International Journal of Statistics and Probability; Vol. 4, No. 3; 2015
ISSN 1927-7032 E-ISSN 1927-7040
Published by Canadian Center of Science and Education

Some Generalized Families of Weibull Distribution: Properties and Applications

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Received: April 1, 2015 Accepted: April 24, 2015 Online Published: July 1, 2015

doi:10.5539/ijsp.v4n3p18 URL: http://dx.doi.org/10.5539/ijsp.v4n3p18

Abstract

The Weibull distribution has been applied in various fields, especially to fit life time data. Some of these applications are limited partly due to the fact that the distribution has monotonically increasing, monotonically decreasing or constant hazard rate. This limitation undoubtedly inspired researchers to develop generalized Weibull distribution that can exhibit unimodal or bathtub hazard rate. In this article, we introduce six new families of *T*-Weibull{*Y*} distributions arising from the quantile function of a random variable *Y*. These six families are: The *T*-Weibull{uniform}, *T*-Weibull{exponential}, *T*-Weibull{log-logistic}, *T*-Weibull{Fréchet}, *T*-Weibull{logistic} and *T*-Weibull{extreme value}. Some properties of these families are discussed and general expressions for the quantile function, the Shannon's entropy, the non-central moments and the mean deviations are provided. Different new members of the *T*-Weibull{*Y*} families are derived and some of their properties are discussed. Two real data sets are used to illustrate the potential usefulness of the *T*-Weibull{*Y*} distributions and the results are compared with the results from some existing distributions.

Keywords: moments, quantile function, Shannon's entropy, generalized distribution, *T-X* family

1. Introduction

The Weibull distribution is a well-known distribution named after the Swedish physicist, Waloddi Weibull, who first published its application to fly ash and strength of material in 1951. It was popularized by Kao (1956) when he applied it to the failure of electronic components and systems, and since then, it has been extensively used for analyzing lifetime data. The probability density function (PDF) of a two-parameter Weibull distribution (Johnson et al., 1994) is

$$f(x)=\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}e^{-(x/\lambda)^{k}}, \tag{1.1}$$

where $\lambda>0$ is the scale parameter and $k>0$ is the shape parameter. The cumulative distribution function (CDF) corresponding to (1.1) is

$$F(x)=1-e^{-(x/\lambda)^{k}}. \tag{1.2}$$

The Weibull distribution has exponential and Rayleigh distributions as special cases. It has been extensively used for modeling phenomenon with monotone failure rates. When modeling monotone hazard rates, the Weibull distribution may be an initial choice because of its negatively and positively skewed density shapes. However, it does not provide reasonable parametric fit for modeling phenomenon with non-monotone failure rates such as the bathtub shaped and the unimodal failure rates.

In the last few years, new classes of distributions were proposed by extending the Weibull distribution to cope with bathtub and unimodal failure rates. A review of some of these models includes the exponentiated Weibull distribution which was first introduced by Mudholkar and Srivastava (1993) to analyze bathtub failure data. Mudholkar and Hutson (1996) studied the properties of the exponentiated Weibull distribution and showed that its failure rate can be bathtub, upside-down, monotonically increasing or monotonically decreasing. These authors applied the exponentiated Weibull distribution to lifetime data sets and compared the fit with other distributions. Nassar and Eissa (2003) studied some of the statistical properties of exponentiated Weibull distribution such as moments, mode, and the hazard function. Singh et al. (2002) and Nassar and Eissa (2004) studied and compared the maximum likelihood and Bayesian methods of estimation for the exponentiated Weibull distribution.

Lai et al. (2003) proposed the modified Weibull distribution by introducing another shape parameter to the Weibull distribution in (1.1). The distribution has an advantage of being able to model data with bathtub shaped failure rate. Cooray (2006) introduced the odd Weibull distribution and showed that it can exhibit monotonic, unimodal and bathtub shaped failure rates, a good property in fitting real life applications.

The beta-generated family was first proposed by Eugene et al. (2002) and further studied by Jones (2004) and Zografos and Balakrishnan (2009). The beta distribution with PDF $b(t)$ is used to generate this class of distributions, and the CDF of beta-generated distribution is defined by

$$G(x) = \int_0^{F(x)} b(t)dt , \tag{1.3}$$

where $F(x)$ is the CDF of any random variable *X*. The beta-Weibull distribution by Famoye et al. (2005) and beta-modified Weibull distribution by Silva et al. (2010) are members of this class. Jones (2009) and Cordeiro and de Castro (2011) introduced another family, the Kumaraswamy-generated (*Kum*-G) distributions using Kumaraswamy PDF instead of beta PDF used in (1.3). Under this family, the *Kum*-Weibull distribution by Cordeiro et al. (2010) is a generalization of the Weibull distribution.

Both beta-generated and Kumaraswamy-generated families use distributions with support between 0 and 1 as generators. Recently, Alzaatreh et al. (2013a) proposed a general method that allows the use of any continuous PDF as a generator.

Let $r(t)$ be the PDF of a random variable $T \in [a,b]$, $-\infty \le a < b \le \infty$. Let $W(F(x))$ be a monotonic and absolutely continuous function of the CDF $F(x)$ of any random variable *X*. The CDF of a new family of distributions defined by Alzaatreh et al. (2013a) is

$$G(x) = \int_a^{W(F(x))} r(t)dt = R\left\{W(F(x))\right\} , \tag{1.4}$$

where $R(.)$ is the CDF of the random variable *T*. This family is named as *T*-*X* family. Different $W(.)$ functions generate different families of *T*-*X* distributions. By taking $W(F(x)) = -\log\{1-F(x)\}$ in (1.4), Alzaatreh et al. (2013a) studied some properties of the family in (1.4). A number of new distributions under this family were developed and studied including: Weibull-Pareto distribution by Alzaatreh et al. (2013b), gamma-Pareto distribution by Alzaatreh et al. (2012) and gamma-normal distribution by Alzaatreh et al. (2014a). A large number of distributions can be generated by using $W(.)$ function defined on the CDF $F(x)$ by applying any two existing univariate distributions based on this method.

Aljarrah et al. (2014) extended the *T*-*X* family by defining $W(.)$ in (1.4) as a quantile function of an existing random variable *Y*. A unified definition of this family was given in Alzaatreh et al. (2014b) and named as *T*-*R*{*Y*}. Let *T*, *R* and *Y* be random variables with CDF $F_T(x) = P(T \le x)$, $F_R(x) = P(R \le x)$ and $F_Y(x) = P(Y \le x)$. The corresponding quantile functions are $Q_T(p)$, $Q_R(p)$ and $Q_Y(p)$, and if the densities exist, then they can be denoted by $f_T(x)$, $f_R(x)$ and $f_Y(x)$, respectively. Following this notation, the CDF of *T*-*R*{*Y*} is given by

$$F_X(x) = \int_a^{Q_Y(F_R(x))} f_T(t)dt = F_T\{Q_Y(F_R(x))\} , \tag{1.5}$$

and the PDF corresponding to (1.5) is given by

$$f_X(x) = \frac{f_R(x)}{f_Y\left\{Q_Y\left(F_R(x)\right)\right\}} f_T\left\{Q_Y\left(F_R(x)\right)\right\} . \tag{1.6}$$

The purpose of this article is to propose several different generalizations of Weibull family, by using the *T*-*R*{*Y*} framework. If *R* is a Weibull random variable, then we propose the *T*-Weibull{*Y*} family of distributions by taking $f_R(x)$ and $F_R(x)$ in (1.6) to be the PDF and the CDF of the Weibull distribution given in (1.1) and (1.2) respectively.

The remainder of this article is organized as follows. Section 2 defines several new generalizations of Weibull distribution. In section 3, some general properties of the proposed families are studied. Section 4 presents some members of the *T*-Weibull{*Y*} family and some of their properties. In section 5, two real data sets are used as applications to illustrate the flexibility of *T*-Weibull{*Y*} family. Finally, the article is concluded in section 6.

2. Some *T*-Weibull{*Y*} Families Based on Different Quantile Functions $Q_Y(p)$

In this section we define the families of generalized Weibull distribution, *T*-Weibull{*Y*}, based on the quantile functions defined in Table 1.

Table 1. Different choices of the quantile functions

Random variable *Y*	The quantile function
Uniform	$Q_Y(p) = p$
Exponential	$Q_Y(p) = -b\log(1-p),\ b > 0$
Log-logistic	$Q_Y(p) = \theta[p/(1-p)]^{1/\beta},\ \theta, \beta > 0$
Fréchet	$Q_Y(p) = \theta[-\log(p)]^{-1/b},\ \theta, b > 0$
Logistic	$Q_Y(p) = \theta\log[p/(1-p)],\ \theta > 0$
Extreme value	$Q_Y(p) = b\log[-\log(1-p)],\ b > 0$

a) *T*-Weibull{uniform}: This family is generated by using the quantile function of the uniform distribution in Table 1. In this case, the support of the random variable *T* is [0, 1]. Given that $Q_Y(F_R(x)) = F_R(x)$, the CDF of this family as defined in (1.5) is given by

$$F_X(x) = F_T\{F_R(x)\}, \tag{2.1}$$

and the corresponding PDF is

$$f_X(x) = f_R(x) f_T\{F_R(x)\}. \tag{2.2}$$

The beta-Weibull distribution (Famoye et al., 2005) and Kumaraswamy-Weibull distribution (Cordeiro et al., 2010) are members of this family.

b) *T*-Weibull{exponential}: This family is generated by using the quantile function of the exponential distribution in Table 1. In this case, the support of the random variable *T* is $(0,\infty)$. Given that $Q_Y(F_R(x)) = -b\log\left(1-F_R(x)\right)$, the CDF of this family as defined in (1.5) is given by

$$F_X(x) = F_T\left\{-b\log\left(1-F_R(x)\right)\right\}, \tag{2.3}$$

and the corresponding PDF is

$$f_X(x) = \frac{bf_R(x)}{1-F_R(x)} f_T\left\{-b\log\left(1-F_R(x)\right)\right\}. \tag{2.4}$$

Since $f_R(x)/(1-F_R(x))$ and $-\log(1-F_R(x))$ are, respectively, the hazard and cumulative hazard functions of the Weibull distribution, this family of distributions can be considered as a family of distributions arising from weighted hazard function of Weibull distribution. Gamma-Weibull distribution proposed by Alzaatreh et al. (2014a) is a member of this family.

c) *T*-Weibull{log-logistic}: This family is generated by using the quantile function of the log-logistic distribution in Table 1. In this case, the support of the random variable *T* is $(0,\infty)$. Given that $Q_Y(F_R(x)) = \theta\left[F_R(x)/\left(1-F_R(x)\right)\right]^{1/\beta}$, the CDF of this family as defined in (2.5) is given by

$$F_X(x) = F_T\left\{\theta\left[F_R(x)/\left(1-F_R(x)\right)\right]^{1/\beta}\right\}, \tag{2.5}$$

and the corresponding PDF is

$$f_X(x) = \frac{(\theta/\beta)f_R(x)}{F_R^{(\beta-1)/\beta}(x)\left(1-F_R(x)\right)^{(\beta+1)/\beta}}\, f_T\left\{\theta\left(\frac{F_R(x)}{1-F_R(x)}\right)^{1/\beta}\right\}. \tag{2.6}$$

When $\theta=\beta=1$, the family in (2.6) reduces to

$$g(x) = \frac{f_R(x)}{\left(1-F_R(x)\right)^2}\, f_T\left\{\frac{F_R(x)}{1-F_R(x)}\right\}. \tag{2.7}$$

The PDF in (2.7) can be written in terms of hazard and survival functions of Weibull distribution. The odd Weibull distribution (Cooray, 2006) and Weibull-Weibull distribution (Bourguignon et al., 2014) are members of this family by taking the random variable *T* in (2.7) to be standard log-logistic and Weibull respectively.

d) *T*-Weibull{Fréchet}: This family is generated by using the quantile function of the Fréchet distribution in Table 1. In this case, the support of the random variable *T* is $(0,\infty)$. Given that $Q_Y(F_R(x)) = \theta\left[-\log F_R(x)\right]^{-1/b}$, the CDF of this family as defined in (1.5) is given by

$$F_X(x) = F_T\left\{\theta\left[-\log F_R(x)\right]^{-1/b}\right\}, \tag{2.8}$$

and the corresponding PDF is

$$f_X(x) = \frac{(\theta/b)f_R(x)}{F_R(x)\left(-\log F_R(x)\right)^{(b+1)/b}}\, f_T\left\{\theta\left[-\log F_R(x)\right]^{-1/b}\right\}. \tag{2.9}$$

e) *T*-Weibull{logistic}: This family is generated by using the quantile function of the logistic distribution in Table 1. In this case, the support of the random variable *T* is $(-\infty,\infty)$. Given that $Q_Y(F_R(x)) = \theta\log\left[F_R(x)/\left(1-F_R(x)\right)\right]$, the CDF of this family as defined in (1.5) is given by

$$F_X(x) = F_T\left\{\theta\log\left[F_R(x)/\left(1-F_R(x)\right)\right]\right\}, \tag{2.10}$$

and the corresponding PDF is

$$f_X(x) = \frac{\theta f_R(x)}{F_R(x)\left(1-F_R(x)\right)}\, f_T\left\{\theta\log\left[F_R(x)/\left(1-F_R(x)\right)\right]\right\}. \tag{2.11}$$

Normal-Weibull distribution (Aljarrah et al., 2014) and Gumbel-Weibull distribution (Al-Aqtash et al., 2014) are members of this family.

f) *T*-Weibull{extreme value}: This family is generated by using the quantile function of the extreme value distribution in Table 1. In this case, the support of the random variable *T* is $(-\infty,\infty)$. Given that $Q_Y(F_R(x)) = b\log\left[-\log\left(1-F_R(x)\right)\right]$, the CDF of this family as defined in (1.5) is given by

$$F_X(x) = F_T\left\{b\log\left[-\log\left(1-F_R(x)\right)\right]\right\}, \tag{2.12}$$

and the corresponding PDF is

$$f_X(x) = \frac{bf_R(x)}{-[1-F_R(x)]\log[1-F_R(x)]} f_T\left\{b\log\left[-\log\left(1-F_R(x)\right)\right]\right\}. \tag{2.13}$$

The support of any distribution under the families (a) to (f) is $(0,\infty)$ and the parameter set comes from Weibull distribution and the random variables *T* and *Y*. An extension of these families is by using $F_R^{\alpha}(x)$ in place of $F_R(x)$ to obtain the exponentiated *T*-Weibull{*Y*} families. Among all families (a) to (f), the family of distributions in (b) appeared to have been studied more widely and has many references in the literature. For more details about this family, one may refer to Pinho et al. (2012), Zografos and Balakrishnan (2009) and Alzaatreh et al. (2012, 2013a, 2013b, 2014a).

3. General Properties of *T*-Weibull{*Y*} Families of Distributions

In this section, we present some general properties of the *T*-Weibull{*Y*} families of distributions, including transformations, quantile functions, implicit formula for the mode(s), Shannon's entropies, moments and mean deviations.

3.1 Transformation and Quantile Function

The relationship between the random variable *X* of *T*-Weibull{*Y*} distribution (where *Y* represents uniform, exponential, log-logistic, Fréchet, logistic or extreme value) and the generator random variable *T* is given in the following lemma.

Lemma 1: Let *T* be a random variable with PDF $f_T(x)$.

a) If *T* is defined on [0, 1], then the random variable $X = \lambda\left(-\log(1-T)\right)^{1/k}$ belongs to the *T*-Weibull{uniform} family in (2.1).

b) If *T* is defined on $(0,\infty)$, then the random variable

  i. $X = \lambda(T/b)^{1/k}$ belongs to the *T*-Weibull{exponential} family in (2.3).

  ii. $X = \lambda\left[\log\left((T/\theta)^{\beta}+1\right)\right]^{1/k}$ belongs to the *T*-Weibull{log-logistic} family in (2.5).

  iii. $X = \lambda\left[-\log\left(1-e^{-(T/\theta)^{-b}}\right)\right]^{1/k}$ belongs to the *T*-Weibull{Fréchet} family in (2.8).

c) If *T* is defined on $(-\infty,\infty)$, then the random variable

  i. $X = \lambda\left[\log\left(e^{T/\theta}+1\right)\right]^{1/k}$ belongs to the *T*-Weibull{logistic} family in (2.10).

  ii. $X = \lambda e^{T/(bk)}$ belongs to the *T*-Weibull{extreme value} family in (2.12).

Proof: The results follow directly from the way we defined the *T* random variable in (2.1), (2.3), (2.5), (2.8), (2.10) and (2.12). □

Lemma 1 gives the relationships between the random variable *X* and the random variable *T*. These relationships can be used to generate random samples from *X* by using *T*. For example, one can simulate the random variable *X* which follows the distribution of *T*-Weibull{uniform} family in (2.2) by first simulating random variable *T* from the PDF $f_T(x)$ and then computing $X = \lambda(-\log(1-T))^{1/k}$, which has the CDF $F_X(x)$.

The following Lemma gives a general formula for the quantile function of the *T*-Weibull{*Y*} families.

Lemma 2: The quantile function for the (i) *T*-Weibull{uniform}, (ii) *T*-Weibull{exponential}, (iii) *T*-Weibull{log-logistic}, (iv) *T*-Weibull{Fréchet}, (v) *T*-Weibull{logistic} and (vi) *T*-Weibull{extreme value} distributions, are respectively,

(i) $Q_X(p) = \lambda[-\log(1-Q_T(p))]^{1/k}$,

(ii) $Q_X(p) = \lambda(Q_T(p)/b)^{1/k}$,

(iii) $Q_X(p) = \lambda\{\log[(Q_T(p)/\theta)^{\beta} + 1]\}^{1/k}$,

(iv) $Q_X(p) = \lambda[-\log(1 - e^{(Q_T(p)/\theta)^{-b}})]^{1/k}$,

(v) $Q_X(p) = \lambda\{\log(e^{Q_T(p)/\theta} + 1)\}^{1/k}$,

(vi) $Q_X(p) = \lambda e^{Q_T(p)/(bk)}$.

Proof: The results follow directly by solving $F_X(Q_X(p)) = p$ for $Q_X(p)$, where $F_X(.)$ is the CDF given in (2.1), (2.3), (2.5), (2.8), (2.10) and (2.12). Note that $F_T^{-1}(p) = Q_T(p)$. □

3.2 Mode(s)

The following theorem presents an implicit formula for the mode(s) of the *T*-Weibull{*Y*} families, where $\overline{F}_R(x) = 1 - F_R(x)$ is the survival function of the Weibull distribution.

Theorem 1: The mode(s) of the (i) *T*-Weibull{uniform}, (ii) *T*-Weibull{exponential}, (iii) *T*-Weibull{log-logistic}, (iv) *T*-Weibull{Fréchet}, (v) *T*-Weibull{logistic} and (vi) *T*-Weibull{extreme value} distributions, respectively, are the solutions of the equations

(i)
$$x = \begin{cases} \lambda\left[\dfrac{k}{k-1}\left(\dfrac{-f_T'(F_R(x))}{f_T(F_R(x))}\overline{F}_R(x) + 1\right)\right]^{-1/k}, & k \neq 1 \\ \lambda\log\left[\dfrac{f_T'(F_R(x))}{f_T(F_R(x))}\right], & k = 1, \end{cases} \tag{3.1}$$

(ii)
$$x = \begin{cases} \lambda\left[\dfrac{-bk\ f_T'\left[-b\log(\overline{F}_R(x))\right]}{(k-1)\,f_T\left[-b\log(\overline{F}_R(x))\right]}\right]^{-1/k}, & k \neq 1 \\ \text{Scalar multiple of the mode(s) of } f_T(x), & k = 1, \end{cases} \tag{3.2}$$

(iii)
$$x = \begin{cases} \lambda\left[\dfrac{k}{(k-1)F_R(x)}\left(\dfrac{-f_T'(W_1(x))}{\beta f_T(W_1(x))}W_1(x) - F_R(x) + \dfrac{\beta-1}{\beta}\right)\right]^{-1/k}, & k \neq 1 \\ \lambda\log\left[\dfrac{1}{F_R(x)\overline{F}_R(x)}\left(\dfrac{f_T'(W_1(x))}{\beta f_T(W_1(x))}W_1(x) + 2F_R(x) - \dfrac{\beta-1}{\beta}\right)\right], & k = 1, \end{cases} \tag{3.3}$$

where $W_1(x) = \theta\left(F_R(x)/\overline{F}_R(x)\right)^{1/\beta}$.

(iv)
$$x = \begin{cases} \lambda\left\{\dfrac{k}{k-1}\left[\dfrac{\overline{F}_R(x)}{F_R(x)\log F_R(x)}\left(\dfrac{f_T'(W_2(x))}{bf_T(W_2(x))}W_2(x) + \dfrac{\log F_R(x)}{\overline{F}_R(x)} + \dfrac{b+1}{b}\right) + 1\right]\right\}^{-1/k}, & k \neq 1 \\ \lambda\log\left[\dfrac{1}{F_R(x)\log F_R(x)}\left(\dfrac{-f_T'(W_2(x))}{bf_T(W_2(x))}W_2(x) - \log F_R(x) - \dfrac{b+1}{b}\right)\right], & k = 1, \end{cases} \tag{3.4}$$

where $W_2(x) = \theta\left(-\log F_R(x)\right)^{-1/b}$.

(v) $$x=\begin{cases}\lambda\left[\dfrac{k}{(k-1)F_R(x)}\left(\dfrac{-\theta f_T'(W_3(x))}{f_T(W_3(x))}-F_R(x)+1\right)\right]^{-1/k}, & k\neq 1\\ \lambda\log\left[\dfrac{1}{F_R(x)\overline{F}_R(x)}\left(\dfrac{\theta f_T'(W_3(x))}{f_T(W_3(x))}+2F_R(x)-1\right)\right], & k=1,\end{cases} \tag{3.5}$$

where $W_3(x)=\theta\log[F_R(x)/\overline{F}_R(x)]$.

(vi) $$x=\begin{cases}\lambda\left[\dfrac{k}{(k-1)\log\left(\overline{F}_R(x)\right)}\left(\dfrac{bf_T'(W_4(x))}{f_T(W_4(x))}-1\right)\right]^{-1/k}, & k\neq 1\\ \lambda\log\left[\dfrac{1}{\overline{F}_R(x)\log\left(\overline{F}_R(x)\right)}\left(\dfrac{-bf_T'(W_4(x))}{f_T(W_4(x))}+\log\left(\overline{F}_R(x)\right)+1\right)\right], & k=1,\end{cases} \tag{3.6}$$

where $W_4(x)=b\log[-\log(\overline{F}_R(x))]$.

Proof: We first show (i) by using the fact that

$$f_R'(x)=\begin{cases}\dfrac{f_R^2(x)}{\overline{F}_R(x)}\left(-1+\dfrac{k-1}{k}\left(\dfrac{x}{\lambda}\right)^{-k}\right), & k\neq 1\\ -f_R^2(x)e^{x/\lambda}, & k=1.\end{cases} \tag{3.7}$$

The derivative with respect to x of (2.2) can be simplified to $f_R'(x)=f_R^2(x)m(x)$, where

$$m(x)=\begin{cases}f_T'(F_R(x))+\dfrac{f_T(F_R(x))}{\overline{F}_R(x)}\left(-1+\dfrac{k-1}{k}\left(\dfrac{x}{\lambda}\right)^{-k}\right), & k\neq 1\\ f_T'(F_R(x))-f_T(F_R(x))e^{x/\lambda}, & k=1.\end{cases} \tag{3.8}$$

By setting $f_X'(x)$ to 0, the mode of $f_X(x)$ is the solution of the equation $m(x)=0$. On solving $m(x)=0$ in (3.8) gives the result of the theorem in part (i). The results of (ii)-(vi) can be shown by using a similar technique in (i). □

3.3 Shannon's Entropy

The entropy of a random variable is a measure of the variation of uncertainty. Shannon's (1948) entropy of the random variable *X* with density $g(x)$ is defined as $E\left[-\log\left(g(X)\right)\right]$.

Theorem 2: The Shannon's entropies for the (i) *T*-Weibull{uniform}, (ii) *T*-Weibull{exponential}, (iii) *T*-Weibull{log-logistic}, (iv) *T*-Weibull{Fréchet}, (v) *T*-Weibull{logistic} and (vi) *T*-Weibull{extreme value} distributions, respectively, are given by

(i) $$\eta_X=\log(\lambda/k)-\left((k-1)/k\right)E\{\log[-\log(1-T)]\}+(1/\lambda^k)\mu_k'+\eta_T, \tag{3.9}$$

(ii) $$\eta_X=\log(\lambda/bk)-\left((k-1)/k\right)E[\log(T/b)]+\left((1-b)/b\right)\mu_T+\eta_T, \tag{3.10}$$

(iii) $$\eta_X=\log\left(\beta\lambda/(\theta k)\right)-\left((k-1)/k\right)E\{\log[\log((T/\theta)^{\beta}+1)]\}-\left((\beta-1)/\beta\right)E[\log((T/\theta)^{-\beta}+1)]$$

$$-(1/\beta\lambda^k)\mu_k'+\eta_T, \tag{3.11}$$

(iv) $\eta_X = \log\left(\beta\lambda/(\theta k)\right) - \left((k-1)/k\right)E\{\log[-\log(1-e^{-(T/\theta)^{-b}})]\} - (b+1)E[\log(T/\theta)] - E[(T/\theta)^{-b}]$

$$+(1/\lambda^k)\mu'_k + \eta_T, \qquad (3.12)$$

(v) $\eta_X = \log\left(\lambda/(bk)\right) - \left((k-1)/k\right)E\{\log[\log(e^{T/\theta}+1)]\} - E[\log(e^{-T/\theta}+1)] + \eta_T$, (3.13)

(vi) $\eta_X = \log\left(\lambda/(bk)\right) + \left(1/(bk)\right)\mu_T + \eta_T$, (3.14)

where μ'_k is the k^{th} non-central moment of the *T*-Weibull{*Y*} distribution (where *Y* represents uniform, exponential, log-logistic, Fréchet, logistic or extreme value), μ_T and η_T are the mean and Shannon's entropy of the random variable *T*.

Proof: We first show (3.9). By definition,

$$\eta_X = E\left(-\log[f_X(X)]\right) = -E\left(\log f_R(X)\right) + E\left(-\log f_T(F_R(X))\right). \qquad (3.15)$$

From (2.1), the random variable $T(=F_R(X))$ has PDF $f_T(x)$, so

$$E\left(-\log f_T(F_R(X))\right) = E\left(-\log f_T(T)\right) = \eta_T. \qquad (3.16)$$

Since $\log f_R(x) = \log(\lambda/k) + (k-1)\log(x/\lambda) - (x/\lambda)^k$, we obtain $E\left(\log f_R(X)\right) = \log(\lambda/k) + (k-1)E\left(\log(X/\lambda)\right) - E\left((X/\lambda)^k\right)$. By Lemma 1(i), $X = \lambda\left(-\log(1-T)\right)^{1/k}$ follows the *T*-Weibull{uniform} distribution, and this implies that

$$E\left(\log f_R(X)\right) = \log(\lambda/k) + \left((k-1)/k\right)E\left(\log[-\log(1-T)]\right) - (1/\lambda^k)\mu'_k. \qquad (3.17)$$

The result in (3.9) follows from equations (3.15)- (3.17). By applying similar technique we can show the results in (3.10)- (3.14). □

3.4 Moments

In this subsection, we will use the relationship given in Lemma 1 between the generator random variable *T* and the random variable *X* that follows any of the *T*-Weibull{*Y*} families to find the non-central moment of the random variable *X*. For example, let us consider a random variable *X* that follows the distribution of *T*-Weibull{exponential} family in (2.2). The r^{th} non-central moment of *X* can be expressed as

$$E\left(X^r\right) = E\left[\left(\lambda(-\log(1-T))^{1/k}\right)^r\right] = \lambda^r E\left[\left(-\log(1-T)\right)^{r/k}\right].$$

A useful expansion of the expression $\left(-\log(1-T)\right)^{r/k}$ can be derived using the formula (http://functions.wolfram.com/Elementary Functions/ Log/06/01/04/03/):

$$\left(-\log(1-z)\right)^a = a\sum_{i=0}^{\infty}\binom{i-a}{i}\sum_{j=0}^{i}\frac{(-1)^{i+j}}{a-j}\binom{i}{j}p_{j,i}z^{a+i}, \qquad (3.18)$$

where $a>0$ is any real value and $|z|<1$. The constants $p_{j,i}$ can be calculated recursively by using

$$p_{j,i} = \frac{1}{i}\sum_{m=1}^{i}\frac{(jm-i+m)(-1)^m}{m+1}p_{j,i-m}, \text{ for } i=1,2,3,\ldots, \text{ and } p_{j,0}=1. \qquad (3.19)$$

In the following theorem we are able to find general expressions for the non-central moments of all families presented in section 2 except for the two families *T*-Weibull{logistic} and *T*-Weibull{log-logistic}. This is because the formula in (3.18) cannot be used to find a general expansion of the non-central moment of the families *T*-Weibull{logistic} and *T*-Weibull{log-logistic} due to the convergence condition $|z|<1$. However, one may be able to derive the expression of the non-central moments for specific random variable *T* without using the formula in (3.18). Further study is needed for these two families.

Theorem 3: The r^{th} non-central moments of the (i) *T*-Weibull{uniform}, (ii) *T*-Weibull{Fréchet}, (iii) *T*-Weibull{exponential}, (iv) *T*-Weibull{extreme value} distributions, respectively, are given by

$$\text{(i)}\quad E(X^r)=\frac{r\lambda^r}{k}\sum_{i=0}^{\infty}\binom{i-r/k}{i}\sum_{j=0}^{i}\frac{(-1)^{i+j}}{(r/k)-j}\binom{i}{j}p_{j,i}E\left(T^{(r/k)+i}\right), \tag{3.20}$$

$$\text{(ii)}\quad E(X^r)=\frac{r\lambda^r}{k}\sum_{i=0}^{\infty}\binom{i-r/k}{i}\sum_{j=0}^{i}\frac{(-1)^{i+j}}{(r/k)-j}\binom{i}{j}p_{j,i}E\left(e^{-(T/\theta)^{-b}((r/k)+i)}\right), \tag{3.21}$$

$$\text{(iii)}\quad E(X^r)=(\lambda/b^{1/k})^r E\left(T^{r/k}\right), \tag{3.22}$$

$$\text{(iv)}\quad E(X^r)=\lambda^r M_T(r/(bk)), \tag{3.23}$$

where $M_T(r/(bk))=E(e^{(r/(bk))T})$ and $p_{j,i}$ is given in equation (3.19).

Proof: We first show (3.20). Using Lemma 1, the r^{th} non-central moments for *T*-Weibull{uniform} distribution can be written as

$$E(X^r)=E\left\{\left[\lambda\left(-\log(1-T)\right)^{1/k}\right]^r\right\}=\lambda^r E\left[\left(-\log(1-T)\right)^{r/k}\right]. \tag{3.24}$$

Using the series representation in (3.18) for $\left(-\log(1-T)\right)^{r/k}$, we get

$$\left(-\log(1-T)\right)^{r/k}=\frac{r}{k}\sum_{i=0}^{\infty}\binom{i-r/k}{i}\sum_{j=0}^{i}\frac{(-1)^{i+j}}{(r/k)-j}\binom{i}{j}p_{j,i}T^{(r/k)+i}, \tag{3.25}$$

where $p_{j,i}$ is given in (3.19). The result of (3.20) follows by using equation (3.25) in equation (3.24). Using a similar technique, we can obtain the result in (3.21). The results in (3.22) and (3.23) can be derived easily using parts b(i) and c(ii) of Lemma 1, respectively. □

3.5 Mean Deviations

The deviation from the mean or deviation from the median can be used as a measure of spread from the center in a population. For any random variable *X*, let D_μ and D_M be the mean deviation from the mean and the mean deviation from the median, respectively. The following theorem gives general expressions for D_μ and D_M when *X* follows *T*-Weibull{*Y*}, where *Y* represents uniform, Fréchet, exponential, or extreme value distribution.

Theorem 4: The D_μ and D_M for each of (i) *T*-Weibull{uniform}, (ii) *T*-Weibull{Fréchet}, (iii) *T*-Weibull{exponential}, (iv) *T*-Weibull{extreme value} distributions, respectively, are given by

$$\text{(i)}\quad D_\mu=2\mu F_X(\mu)-\frac{2\lambda}{k}\sum_{i=0}^{\infty}\binom{i-1/k}{i}\sum_{j=0}^{i}\frac{(-1)^{i+j}}{(1/k)-j}\binom{i}{j}p_{j,i}S_u(\mu,0,k^{-1}+i), \tag{3.26}$$

$$D_M=\mu-\frac{2\lambda}{k}\sum_{i=0}^{\infty}\binom{i-1/k}{i}\sum_{j=0}^{i}\frac{(-1)^{i+j}}{(1/k)-j}\binom{i}{j}p_{j,i}S_u(M,0,k^{-1}+i), \tag{3.27}$$

where $S_{\varphi(u)}(c,\delta,\alpha)=\int_{\delta}^{Q_Y(F_R(c))}\left(\varphi(u)\right)^{\alpha}f_T(u)du$ and $Q_Y\left(F_R(c)\right)=F_R(c)$.

(ii) $$D_{\mu} = 2\mu F_X(\mu) - \frac{2\lambda}{k}\sum_{i=0}^{\infty}\binom{i-1/k}{i}\sum_{j=0}^{i}\frac{(-1)^{i+j}}{(1/k)-j}\binom{i}{j}p_{j,i}S_{e^{-(u/\theta)^{-b}}}(\mu, 0, k^{-1}+i)\,, \tag{3.28}$$

$$D_M = \mu - \frac{2\lambda}{k}\sum_{i=0}^{\infty}\binom{i-1/k}{i}\sum_{j=0}^{i}\frac{(-1)^{i+j}}{(1/k)-j}\binom{i}{j}p_{j,i}S_{e^{-(u/\theta)^{-b}}}(M, 0, k^{-1}+i)\,, \tag{3.29}$$

where $Q_Y\left(F_R(c)\right) = \theta\left(-\log F_R(c)\right)^{-1/b}$.

(iii) $D_{\mu} = 2\mu F_X(\mu) - 2(\lambda / b^{1/k})S_u(\mu, 0, 1/k)$, $D_M = M - 2(\lambda / b^{1/k})S_u(M, 0, 1/k)$,

where $Q_Y\left(F_R(c)\right) = -b\log\left(1 - F_R(c)\right)$.

(iv) $D_{\mu} = 2\mu F_X(\mu) - 2\lambda S_{e^u}(\mu, -\infty, 1/(bk))$, $D_M = M - 2\lambda S_{e^u}(M, -\infty, 1/(bk))$.

where $Q_Y\left(F_R(c)\right) = b\log\left(-\log(1 - F_R(c))\right)$.

Proof: The D_{μ} and D_M are defined by

$$D_{\mu} = E(|X-\mu|) = \int_{-\infty}^{\infty}|x-\mu|\ f_X(x)dx = \int_{-\infty}^{\mu}(\mu - x)f_X(x)dx + \int_{\mu}^{\infty}(x-\mu)f_X(x)dx$$

$$= 2\mu F_X(\mu) - 2\int_{-\infty}^{\mu}xf_X(x)dx\,. \tag{3.30}$$

$$D_M = E(|X-M|) = \int_{-\infty}^{\infty}|x-M|\ f_X(x)dx = \int_{-\infty}^{M}(M - x)f_X(x)dx + \int_{M}^{\infty}(x-M)f_X(x)dx$$

$$= \mu - 2\int_{-\infty}^{M}xf_X(x)dx\,. \tag{3.31}$$

In order to find the two integrals in (3.30) and (3.31), let $I(c) = \int_{-\infty}^{c}xf_X(x)dx$. We now prove the results in (3.26) and (3.27) for *T*-Weibull{uniform} family. By using (2.2), we re-write $I(c)$ as

$$I(c) = \int_{-\infty}^{c}xf_X(x)dx = \int_{-\infty}^{c}xf_R(x)f_T\left(F_R(x)\right)dx\,. \tag{3.32}$$

Using the substitution $u = F_R(x)$ in (3.32), $I(c)$ can be expressed as

$$I(c) = \int_{0}^{F_R(c)}\lambda\left(-\log(1-u)\right)^{1/k}\ f_T(u)du\,. \tag{3.33}$$

By using a similar technique shown in (3.25) to write an expansion of $\left(-\log(1-u)\right)^{1/k}$, we get

$$I(c) = \frac{\lambda}{k}\sum_{i=0}^{\infty}\binom{i-1/k}{i}\sum_{j=0}^{i}\frac{(-1)^{i+j}}{(1/k)-j}\binom{i}{j}p_{j,i}S_u(c, 0, k^{-1}+i)\,, \tag{3.34}$$

where $p_{i,j}$ is defined in (3.19) and $S_{\varphi(u)}(c,a,\alpha)=\int_a^{Q_Y(F_R(c))}(\varphi(u))^{\alpha} f_T(u)du$. Now the result in (3.26) and (3.27) follow by using (3.34) in (3.30) and (3.31). Applying a similar approach used to obtain the results in (3.26) and (3.27), we can show the results in (3.28) and (3.29). Parts (iii) and (iv) can be derived by using parts b(i) and c(ii) of Lemma 1, respectively. □

4. Examples on *T*-Weibull{*Y*} Families of Distributions

In this section, we give an example for each of the *T*-Weibull{*Y*} families presented in section 2 for different choices of the random variable *T*. Since a lot of work has been done on the *T*-*X*{exponential} family where many new distributions were developed; no example will be presented when *Y* is an exponential random variable. In the following five examples, we take the distribution of the random variable *T* as (a) Weibull-uniform{log-logistic} (WULL) defined by Aljarrah et al. (2014), (b) Lomax, (c) Exponential, (d) Cauchy and (e) Weibull-logistic (WL) defined by Alzaatreh et al. (2013a). These five distributions are new generalized Weibull distributions. Many more new generalized Weibull distributions can be derived using different *T* and *Y* random variables.

4.1 WULL-Weibull{uniform}

If a random variable *T* follows the Weibull-uniform{log-logistic} (WULL) distribution with PDF $f_T(x)=\{c/[\gamma(1-x)^2]\}\{x/[\gamma(1-x)]\}^{c-1}\exp\{-[x/(\gamma(1-x))]\}$, $0<x<1$, $c,\gamma>0$, then the PDF of WULL-Weibull{uniform} (WUW{U}) distribution is obtained from equation (2.2) as

$$f_X(x)=\frac{ck}{\gamma\lambda}\left(\frac{x}{\lambda}\right)^{k-1}\left(\frac{e^{(x/\lambda)^k}-1}{\gamma}\right)^{c-1}\exp\left\{-\left(\frac{e^{(x/\lambda)^k}-1}{\gamma}\right)^{c}\right\},\quad x>0,c,k,\lambda,\gamma>0. \qquad (4.1)$$

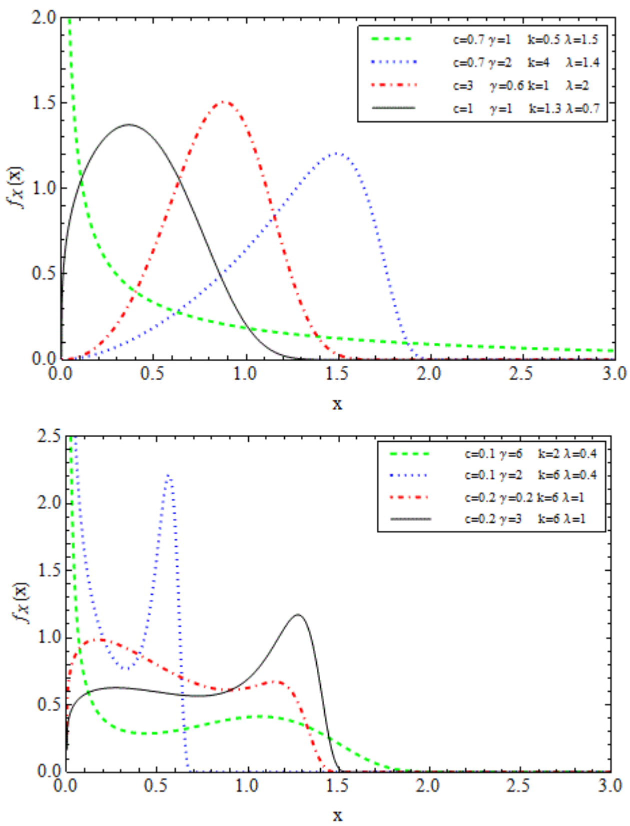

Figure 1. The PDFs of WULL-Weibull{uniform} for various values of c,γ,k and λ

Plots of WUW{U} density for different parameter values are given in Figure 1. The graphs in Figure 1 show that the WUW{U} distribution can be monotonically decreasing (reversed J-shape), left skewed, right skewed, unimodal or bimodal.

4.2 Lomax-Weibull{log-logistic} Distribution

Let $\theta=\beta=1$ in the PDF of the *T*-Weibull{log-logistic} family in (2.6). On taking *T* to be a Lomax random variable with PDF $f_T(x)=(\alpha/\theta)\left(1+(x/\theta)\right)^{-(\alpha+1)}$, $x\geq 0$, $\alpha,\theta>0$ and using equation (2.6), we get Lomax-Weibull{log-logistic} (LW{LL}) distribution with PDF

$$f_X(x)=\frac{k\alpha}{\theta\lambda}\left(\frac{x}{\lambda}\right)^{k-1}e^{(x/\lambda)^k}\left[1+(e^{(x/\lambda)^k}-1)/\theta\right]^{-(\alpha+1)},\ x>0,\alpha,\theta,k,\lambda>0. \qquad (4.2)$$

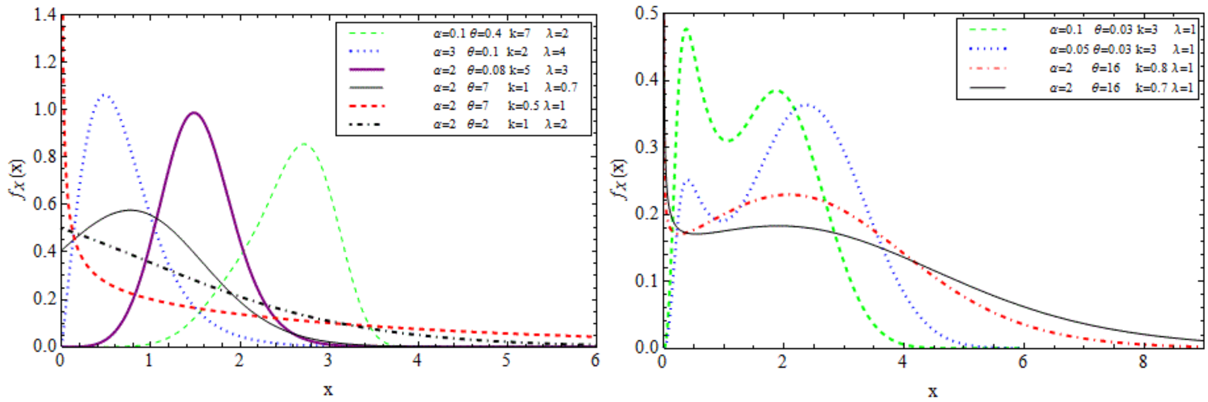

Figure 2. The PDFs of Lomax-Weibull{log-logistic} for various values of α,θ,k and λ

Plots of LW{LL} density function for different parameter values are given in Figure 2. The graphs in Figure 2 show that the LW{LL} distribution can be monotonically decreasing (reversed J-shape), left skewed, right skewed, unimodal or bimodal.

4.3 Exponential-Weibull{Fréchet} Distribution

Let $\theta=b=1$ in the PDF of the *T*-Weibull{logistic} family in (2.9). On taking *T* to be an exponential random variable with PDF $f_T(x)=(1/\beta)e^{-x/\beta}$, $x>0$, $\beta>0$ and using equation (2.9), the PDF of exponential-Weibull{Fréchet} (EW{F}) distribution is obtained as

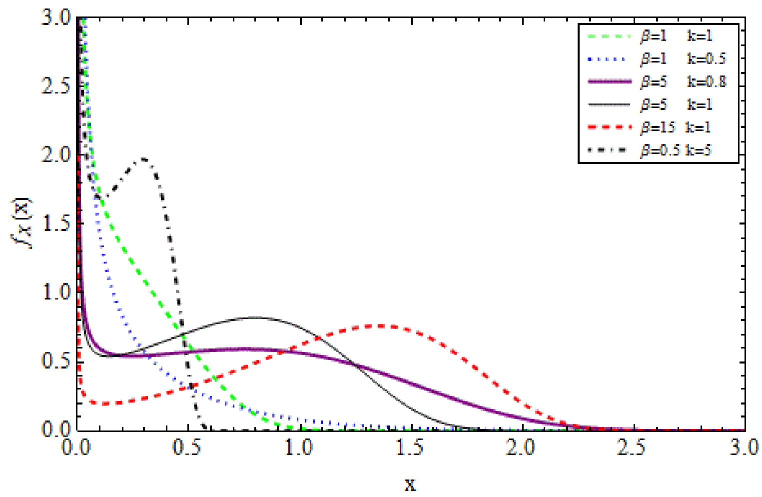

Figure 3. The PDFs of exponential-Weibull{Fréchet} for various values of β and k

$$f_X(x) = \frac{k(x/\lambda)^{k-1}}{\beta\lambda\left(e^{(x/k)^k}-1\right)\left[\log\left(1-e^{-(x/k)^k}\right)\right]^2} \exp\left\{\left[\beta \log\left(1-e^{-(x/k)^k}\right)\right]^{-1}\right\}, x>0, k, \lambda, \beta > 0. \tag{4.3}$$

Plots of EW{F} density for different parameter values of β and k are given in Figure 3 for $\lambda = 0.5$ since changing the scale parameter λ only stretches out or shrinks the distribution. The graphs in Figure 3 show that the EW{F} distribution can be monotonically decreasing (reversed J-shape) or bimodal.

4.4 Cauchy-Weibull{logistic} Distribution

If a random variable T follows a Cauchy distribution with PDF $f_T(x) = \left[\pi\beta\left(1+[(x-\alpha)/\beta]^2\right)\right]^{-1}$, $-\infty < x < \infty, \beta > 0, -\infty < \alpha < \infty$, then the PDF of the four-parameter Cauchy-Weibull{logistic} (CW{L}) distribution is obtained from equation (2.11) as

$$f_X(x) = \frac{\beta k(x/\lambda)^{k-1}}{\pi\lambda\left(1-e^{-(x/\lambda)^k}\right)\left\{\beta^2 + \left[\log\left(e^{(x/\lambda)^k}-1\right)-\alpha\right]^2\right\}}, \quad x>0. \tag{4.4}$$

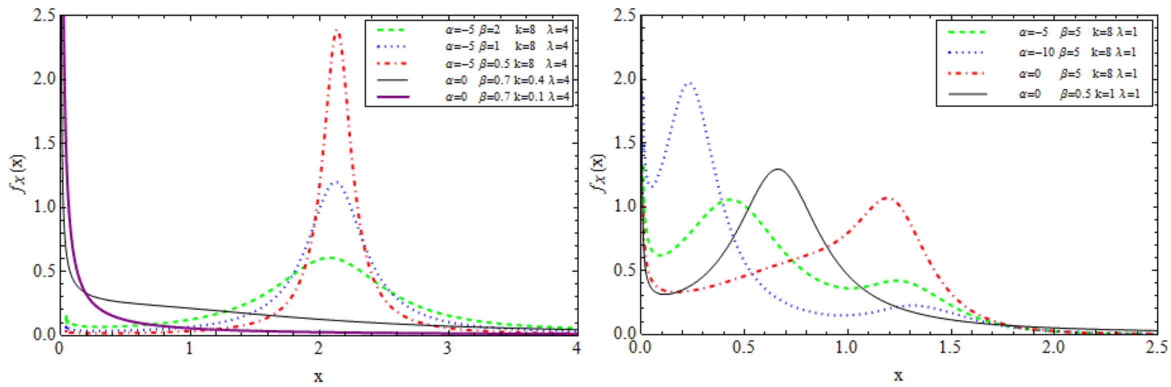

Figure 4. The PDFs of Cauchy-Weibull{logistic} for various values of α, β, k and λ

Setting the parameter $\alpha = 0$, we get the three-parameter CW{L} with PDF

$$f_X(x) = \frac{\beta k(x/\lambda)^{k-1}}{\pi\lambda\left(1-e^{-(x/\lambda)^k}\right)\left\{\beta^2 + \left[\log\left(e^{(x/\lambda)^k}-1\right)\right]^2\right\}}, \quad x>0. \tag{4.5}$$

Plots of CW{L} density for different parameter values are given in Figure 4. The graphs in Figure 4 show that the CW{L} distribution can be unimodal, bimodal or trimodal with different shapes.

4.5 WL-Weibull{extreme value} Distribution

If T follows Weibull-logistic (WL) distribution with PDF $f_T(t) = ce^x / \left(\beta(1+e^x)\right)\left[\log(1+e^x)/\beta\right]^{c-1} \exp\left\{-\left[\log(1+e^x)/\beta\right]^c\right\}$, $-\infty < x < \infty$, $c, \beta > 0$, then the PDF of WL-Weibull{extreme value} (WLW{EV}) distribution is obtained from equation (2.11) as

$$f_X(x) = \frac{ck(x/\lambda)^{k-1}}{\beta\lambda\left(1+(x/\lambda)^k\right)}\left[\log\left(1+(x/\lambda)^k\right)/\beta\right]^{c-1} \exp\left\{-\left[\log\left(1+(x/\lambda)^k\right)/\beta\right]^c\right\}, \tag{4.6}$$

where $x > 0, c, \beta, k, \lambda > 0$.

Plots of WLW{EV} density for different parameter values are given in Figure 5. The graphs in Figure 5 show that the WLW{EV} distribution can be monotonically decreasing (reversed J-shape), left skewed or right skewed. We have tried many combinations of the parameters and the graphical displays indicate that WLW{EV} appears to be unimodal.

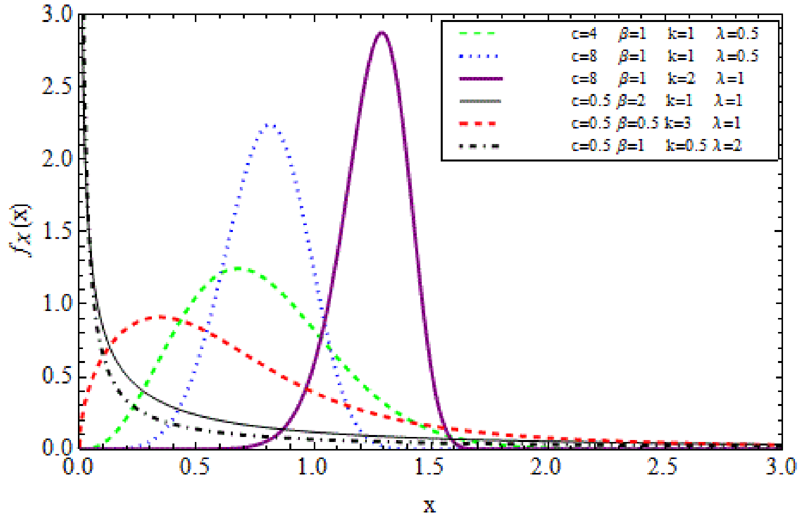

Figure 5. The PDFs of WL-Weibull{extreme value} for various values of c, β, k and λ

5. Applications

In this section we present two applications of Cauchy-Weibull{logistic} distribution using real life data sets. The parameters are estimated by using the maximum likelihood method. The fit is compared with other distributions based on the maximized log-likelihood, the Kolmogorov-Smirnov test (K-S) along with the corresponding p-value and Akaike Information Criterion (AIC). The required numerical evaluations are carried out by using SAS (PROC NLMIXED). In addition, the histograms of the data sets and the PDFs of the fitted distributions are presented for graphical illustration of the goodness of fit.

5.1 Remission Times of Bladder Cancer Patients

In this application we consider the data set (given in Table 2) on the remission times (in months) of a random sample of 128 bladder cancer patients. The distribution of this data set is unimodal and right skewed (skewness = 3.286 and kurtosis = 18.483).

Table 2. Remission times (in months) of bladder cancer patients

0.080	0.200	0.400	0.500	0.510	0.810	0.900	1.050	1.190	1.260
1.350	1.400	1.460	1.760	2.020	2.020	2.070	2.090	2.230	2.260
2.460	2.540	2.620	2.640	2.690	2.690	2.750	2.830	2.870	3.020
3.250	3.310	3.360	3.360	3.480	3.520	3.570	3.640	3.700	3.820
3.880	4.180	4.230	4.260	4.330	4.340	4.400	4.500	4.510	4.870
4.980	5.060	5.090	5.170	5.320	5.320	5.340	5.410	5.410	5.490
5.620	5.710	5.850	6.250	6.540	6.760	6.930	6.940	6.970	7.090
7.260	7.280	7.320	7.390	7.590	7.620	7.630	7.660	7.870	7.930
8.260	8.370	8.530	8.650	8.660	9.020	9.220	9.470	9.740	10.06
10.34	10.66	10.75	11.25	11.64	11.79	11.98	12.02	12.03	12.07
12.63	13.11	13.29	13.80	14.24	14.76	14.77	14.83	15.96	16.62
17.12	17.14	17.36	18.10	19.13	20.28	21.73	22.69	23.63	25.74
25.82	26.31	32.15	34.26	36.66	43.01	46.12	79.05		

The data set was previously studied by Lemonte and Cordeiro (2013) and Zea et al. (2012) as an application to extended-Lomax distribution and beta-exponentiated Pareto distribution (BEPD), respectively. Table 3 provides the MLEs (with the corresponding standard errors in parentheses) of the parameters of the distributions. The goodness of fit statistics of BEPD and beta-Pareto distribution (BPD) are obtained from Zea et al. (2012).

We apply the three-parameter CW{L} distribution in (4.5) and the four-parameter CW{L} distribution in (4.4) to fit the data and the results are reported in Table 3. From Table 3, both the three-parameter and the four-parameter CW{L} distributions provide an adequate fit to the data based on the p-value of the K-S test. The additional parameter α in the four-parameter CW{L} distribution is significantly different from zero at 5% by using the asymptotic Wald test (p-value = 0.035). Furthermore, the four-parameter CW{L} distribution has the smallest AIC and K-S statistics and the largest log-likelihood value, which indicates that the four-parameter CW{L} seems to be superior to the other distributions in Table 3. Figure 6 displays the histogram and the fitted density functions for the remission times of bladder cancer patients' data in Table 2.

Table 3. Parameter estimates for remission times of bladder cancer patients

Distribution	BEPD	BPD	Three-parameter CW{L}	Four-parameter CW{L}
Parameter estimates	$\hat{a}$ = 0.348	$\hat{a}$ = 4.805	$\hat{\beta}$ = 1.5623	$\hat{\alpha}$ = –2.3040
	(0.097)	(0.055)	(0.3614)	(1.0937)
	$\hat{b}$ = 159831	$\hat{b}$ = 100.502	$\hat{k}$ = 1.7888	$\hat{\beta}$ = 2.0205
	(183.7501)	(0.251)	(0.3215)	(0.4585)
	$\hat{k}$ = 0.051	$\hat{k}$ = 0.011	$\hat{\lambda}$ = 7.3836	$\hat{k}$ = 3.0673
	(0.019)	(0.001)	(0.6402)	(0.7319)
	$\hat{\beta}$ = 0.080	$\hat{\beta}$ = 0.080		$\hat{\lambda}$ = 12.663
	$\hat{\alpha}$ = 8.612			(2.6326)
	(2.093)			
Log-likelihood	–432.41	–480.446	–419.233	–416.0965
AIC	874.819	968.893	844.5	840.2
K-S statistics (p-value)	0.142 (0.0121)	0.217 (1.105E-5)	0.08256 (0.3473)	0.06672 (0.6189)

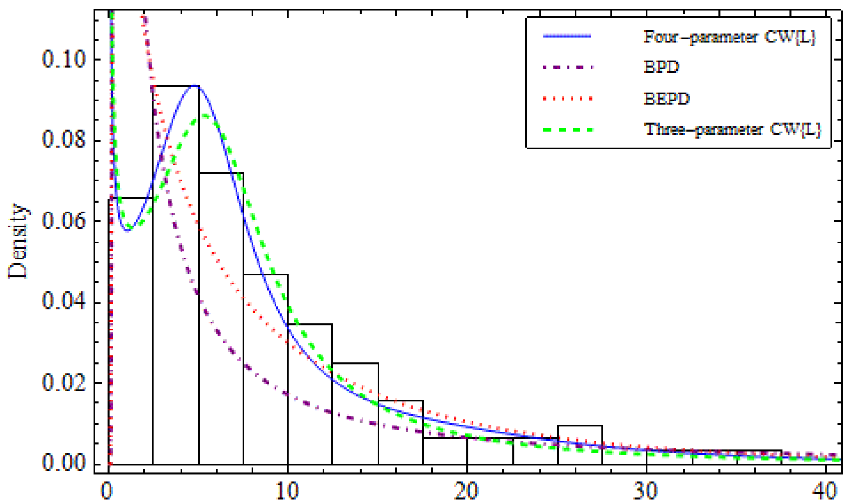

Figure 6. The fitted PDFs for the remission time of bladder cancer data

5.2 Breaking Stress of Carbon Fibers Data

In this application we consider a data set in Table 4 from Nicholas and Padgett (2006), on the breaking stress of carbon fibers of 50 mm in length (n = 66). The distribution of this data set is unimodal and slightly left skewed (skewness = –0.131 and kurtosis = 3.223).

The data set was used by Cordeiro and Lemonte (2011) to illustrate the application of a four-parameter beta-Birnbaum-Saunders distribution (BBSD) and it was compared to the two-parameter Birnbaum-Saunders distribution (BSD) (Birnbaum and Saunders, 1969).

Table 4. Breaking stress of carbon fibers data

3.70	2.74	2.73	2.50	3.60	3.11	3.27	2.87	1.47
3.11	3.56	4.42	2.41	3.19	3.22	1.69	3.28	3.09
1.87	3.15	4.90	1.57	2.67	2.93	3.22	3.39	2.81
4.20	3.33	2.55	3.31	3.31	2.85	1.25	4.38	1.84
0.39	3.68	2.48	0.85	1.61	2.79	4.70	2.03	1.89
2.88	2.82	2.05	3.65	3.75	2.43	2.95	2.97	3.39
2.96	2.35	2.55	2.59	2.03	1.61	2.12	3.15	1.08
2.56	1.80	2.53						

The three-parameter CW{L}, four-parameter CW{L} and flexible Weibull distribution (FWD) (Bebbington et al., 2007) are applied to fit the data. The maximum likelihood estimates (with the corresponding standard errors in parentheses) of the three-parameter CW{L} distribution and FWD parameters, the log-likelihood and the AIC values, the K-S test statistic and the p-value for the K-S statistic are reported in Table 5. The estimates and goodness of fit statistics for BBSD and BSD are taken from Cordeiro and Lemonte (2011).

Table 5. Parameter estimates for the carbon fibers data

Distribution	BBSD	BSD	FWD	Three-parameter CW{L}
Parameter estimates	$\hat{\alpha}$ = 1.045 (0.004) $\hat{\beta}$ = 57.600 (0.331) $\hat{a}$ = 0.193 (0.026) $\hat{b}$ = 1876 (605.05)	$\hat{\alpha}$ =0.437 (0.0381) $\hat{\beta}$ = 2.515 (0.1312)	$\hat{\alpha}$ = 0.5139 (0.0482) $\hat{\beta}$ = 4.6098 (0.5301)	$\hat{\beta}$ = 2.1437 (0.7221) $\hat{k}$ = 7.9321 (1.8887) $\hat{\lambda}$ = 2.953 (0.1083)
Log-likelihood	–91.355	–100.19	–90.2451	–86.989
AIC	190.71	204.38	184.5	180.0
K-S statistics (p-value)	0.138 (0.160)	0.184 (0.022)	0.1264 (0.2415)	0.0570 (0.9827)

In fitting the data, we notice that the goodness of fit statistics between the four-parameter CW{L} and the three-parameter CW{L} are similar. In addition, the parameter α in the four-parameter CW{L} distribution is not significant (asymptotic Wald test p-value = 0.6935) at the 5% level. Therefore, only the three-parameter CW{L} is included in Table 5.

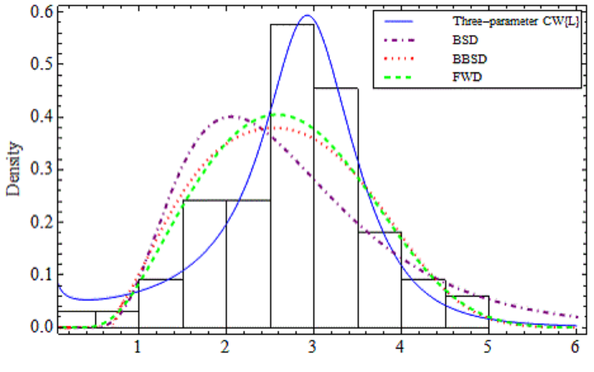

Figure 7. The fitted PDFs for the carbon fibers data

The results in Table 5 indicate that the BBSD, FWD and the three-parameter CW{L} distribution give an adequate fit to the data based on the p-value of the K-S test. However, the three-parameter CW{L} has the smallest K-S and AIC statistics and the largest log-likelihood value among the fitted distributions. Thus, the three-parameter CW{L} seems to be the best for fitting the data compared to the other distributions in Table 5. Figure 7 displays the histogram and the fitted density functions for the data.

6. Summary and Conclusion

Following the method of *T*-*R*{*Y*} family by Aljarrah et al. (2014), we derive a generalization of Weibull distribution. Different families can be developed based on the choice of the quantile function of the random variable *Y*. In this article six types of generalized Weibull families from the quantile functions of uniform, exponential, log-logistic, Fréchet, logistic and extreme value distributions are proposed. Some of their general and statistical properties are studied.

Five new generalized Weibull distributions in *T*-Weibull{*Y*} families are defined, namely, WULL-Weibull{uniform}, the Lomax-Weibull{log-logistic}, the exponential-Weibull{Fréchet}, the Cauchy-Weibull{logistic} and WL-Weibull{extreme value}. These distributions are more flexible than the Weibull distribution and are expected to provide adequate fit to data sets from more complex phenomena. The usefulness of the new families is illustrated by applying one of the members, Cauchy-Weibull{logistic} distribution, to fit two real data sets. The estimation of the model parameters is by the method of maximum likelihood. There are still a lot of works to be done in this area, including deriving more members of *T*-Weibull{*Y*} families and investigating some of their properties. It is hoped that this article would motivate and encourage developments of further generalizations of the two-parameter Weibull distribution and their applications to different real world phenomena.

Acknowledgment

The authors would like to thank the editor and anonymous referees for their valuable comments and suggestions that improved the quality of the paper.

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