Keywords: compound geometric distributions, Goldie-Steutel theorem, hazard functions, infinitely divisible distributions, Kaluza sequences, moment sequences, renewal sequences

1. Introduction

By Kaluza (1928, Proposition 1), it follows that all logconvex, and hence completely monotone, sequences $\{c(n) : n = 0, 1, ...\}$, with $c(0) = 1$, are renewal. Consequently, it is seen that each probability distribution on $\{0, 1, ...\}$ with logconvex probability function, or, in particular, that relative to a mixture of (standard) geometric distributions is compound geometric and hence infinitely divisible. (The result on geometric distributions referred to here also holds if we allow the degenerate distribution at 0 to be referred to as geometric (with mean 0)). Applying Kaluza's (1928) result, in conjunction with the closure property of the class of infinitely divisible (i.d.) distributions, what is shown by Steutel (1970, p.89) effectively tells us that every (probability) distribution function (d.f.) F concentrated on $[0, \infty)$, satisfying $F(.) = F(0) + (1 - F(0))G(.)$, with G absolutely continuous having density that is logconvex on $(0, \infty)$, is i.d. For further results and observations on Kaluza- Steutel results, see Sapatinas et al (2011); Kingman (1972, Section 1.5) has also made some illuminating observations on Kaluza's result.

A specialized version of the result involving G met above, in the case when the density relative to G is completely monotone (or, equivalently, when G is the d.f. of a mixture of exponential distributions), was proved earlier using two distinct approaches by Goldie (1967) and Steutel (1967), respectively, with the proof given by the former based implicitly on Kaluza's result. Steutel and van Harn (2004) and Sapatinas et al (2011) have unified the literature on Kaluza-Steutel and Goldie-Steutel results and have shed further light on aspects of the results of relevance to these, such as Theorem 2.3.1 of Steutel (1970); incidentally the latter theorem of Steutel referred to here implies, in view of the closure property of the class of i.d. distributions, that, if X and Y are independent random variables, with Y exponential and X real (not necessarily nonnegative), then XY (i.e. its distribution) is i.d. .

Cox (1962 (page 5), 1972), Barlow and Proschan (1965, 1975) and Kotz and shanbhag (1980), amongst others, have given representations for survivor functions (relative to univariate probability distributions), under some constraints or otherwise, in terms of hazard functions or measures. Under appropriate assumptions, from these representations, one can obtain the related representations for survivor functions in terms of the mean residual life functions, see, e.g., Cox (1962; Exercise 1, Appendix II) or Meilijson (1972).

In the present article, we characterize nondegenerate d.f.'s F concentrated on $[0, \infty)$, satisfying $F(.) = F(0) + (1 - F(0))G(.)$, with G absolutely continuous, having a completely monotone hazard function, on $(0, \infty)$, and, also, nondegenerate probability distributions on $\{0, 1, ...\}$ with completely monotone hazard sequences. We also make some relevant observations on these results through some interesting remarks and examples. That the research material covered in this work addresses the problems linked with the Goldie-Steutel result is hence obvious.

Before we go through our main results, we give here the following crucial definitions and tools used throughout this article.

Definition 1 (Feller 1966, Page 173):

A distribution function F is infinitely divisible if and only if (iff), for each positive integer n, it can be represented as the distribution of the sum

$$S_n = X_{1,n} + ... + X_{n,n}$$

of n independent random variables with a common distribution F_n.

Definitions 2 (Feller 1966, Pages 224 and 415):

A moment sequence $\{\mu_n\}$ of some d.f. concentrated on $[0, 1]$ is called completely monotone if $(-1)^r \Delta^r \mu_k \geq 0$ for all combinations r, k where Δ is the difference operator.

Another definition relative to that mentioned above is as follows: a function f concentrated on $(0, \infty)$ is said to be completely monotone if f has derivatives of all orders and satisfies

$$(-1)^n f^n(x) \geq 0, \qquad for\ all\ x > 0\ and\ n = 0, 12, ...,$$

According to the above definitions, Feller (1966) proved the following theorem.

Theorem 1 (Feller 1966, Page 425):

The function ω is the Laplace transform of an infinitely divisible probability distribution iff

$$\omega(x) = \exp\{H(x)\}$$

where H has a completely monotone derivative and $H(0) = 0$.

2. The Main Results

Kotz and Shanbhag (1980) defined relative to each univariate d.f. F, for the survivor function $\overline{F}$ so that for each real x, $\overline{F}(x) = 1 - F(x-)$. In the case of F concentrated on $[0, \infty)$, for simplicity, one may refer to the restrictions of F and $\overline{F}$ to $[0, \infty)$, respectively, as a d.f. on $[0, \infty)$ and the corresponding survivor function; note that in this latter case, $\overline{F}(0) = 1$ and $\overline{F}(x) = 1 - F(x-)$ if $x > 0$. In many places in the literature, this concept is adopted and we assume in this article that there is no confusion or ambiguity if we do the same.

We now give below our main results; for the relevant definitions of completely monotone sequences and functions that we have followed in our analysis, we refer the reader to Feller (1966, pages 224 and 415), respectively. It may also be worth pointing out in this place that the condition of $F(x) < 1$ for all x that we have used in the following results is equivalent to the one that the survivor function $\overline{F}$ in each of these cases is non-vanishing.

Theorem 2.1.:

Let F be a d.f. on $[0, \infty)$ with $F(x) < 1$ for all x. Then it satisfies

$$F(x) = F(0) + (1 - F(0))G(x), \qquad x \in [0, \infty), \tag{2.1}$$

where G is an absolutely continuous d.f. with completely monotone (version of) hazard function on $(0, \infty)$ iff the survivor function, $\overline{F}$, relative to F, is so that, for some Laplace-Stieltjes transform ω of an i.d. distribution on $[0, \infty)$, $\overline{F}(x) \propto \omega(x)$ if $x \in (0, \infty)$.

Proof. Under the stated conditions, the standard representation for the survivor function, relative to G in (2.1), in terms of the corresponding hazard function tells us clearly that (2.1) is equivalent to that

$$F(x) = 1 - (1 - F(0)) \exp\{-H(x)\}, \qquad x \in [0, \infty), \tag{2.2}$$

where $H(0) = 0$ and the restriction of H to $(0, \infty)$ is differentiable with completely monotone derivative. In view of Feller (1966, Theorem XIII.7.1), stated in the previous section, it is hence obvious that the theorem holds. □ ▫

Corollary 2.1.: *Any absolutely continuous d.f. with F on $[0, \infty)$ such that $F(x) < 1$ for all x, has a completely monotone hazard function on $(0, \infty)$ iff the corresponding survivor function agrees with the Laplace-Stieltjes transform of an i.d. distribution on $[0, \infty)$.*

Proof. The result is immediate since, in this case F, the d.f. of the distribution, satisfies (2.1) with $F(0) = 0$.□ ▫

In view of Theorem 2.1 and Corollary 2.1, specific observations can be presented the following two remarks.

Remark 2.1.: *The distributions that are characterized by the criterion in Corollary 2.1 are indeed mixtures of exponential distributions with mixing distributions that are i.d. The class of such distributions obviously does not include some of the distributions that are involved in the Goldie-Steutel result, such as those that are mixtures of exponential distributions with mixing distributions non-i.d. Amongst the non-i.d. mixing distributions, we have beta concentrated on* $(0, 1)$ *or, more generally, every non-degenerate distribution concentrated on an interval* (a, b) *with* $0 < a < b < \infty$*. On the other hand, there are well known mixtures of exponential distributions such as Weibull and Pareto, amongst others, with d.f.'s* F_r *,* $r = 1, 2$*, satisfying* $F_1(x) = 1 - \exp\{-\lambda x^{\beta}\}$*,* $x > 0$ *with* $\lambda > 0$ *and* $\beta \in (0, 1]$*, and* $F_2(x) = 1 - (1 + \lambda x)^{-\beta}$ *,* $x > 0$*, with* $\lambda, \beta > 0$ *respectively, for which the criterion referred to is met. (Note that, in the case of Weibull with parameter* $\beta < 1$*, the mixing distribution is stable concentrated on* $[0, \infty)$ *with exponent* β*, and, in the case of Pareto, the mixing distribution is gamma).*

Remark 2.2.: *In view of Fubini's theorem, amongst other things, for* $0 < \alpha < 1$*,*

$$
\begin{aligned}
\int_x^{\infty} (\exp\{-y\}\, y^{\alpha-1}) dy &= \exp\{-x\} \int_0^{\infty} (\exp\{-y\}\, (y + x)^{\alpha-1}) dy \\
&\propto \exp\{-x\} \int_0^{\infty} \left(\int_0^{\infty} \exp\{-y - (y + x)z\}\, z^{-\alpha}\right) dz)\, dy \\
&= \exp\{-x\} \int_0^{\infty} \left(\int_0^{\infty} \exp\{-(1 + z)y - xz\}\, dy\right) z^{-\alpha})\, dz \\
&= \exp\{-x\} \int_0^{\infty} (\exp\{-xz\}\, (1 + z)^{-1}\, z^{-\alpha})\, dz\, ,\ x > 0
\end{aligned}
$$

which implies essentially that the survivor function relative to gamma distribution with index (referred to also as shape) parameter α *agrees with the Laplace transform of an absolutely continuous distribution with completely monotone density and, hence, by the Goldie-Steutel result, agrees with the Laplace transform of an i.d. distribution. By Theorem 2.1, we have hence the hazard function relative to a gamma distribution of the type considered to be completely monotone. This latter result can also be obtained as a by-product of the standard result met in Shanbhag and Sreehari (1977) that any gamma random variable, in the case* $\alpha < 1$*, is distributed as the product of two independent random variables X and Y, where X is exponential and* $Y \sim$ *beta*$(\alpha, 1 - \alpha)$*.*

Theorem 2.2.:
Let F be a d.f. on $\{0, 1, ...\}$ *with* $F(x) < 1$ *for all x. Then (in obvious notation) the corresponding sequence* $\{h(x) : x = 0, 1, ...\}$ *of the relevant hazard measure-values, referred to as hazard sequence, is completely monotone on* $\{0, 1, ...\}$ *iff the survivor function,* $\overline{F}$*, corresponding to F satisfies*

$$\overline{F}(x) = \prod_{n=0}^{x-1} (1 - \alpha m_n), \qquad x = 1, 2, ... \quad , \tag{2.3}$$

with $\alpha \in (0, 1)$ *and* $\{m_n : n = 0, 1, ...\}$ *as the moment sequence relative to a distribution concentrated on* $[0, 1]$*.*

Proof. In view of Hausdroff's theorem appearing as Theorem VII.3.2 in Feller (1966), it follows that the hazard sequence $\{h(x) : x = 0, 1, ...\}$ relative to F is completely monotone iff

$$h(x) = \alpha\, m_x\, , \qquad x = 0, 1, ... \quad , \tag{2.4}$$

with α and $\{m_x\}$ as in (2.3). Consequently, it follows that the hazard sequence in question is completely monotone iff, with notation as above

$$\overline{F}(x - 1) - \overline{F}(x) = \alpha\, m_{x-1}\, \overline{F}(x - 1), \qquad x = 1, 2, ... \quad , \tag{2.5}$$

and, hence, iff

$$\overline{F}(x) = \overline{F}(x - 1)(1 - \alpha\, m_{x-1})\, , \qquad x = 1, 2, ... \quad , \tag{2.6}$$

Since $\overline{F}(0) = 1$, it follows recursively that (2.6) implies (2.3) and vice versa. Hence, we have the theorem. (One may also apply the relevant representation for $\overline{F}$ in terms of the hazard sequence, met in the literature, to see that the theorem holds, since it implies that (2.4) is equivalent to (2.3).) □ □

Corollary 2.2.: *If F is as in Theorem 2.2 and its hazard sequence $\{h(x)\}$ is completely monotone, then the corresponding survival and probability sequences $\{\overline{F}(x) : x = 0, 1, ...\}$ and $\{\overline{F}(x) - \overline{F}(x + 1) : x = 0, 1, ...\}$, respectively, are completely monotone (and hence correspond to mixtures of geometric distributions).*

Proof. Since

$$\overline{F}(x) - \overline{F}(x + 1) = \overline{F}(x) \, . \, h(x) \, , \qquad x = 0, 1, ... \qquad ,$$

it follows that

$$(-1)^{n+1} \, \Delta^{n+1} \, \overline{F}(x) = \sum_{r=0}^{n} \binom{n}{r} [(-1)^r \, \Delta^r \, \overline{F}(x)][(-1)^{n-r} \, \Delta^{n-r} \, h(x)] \, , \quad x = 0, 1... \, ; n = 0, 1, ...,$$

where Δ is the difference operator, and hence, by induction, that the survivor sequence $\{\overline{F}(x)\}$ is completely monotone. This, in turn, implies, in view of $\overline{F}(x) - \overline{F}(x+1) = -\Delta \, \overline{F}(x)$, $x = 0, 1, ...$, that the corresponding sequence referred to in the assertion is completely monotone. . □ □

Based on our findings above, we present the following remarks.

Remark 2.3.: *If the distribution relative to $\{m_n \; n = 0, 1, ...\}$ of (2.3) has at least one support point in $(0, 1)$, then since (2.3) implies (2.6) and hence that*

$$\begin{aligned} \overline{F}(x) &= \overline{F}(x + 1)/(1 - \alpha \, m_x) \\ &= (\overline{F}(x + 1)/\overline{F}(1))((1 - \alpha)(1 + \alpha m_x + \alpha^2 m_x^2 + ...)) \, , x = 0, 1, ... \quad , \end{aligned}$$

it follows, by the relevant moment argument, that any mixture of geometric distributions with mixing distribution concentrated on $\{0\} \bigcup [c, 1)$, where $0 < c < 1$, can not satisfy the relevant version of (2.3); note that we allow here the degenerate distribution at 0 to be referred to as geometric. Also, it can be seen that (2.3), with $\{m_x\}$ as the moment sequence of a distribution with support $\{1\}$ or $\{0, 1\}$, holds iff

$$\overline{F}(x) = (1 - \alpha) \, (1 - \alpha \, p)^{x-1} \, , x = 1, 2, ... \qquad ,$$

with α as in (2.3) and $p \in (0, 1]$; the distribution characterized in this latter case reduces to geometric if $p = 1$ and a mixture of two geometric distributions with one of them degenerate at 0 if $p < 1$.

Remark 2.4.: *If F is the mixture of geometric distribution with beta mixing distribution, relative to parameter vector (α,β), then*

$$\overline{F}(x + 1)/\overline{F}(x) = (\alpha + x)/(\alpha + \beta + x) = 1 - \beta(\alpha + \beta + x)^{-1} \qquad , x = 0, 1, ... \qquad ,$$

which clearly satisfies (2.6) with $\alpha/(\alpha + \beta)$ in place of α and $m_x = (\alpha + \beta)/(\alpha + \beta + x)$, $x = 0, 1, ...$, the moment of the beta distribution with parameter vector $(\alpha + \beta, 1)$. Since (2.6) is equivalent to (2.3), we are then led by Theorem 2.2 to further cases of discrete distributions with completely monotone hazard sequences.

Remark 2.5.: *Feller (1966, Theorem XIII.7.1) used in Theorem 2.1 and Hausdorff's theorem used in the proof of Theorem 2.2 have proofs based on a certain version of ICFE, see, e.g, Rao and Shanbhag (1994, pp.72-75) and also, for a more recent account, Rao and Shanbhag (2014). On the other hand, for a systematic account of the historical literature on these theorems, we may refer the reader to Widder (1946). It may be worth pointing out here that Shanbhag et.al. (1977), Bondesson (1982) and Steutel and van Harn (2004) contain many interesting results or observations of relevance to the material covered in this article.*

Remark 2.6.: *If F is an absolutely continuous d.f. on $[0, \infty)$ or a discrete non-degenerate d.f. on $\{0, 1, ...\}$, with finite right extremity b, then the general representation for a survivor function in terms of the corresponding hazard measure, given by Kotz and Shanbhag (1980) tells us that the hazard function relative to F can not be decreasing. Since completely monotone functions of $(0, \infty)$ and sequences on $\{0, 1, ...\}$ are decreasing , this sheds further light on the role of the assumption that the survivor function be non-vanishing in the results that we have presented above.*

3. Some Concluding Observations

We may now take the opportunity to make some concluding observations on the key findings of the previous section, including in particular, those on the criteria for certain hazard functions and hazard sequences to be completely monotone. Obviously, there are some similarities as well as some differences between the criteria in the two cases. Some of the examples met in the remarks given earlier enlighten us in this matter. The remarks that appear below shed further light on the mechanisms of these criteria.

Remark 3.1.: *From the first observation in Remark 2.3, it follows that any mixture of geometric distributions with mixing distribution non-degenerate and concentrated on* $[c, 1)$ *where* $0 < c < 1$*, can not have the corresponding hazard sequence to be completely monotone; the mixing distribution appearing here is so that the logarithm of the corresponding random variable is bounded and non-degenerate. This information is essentially in the same spirit as that in Remark 2.1 for the hazard function relative to a mixture of exponential distributions with mixing distribution so that it corresponds to a bounded random variable.*

Remark 3.2.: $\overline{F}(x) = \exp\{-x + c(\exp\{-x\} - 1)\}$, $x \in [0, \infty)$*, with* $c > 0$*, is a survivor function on* $[0, \infty)$ *and has a completely monotone hazard function on* $(0, \infty)$*. However, its restriction to* $\{0, 1, ...\}$ *as a survivor function on* $\{0, 1, ...\}$ *does not meet the criterion for having a completely monotone hazard sequence , since in this latter case, a slight variation of an argument applied in Remark 2.3 implies that this is so; to see this, note that*

$$\begin{aligned} \ln(\overline{F}(x)/\overline{F}(x + 1)) &= 1 + c\ \exp\{-x\}(1 - e^{-1}) = -\ln(1 - \alpha\, m_x) \\ &= \alpha\, m_x + \alpha^2\, (m_x^2/2) + \alpha^3\, (m_x^3/3) + ... , x = 0, 1, ... , \end{aligned}$$

lead us to a contradiction, in view of the moment argument.

Remark 3.3.: *We may note that any non-trivial mixture (i.e. with mixing distribution non-degenerate) of the degenerate distribution at the origin and an exponential distribution is not purely absolutely continuous, and hence the criterion relative to hazard functions appearing in Corollary 2.1 is seen to be not applicable to it, while, by Remark 2.3, it is obvious that the discrete version of the mixture in question has indeed a completely monotone hazard sequence.*

Remark 3.4.: *Extending the argument applied in Remark 3.2, it can further be seen that if the survivor function of a probability distribution on* $\{0, 1, ...\}$ *agrees with the restriction to* $\{0, 1, ...\}$ *of the Laplace-Stieltjes transform of an i.d. distribution on* $[0, \infty)$*, meeting, additionally, the condition that the corresponding Lévy measure be non-null and concentrated on a bounded interval, then the survivor function can not satisfy (2.3); this implies that the hazard sequence in this case can not be completely monotone. It is also now clear that the product of finitely many survivor functions of this form is a survivor function of the same form, for which the hazard sequence is not completely monotone.*

Remark 3.5.: *It is obvious that the class of survivor functions (on* $[0, \infty)$*), meeting the criterion in Theorem 2.1 is closed under the operation of multiplication. This is also so for the class of survivor functions meeting the criterion in Corollary 2.1; note that, in this latter case , the criterion is for the hazard function, under appropriate assumptions, to be completely monotone. However, the class of survivor functions on* $\{0, 1, ...\}$ *for which the criterion in Theorem 2.2 is met, does not possess the closure property referred to . To illustrate this, one may consider, for example, a survivor function for which (2.3) is met with the moment sequence* $\{m_x : x = 0, 1, ...\}$ *as that corresponding to a degenerate distribution with support in* $(0, 1)$ *, or, as* $\{(\eta/(\eta + x))^\alpha : x = 0, 1, ...\}$*, where* $\eta > 0$ *and* $\alpha \in (0, \infty)$*, and verify, via a moment argument, that the squares of these survivor functions, as survivor functions themselves, are not of the form identified by (2.3), and hence, can not correspond to completely monotone hazard sequences. One may note in this connection, applying the binomial theorem, with minor manipulation, that, if* $\beta \in (0, 1]$*, the restriction to* $\{0, 1, ...\}$ *of the survivor function relative to the Pareto distribution met in Remark 2.1, as a survivor function on* $\{0, 1, ...\}$*, satisfies (2.3), since (in obvious notation)*

$$\overline{F}(x + 1)/\overline{F}(x) = (1 - (\lambda/(1 + \lambda(x + 1)))^\beta , x = 0, 1, ...;$$

using the binomial theorem again, it can also be seen that the assertion met here does not hold if we take in place of the condition "$\beta \in (0, 1]$*" the one that "*$\beta \in (1, 2]$*".*

Remark 3.6.: *One may find it interesting to see that under the stated assumptions (in Theorem 2.2), (2.3) implies that (in the stated notation) the function* $\overline{F}(x)/\overline{F}(x + 1),\ x = 1, 2, ...,$ *is either constant or proportional to the restriction to* $\{1, 2, ...\}$ *of the Laplace-Stieltjes transform of a non-degenerate compound geometric distribution, concentrated on* $[0, \infty)$ *(not necessarily, just with integral support points).*

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