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Characterizations of the Weibull-X and Burr XII Negative Binomial Families of Distributions

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Abstract

In this paper, we establish certain characterizations of the Weibull-X family of distributions proposed by Alzaatreh et al. (2013) as well as of the Burr XII Negative Binomial distribution, introduced by Ramos et al. (2015). These characterizations are based on two truncated moments, hazard rate function and conditional expectation of functions of random variables.

1. Introduction

In designing a stochastic model for a particular modeling problem, an investigator will be vitally interested to know if their model fits the requirements of a specific underlying probability distribution.To this end, the investigator will rely on the characterizations of the selected distribution. Generally speaking, the problem of characterizing a distribution is an important problem in various fields and has recently attracted the attention of many researchers. Consequently, various characterization results have been reported in the literature. These characterizations have been established in many different directions. There exists a plethora of studies dealing with the characterizations of the Weibull distribution in the literature. Scholz (1990) gives a characterization in terms of the quantiles and Kagan et al. (1999) proposes a characterization based on the Fisher information and censored experiments. For additional characterization results we refer the interested reader to Nigam and El-Hawary (1996), Mohie El-Din et al. (1991) and Chaudhuri and Chandra (1990). Mukherjee and Roy (1987) deal with four miscellaneous characterizations of two-parameter Weibull distribution. Khan and Beg (1987), Khan and Abu-Salih (1988) study characterization of Weibull distribution through some conditional moments. Ali (1997) shows that Weibull distribution can be characterized by the product moments of any two order statistics. For a comprehensive list of Weibull distribution characterizations, see Murthy et al. (2004) and the references therein. This vast array of Weibull distribution characterizations justifies somewhat the importance of the characterizations of the Weibull-X Family (WXF) of distributions, simply because of the fact that WXF augments the Weibull distribution in terms of its flexibility to model various observed phenomena.

In this paper, we consider several different approaches to characterize the WXF as well as the Burr XII Negative Binomial (BXIINB) distributions. These characterizations are based on: (i) a simple relationship between two truncated moments; (ii) conditional expectations of a single function of the random variable; (iii) hazard function; (iv) other characterizations. Needless to say that, these approaches are markedly different from those mentioned above. We like to mention that the characterization (i) which is expressed in terms of the ratio of truncated moments is stable in the sense of weak convergence. It also serves as a bridge between a first order differential equation and probability and it does not require a closed form for the distribution function. The rest of the paper is organized in the following way. In section 2, we present the WXF and BXIINB families of distributions. Section 3 deals with the characterizations $(i)-(iv)$ mentioned above. In section 4, we provide some concluding remarks.

2. The WXF and BXIINB Families of Distributions

Let $r(t)$ be the probability density function (pdf) of a random variable $T \in [a, b]$ and let $G(x)$ be the cumulative distribution function (cdf). Assume the link function $W(\cdot) : (0, 1) \longrightarrow [a, b]$ satisfies the following conditions:

$$\begin{cases} (i) & W(\cdot)\,\text{is differentiable and monotonically non-decreasing,} \\ (ii) & W(0) \to a \quad \text{and} \quad W(1) \to b \quad . \\ \end{cases} \tag{1}$$

Alzaatreh et al. (2013) defined the *cdf* of the *T-X family* of distributions by

$$F(x) = \int_a^{W(G(x))} r(t)dt, \tag{2}$$

where $W(x)$ satisfies the conditions in (1).

If $T \in (0, \infty)$, X is a continuous random variable and $W(x) = -\log(1 - x)$, then the pdf corresponding to (2) is given by

$$f(x) = \frac{g(x)}{1 - G(x)}\, r\Big(-\log\left[1 - G(x)\right]\Big) = h_g(x)\, r\Big(H_g(x)\Big), \tag{3}$$

where $h_g(x) = \frac{g(x)}{1-G(x)}$ and $H_g(x) = -\log[1 - G(x)]$ are the hazard and cumulative hazard rate functions corresponding to the baseline pdf $g(x) = \frac{d}{dx}G(x)$, respectively.

Let T have a Weibull distribution with parameters $c > 0$ and $\gamma > 0$, then from (3), the *pdf* of the *Weibull-X* family is given by

$$f(x) = \frac{c}{\gamma}\, h_g(x) \left[\frac{H_g(x)}{\gamma}\right]^{c-1} \mathrm{e}^{-\left[\frac{H_g(x)}{\gamma}\right]^c}; \quad x \in SuppG, \tag{4}$$

where $SuppG$ denotes the support of the random variable X with cdf G. WOLG we assume that $SuppG = (0, \infty)$.

The cdf and hazard rate function corresponding to (4) are, respectively, given by

$$F(x) = 1 - \mathrm{e}^{-\left[\frac{H_g(x)}{\gamma}\right]^c}, \tag{5}$$

and

$$h_f(x) = \frac{c}{\gamma}\, h_g(x) \left[\frac{H_g(x)}{\gamma}\right]^{c-1}. \tag{6}$$

For certain general properties of the WXF, we refer the interested reader to Alzaatreh et al. (2013; 2014).

Remark 2.1. Recently, Nadarajah and Rocha (2014) have developed the necessary code for the WXF including the density, distribution, quantile functions as well as random generation. The R package named, "Newdistns" is freely available at
"http://cran.r-project.org/web/packages/Newdistns/index.html'.

Remark 2.2. The WXF of distributions is closed under minimization, i.e., if X_i , $i = 1, 2$ are independent and follow $WXF(c, \gamma_i)$, then $\min\{X_1, X_2\} \sim WXF$.

Proof. For any $x \in (0, \infty)$,

$$\begin{aligned} P(\min\{X_1, X_2\} > x) &= P(X_1 > x\,,\ X_2 > x) \\ &= P(X_1 > x)\, P(X_2 > x) \\ &= \exp\left(-\left[H_g(x)\right]^c \left(\gamma_1^{-c} + \gamma_2^{-c}\right)\right), \end{aligned}$$

which establishes the fact that $\min\{X_1, X_2\} \sim WXF$. □

Note: If $\gamma_1 = \gamma_2 = \gamma$, i.e. if X_1 and X_2 are *i.i.d.* ,then $\min\{X_1, X_2\} \sim WXF\left(c, 2^{1/c}\gamma\right)$.

In modern times, there has been a lot of work in developing new class of lifetime distributions via compounding of some discrete distributions with an appropriate lifetime distributions. One such distribution arising from Burr XII and Negative Binomial by compounding has been studied in detail by Ramos et al (2015). The apparent wide applicability of such models demands a further investigation in terms of its characterization. The pdf and cdf of the BXIINB proposed by Ramos et al. (2015) are given, respectively, by

$$f(x) = \frac{s\beta cka^{-c}}{\left[(1-\beta)^{-s}-1\right]}x^{-c-1}\left[1+\left(\frac{x}{a}\right)^{c}\right]^{-k-1}\times\left\{1-\beta\left[1+\left(\frac{x}{a}\right)^{c}\right]^{-k}\right\}^{-s-1}, \tag{7}$$

and

$$F(x) = \frac{(1-\beta)^{-s}-\left\{1-\beta\left[1+\left(\frac{x}{a}\right)^{c}\right]^{-k}\right\}^{-s}}{\left[(1-\beta)^{-s}-1\right]}, \quad x>0, \tag{8}$$

where a, s, c, k all positive and $\beta \in (0,1)$ are parameters.

3. Characterization Results

In this section, we present characterizations of WXF and BXIINB distributions via several subsections: (i) based on a simple relationship between two truncated moments; (ii) in terms of conditional expectations of a single function of the random variable; (iii) based on the hazard function and (iv) other characterizations.

3.1 Characterizations Based on Two Truncated Moments

In this subsection we present characterizations of WXF and BXIINB distributions in terms of a simple relationship between two truncated moments. Our characterization results presented here will employ an interesting result due to Glänzel (1987) (Theorem 3.1 below). The advantage of the characterizations given here is that, cdf F need not require to have a closed form and is given in terms of an integral whose integrand depends on the solution of a first order differential equation, which can serve as a bridge between probability and differential equation.

Theorem 3.1. *Let* $(\Omega, , \mathbf{P})$ *be a given probability space and let* $H = [a, b]$ *be an interval for some* $a < b$ $(a = -\infty$, $b = \infty$ *might as well be allowed*$)$. *Let* $X : \Omega \to H$ *be a continuous random variable with the distribution function* F *and let* q_1 *and* q_2 *be two real functions defined on* H *such that*

$\mathbf{E}[q_1(X) \mid X \geq x] = \mathbf{E}[q_2(X) \mid X \geq x]\,\eta(x), \quad x \in H,$

is defined with some real function η . *Assume that* $q_1, q_2 \in C^1(H)$, $\eta \in C^2(H)$ *and* F *is twice continuously differentiable and strictly monotone function on the set* H . *Finally, assume that the equation* $q_2\eta = q_1$ *has no real solution in the interior of* H . *Then* F *is uniquely determined by the functions* q_1 , q_2 *and* η , *particularly*

$$F(x) = \int_a^x C\left|\frac{\eta'(u)}{\eta(u)\,q_2(u)-q_1(u)}\right|\exp(-s(u))\,du,$$

where the function s *is a solution of the differential equation* $s' = \frac{\eta'\,q_2}{\eta\,q_2-q_1}$ *and* C *is a constant, chosen to make* $\int_H dF = 1$.

We like to mention that this kind of characterization based on the ratio of truncated moments is stable in the sense of weak convergence, in particular, let us assume that there is a sequence $\{X_n\}$ of random variables with distribution functions $\{F_n\}$ such that the functions $q_{1,n}$, $q_{2,n}$ and η_n $(n \in \mathbb{N})$ satisfy the conditions of Theorem 3.1 and let $q_{1,n} \to q_1$, $q_{2,n} \to q_2$ for some continuously differentiable real functions q_1 and q_2 . Let, finally, X be a random variable with distribution F . Under the condition that $q_{1,n}(X)$ and $q_{2,n}(X)$ are uniformly integrable and the family $\{F_n\}$ is relatively compact, the sequence X_n converges to X in distribution if and only if η_n converges to η , where

$$\eta(x) = \frac{E[q_1(X) \mid X \geq x]}{E[q_2(X) \mid X \geq x]}.$$

Remark 3.2. (a) In Theorem 3.1, the interval H need not be closed since the condition is only on the interior of H. (b) Clearly, Theorem 3.1 can be stated in terms of two functions q_1 and η by taking $q_2(x) \equiv 1$, which will reduce the condition given in Theorem 3.1 to $E[q_1(X) \mid X \geq x] = \eta(x)$. However, adding an extra function will give a lot more flexibility, as far as its application is concerned.

Proposition 3.3. *Let* $X : \Omega \to (0, \infty)$ *be a continuous random variable and let* $q_1(x) = q_2(x) \left[\frac{H_g(x)}{\gamma}\right]^c$ *and* $q_2(x) = 2e^{-\left[\frac{H_g(x)}{\gamma}\right]^c}$ *for* $x > 0$. *The* pdf *of* X *is* (4) *if and only if the function* η *defined in Theorem 3.1 has the form*

$$\eta(x) = \frac{1}{2} + \left[\frac{H_g(x)}{\gamma}\right]^c, \quad x > 0.$$

Proof. Let X have pdf (4) , then

$$(1 - F(x)) \, \mathbf{E}[q_2(X) \mid X \geq x] = e^{-2\left[\frac{H_g(x)}{\gamma}\right]^c}, \quad x > 0 \, ,$$

and

$$(1 - F(x)) \, \mathbf{E}[q_1(X) \mid X \geq x] = \left(\frac{1}{2} + \left[\frac{H_g(x)}{\gamma}\right]^c\right) e^{-2\left[\frac{H_g(x)}{\gamma}\right]^c}, \quad x > 0,$$

and finally

$$\eta(x) \, q_2(x) - q_1(x) = e^{-\left[\frac{H_g(x)}{\gamma}\right]^c} > 0 \, , \text{ for } x > 0 \, .$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x) \; q_2(x)}{\eta(x) \; q_2(x) - q_1(x)} = \frac{2c}{\gamma} h_g(x) \left[\frac{H_g(x)}{\gamma}\right]^{c-1}, \quad x > 0 \, ,$$

and hence

$$s(x) = 2\left[\frac{H_g(x)}{\gamma}\right]^c, \quad x > 0.$$

Now, in view of Theorem 3.1, X has pdf (4) . □

Corollary 3.4. *Let* $X : \Omega \to (0, \infty)$ *be a continuous random variable and let* $q_2(x)$ *be as in Proposition 3.3. The pdf of* X *is* (4) *if and only if there exist functions* q_1 *and* η *defined in Theorem 3.1 satisfying the differential equation*

$$\frac{\eta'(x) \; q_2(x)}{\eta(x) \; q_2(x) - q_1(x)} = \frac{2c}{\gamma} h_g(x) \left[\frac{H_g(x)}{\gamma}\right]^{c-1}, \quad x > 0.$$

Remark 3.5. (c) The general solution of the differential equation in Corollary 3.4 is

$$\eta(x) = e^{2\left[\frac{H_g(x)}{\gamma}\right]^c} \left[- \int \frac{c}{\gamma} h_g(x) \left[\frac{H_g(x)}{\gamma}\right]^{c-1} e^{-\left[\frac{H_g(x)}{\gamma}\right]^c} q_1(x) \, dx + D \right],$$

for $x > 0$, where D is a constant. One set of appropriate functions is given in Proposition 3.3 with $D = 0$.

(d) Clearly there are other triplets of functions (q_1, q_2, η) satisfying the conditions of Theorem 3.1 We presented one such triplet in Proposition 3.3.

Proposition 3.6. *Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x) = \left\{1 - \beta\left[1 + \left(\frac{x}{a}\right)^c\right]^{-k}\right\}^{s+1}$ and $q_2(x) = q_1(x)\left[1 + \left(\frac{x}{a}\right)^c\right]^{-1}$ for $x > 0$. The pdf of X is (7) if and only if the function η defined in Theorem 3.1 has the form*

$$\eta(x) = \frac{k+1}{k}\left[1 + \left(\frac{x}{a}\right)^c\right], \quad x > 0.$$

Remark 3.7. A Corollary and Remarks similar to Corollary 3.4 and Remarks 3.5 can be stated for BXIINB as well.

3.2 Characterizations Based on Conditional Expectation of a Function of the Random Variable

In this subsection we employ a single function ψ of X and state characterization results in terms of $\psi(X)$.

Proposition 3.8. *Let $X : \Omega \to (a, b)$ be a continuous random variable with cdf F. Let $\psi(x)$ be a differentiable function on (a, b) with $\lim_{x\to a^+} \psi(x) = \delta > 1$ and $\lim_{x\to b^-} \psi(x) = \infty$. Then*

$$E\left[(\psi(X))^\delta \mid X \le x\right] = \delta(\psi(x))^{\delta-1}, \quad x \in (a, b), \tag{9}$$

if and only if

$$\psi(x) = \delta\left[1 + (F(x))^{\frac{1}{\delta-1}}\right]^{-1}, \quad x \in (a, b). \tag{10}$$

Proof. From (9), we have

$$\int_a^x (\psi(u))^\delta f(u)\,du = \delta(\psi(x))^{\delta-1} F(x).$$

Taking derivatives from both sides of the above equation, we arrive at

$$(\psi(X))^\delta f(x) = \delta\left\{(\delta-1)\psi'(x)(\psi(x))^{\delta-2}F(x) + (\psi(x))^{\delta-1} f(x)\right\},$$

from which we have

$$\frac{f(x)}{F(x)} = (\delta-1)\left\{-\frac{\psi'(x)}{\psi(x)} + \frac{\psi'(x)}{\psi(x)-\delta}\right\}.$$

Integrating both sides of this equation from x to b and using the condition $\lim_{x\to b^-} \psi(x) = \infty$, we obtain (10). □

Proposition 3.9. *Let $X : \Omega \to (a, b)$ be a continuous random variable with cdf F. Let $\psi_1(x)$ be a differentiable function on (a, b) with $\lim_{x\to a^+} \psi_1(x) = \delta/2 > 1/2$ and $\lim_{x\to b^-} \psi_1(x) = \delta$. Then*

$$E\left[(\psi_1(X))^\delta \mid X \ge x\right] = \delta(\psi_1(x))^{\delta-1}, \quad x \in (a, b), \tag{11}$$

if and only if

$$\psi_1(x) = \delta\left[1 + (1 - F(x))^{\frac{1}{\delta-1}}\right]^{-1}, \quad x \in (a, b). \tag{12}$$

Proof. Is similar to that of Proposition 3.8. □

Remark 3.10. (e) Taking, e.g., $(a, b) = (0, \infty)$ and $\psi(x) = \delta\left[1 + \left(1 - e^{-\left[\frac{H_g(x)}{\gamma}\right]^c}\right)^{\frac{1}{\delta-1}}\right]^{-1}$, Proposition 3.8 gives a characterization of WXF distribution. (f) Taking, e.g., $(a, b) = (0, \infty)$ and $\psi_1(x) = \delta\left[1 + e^{-\frac{1}{\delta-1}\left[\frac{H_g(x)}{\gamma}\right]^c}\right]^{-1}$, Proposition 3.9 gives a characterization of WXF distribution. (g) Similar conclusions can be drawn for BXIINB distribution.

3.3 Characterizations Based on Hazard Function

It is obvious that the hazard function of twice differentiable distribution function satisfies the first order differential equation

$$\frac{\zeta_F'(x)}{\zeta_F(x)} - \zeta_F(x) = p(x) ,$$

where $p(x)$ is an appropriate integrable function. Although this differential equation has an obvious form since

$$\frac{f'(x)}{f(x)} = \frac{\zeta_F'(x)}{\zeta_F(x)} - \zeta_F(x) , \tag{13}$$

for many univariate continuous distributions (13) seems to be the only differential equation in terms of the hazard function. The goal of the characterization based on hazard function is to establish a differential equation in terms of hazard function, which has as simple form as possible and is not of the trivial form (13) . For some general families of distributions this may not be possible. Here we present a characterization of WXF distribution in terms of the hazard function and one for BXIINB for $s = 1$.

Proposition 3.11. *Let $X : \Omega \to (0,\infty)$ be a continuous random variable with twice differentiable cdf F. The random variable X has pdf (4) if and only if its hazard function $\zeta_F(x)$ satisfies the differential equation*

$$\zeta_F'(x) - g'(x)(g(x))^{-1}\zeta_F(x) = \frac{c}{\gamma^2}\left(\frac{h_g(x)}{1-G(x)}\right)\left[\frac{H_g(x)}{\gamma}\right]^{c-2} \times \left\{c-1-H_g(x)\right\}, \quad x>0. \tag{14}$$

Proof. If X has pdf (4), then clearly (14) holds. Now, if (14) holds, then

$$\frac{d}{dx}\left\{(g(x))^{-1}\zeta_F(x)\right\} = \frac{d}{dx}\left\{\frac{c}{\gamma^2}\left(\frac{1}{1-G(x)}\right)\left[\frac{H_g(x)}{\gamma}\right]^{c-1}\right\},$$

or

$$\frac{f(x)}{1-F(x)} = \zeta_F(x) = \frac{c}{\gamma^2}h_g(x)\left[\frac{H_g(x)}{\gamma}\right]^{c-1}.$$

Integrating both sides of the above equation from 0 to x , we arrive at

$$-\log(1-F(x)) = \left[\frac{H_g(x)}{\gamma}\right]^{c}$$

from which we have

$$1-F(x) = e^{-\left[\frac{H_g(x)}{\gamma}\right]^{c-1}}.$$

□

Proposition 3.12. *Let $X : \Omega \to (0,\infty)$ be a continuous random variable with twice differentiable cdf F. The random variable X has pdf (7) for $s = 1$, if and only if its hazard function $\zeta_F(x)$ satisfies the differential equation*

$$\zeta_F'(x) - (c-1)x^{-1}\zeta_F(x) = cka^{-c}x^{c-1}\frac{d}{dx}\left\{\left[1+\left(\frac{x}{a}\right)^c\right]^{-1}\left\{1-\beta\left[1+\left(\frac{x}{a}\right)^c\right]^{-k}\right\}^{-1}\right\}, \quad x>0.$$

3.4 Other Characterizations

The first three characterizations given below, have appeared in Hamedani's (unpublished as of now) work. We state them here for the sake of completeness and apply them to obtain characterizations of WXF and BXIINB distributions.

Proposition 3.13. *Let* $X : \Omega \to (0, \infty)$ *be a continuous random variable with* cdf F *and* pdf f. *Let* $\psi(x)$ *and* $q(x)$ *be two differentiable functions on* $(0, \infty)$ *such that* $\int_0^\infty \frac{q'(t)}{\psi(t)-q(t)} dt = \infty$. *Then*

$$E[\psi(X) \mid X \geq x] = q(x), \quad x > 0,$$

implies

$$F(x) = 1 - \exp\left\{-\int_0^x \frac{q'(t)}{\psi(t) - q(t)} dt\right\}, \quad x \geq 0.$$

Remark 3.14. (h) Taking, e.g., $\psi(t) = 2q(x)$ and $q(x) = \exp\left\{-\left[\frac{H_g(x)}{\gamma}\right]^c\right\}$, Proposition 3.13 gives a characterization of (4). (i) Clearly there are other pairs of functions $(\psi(t), q(x))$ which satisfy conditions of Proposition 3.13. We used the one in (h) for the simplicity.

Proposition 3.15. *Let* $X : \Omega \to (0, \infty)$ *be a continuous random variable with* cdf F *and* pdf f. *Let* $\psi(x)$ *and* $q(x)$ *be two differentiable functions on* $(0, \infty)$ *such that* $\left(\frac{\psi'(t)-q'(t)}{q(t)}\right) > 0$ *and* $\int_0^\infty \frac{\psi'(t)-q'(t)}{q(t)} dt = \infty$. *Then*

$$E[\psi(X) \mid X \geq x] = \psi(x) - q(x), \quad x > 0,$$

implies

$$F(x) = 1 - \exp\left\{-\int_0^x \frac{\psi'(t) - q'(t)}{q(t)} dt\right\}, \quad x \geq 0.$$

Remark 3.16. (j) Taking, e.g., $\psi(t) = 2q(x)$ and $q(x) = \exp\left\{-\left[\frac{H_g(x)}{\gamma}\right]^c\right\}$, Proposition 3.15 gives a characterization of (4). (k) Clearly there are other pairs of functions $(\psi(t), q(x))$ which satisfy conditions of Proposition 3.15. We used the one in (j) for the simplicity.

Proposition 3.17. *Let* $X : \Omega \to (a, b)$ *be a continuous random variable with* cdf F *and* pdf f. *Let* $\psi(x)$ *be differentiable function on* (a, b) *such that* $\lim_{x \to a^+} \psi(x) = 0$. *Then, for* $0 < \delta < 1$

$$E[\psi(X) \mid X \geq x] = \delta + (1 - \delta)\psi(x), \quad x \in (a, b),$$

if and only if

$$\psi(x) = 1 - (1 - F(x))^{\frac{\delta}{1-\delta}}, \quad x \in (a, b).$$

Remark 3.18. Taking, e.g., $(a, b) = (0, \infty)$ and $\psi(x) = 1 - \left(e^{-\left[\frac{H_g(x)}{\gamma}\right]^c}\right)^{\frac{\delta}{1-\delta}}$, Proposition 3.17 gives a characterization of (4). Similar observation can be made for BXIINB.

Kamps(1995) defined generalized order statistics (gos) in items of their joint pdf. Order statistics, record values and progressive type II order statistics are special cases of gos. The random variables $X(1, n, m, k), X(2, n, m, k), ..., X(n.n.m.k)$ from an absolutely continuous distribution function $F(x)$ with pdf $f(x)$ are called gos if their joint pdf $f_{1,2,...,n}(x_1, x_2, ..., x_n)$ is given as follows.

$f_{1,2,...,n}(x_1, x_2, ..., x_n) = k\Pi_{j=1}^{n-1}\gamma_j(1 - F(x_j))^m f(x_j)(1 - F(x_n)^{k-1} f(x_n), \quad -\infty < x_i < \infty, i = 1, 2, ..., n$, where $\gamma_j = k + (n - j)(m + 1)$, for all $j, 1 \leq j \leq n - 1$, k and n are positive integers and $m \geq -1$.The marginal pdf $f_{r,n,m,k}(x)$ of $X(r, n, m, k)$ is given by

$$f_{r,n,m,k}(x) = \frac{c_r}{\Gamma(r)}(1 - F(x))^{\gamma_r - 1}(g_m(F(x)))^{r-1}, -\infty < x < \infty,$$

where $c_r = \Pi_{j=1}^{r}\gamma_j$ and $g_m(x) = \left\{ \begin{array}{ll} \frac{1}{m+1}(1-(1-x)^{m+1}, & \neq -1 \\ -\log(1-x), & m=-1, 0<x<1. \end{array} \right\}$.

The conditional pdf $f_{r,n,m,k}(x)$ of $X(r+1,n,m,k)|X(rm,n,m,k)=x$ is given by

$$f_{r,n,m,k}(x) = \frac{\gamma_{r+1}(1-F(y))^{\gamma r+1}-1}{(1-F(x))^{\gamma\gamma_{r+1}}} f(y), -\infty < x < y < \infty.$$

For various characterizations of distributions by regressions of generalized order statistics, see Beg et al. (2013). The following proposition gives a characterizations of a general class of distributions of which WXF is a member based on regression property of $X(r+1,n,m,k)|X(r,n,m,k)=x$.

Proposition 3.19. *Let $X:\Omega\to(0,\infty)$ be a continuous random variable with a strictly increasing cdf F and corresponding pdf f. Let $K(x)$ be a strictly increasing function with $\lim_{x\to 0^+} K(x)=0$ and $\lim_{x\to\infty} K(x)=\infty$. Then, $F(x)=1-e^{-K(x)}$ if and only if*

$$E\left[K\left(X\left(r+1,n,m,k\right)\right) \mid X\left(r,n,m,k\right)=x\right] = K(x)+\frac{1}{\gamma_{r+1}}. \tag{15}$$

Proof. The conditional pdf of $X(r+1,n,m,k)\mid X(r,n,m,k)=x$ is given by

$$f_{r+1,r,n,m,k}(y|x) = \gamma_{r+1}\frac{(1-F(y))^{\gamma_{r+1}-1}}{(1-F(x))^{\gamma_{r+1}}} f(y).$$

It is now easy to show that for $F(x)=1-e^{-K(x)}$, equation (15) holds.

Conversely, suppose that (15) holds, then for $x>0$ we have

$$\int_x^\infty \gamma_{r+1}K(y)\frac{(1-F(y))^{\gamma_{r+1}-1}}{(1-F(x))^{\gamma_{r+1}}} f(y)dy = K(x)+\frac{1}{\gamma_{r+1}},$$

or

$$\int_x^\infty \gamma_{r+1}K(y)(1-F(y))^{\gamma_{r+1}-1} f(y)dy = (1-F(x))^{\gamma_{r+1}}\left(K(x)+\frac{1}{\gamma_{r+1}}\right).$$

Differentiating both sides of the above equation with respect to x ,we obtain

$$\begin{aligned} -\gamma_{r+1}K(x)(1-F(x))^{\gamma_{r+1}-1} f(x) &= -\gamma_{r+1} f(x)(1-F(x))^{\gamma_{r+1}-1}\left(K(x)+\frac{1}{\gamma_{r+1}}\right) \\ &\quad +(1-F(x))^{\gamma_{r+1}}K'(x). \end{aligned}$$

Upon simplification of the last equation, we arrive at

$$\frac{f(x)}{1-F(x))} = K'(x). \tag{16}$$

Integrating both sides of equation (16) from 0 to x and using the condition $\lim_{x\to 0^+} K(x)=0$, we obtain

$$F(x)=1-e^{-K(x)}.$$

□

Remark 3.20. Taking $K(x)=\left[\frac{H_g(x)}{\gamma}\right]^c$, Proposition 3.19 provides a characterization of WXF distribution.

4. Summary and Conclusions

Characterization of probability distributions have appeared from time to time in literature, most of the theorems deal with characteristic functions or are based on independence of a pair of suitable statistics. Lukacs (1956) has given an extensive bibliography of such theorems. In modeling various type of lifetime data, applications in the area of reliability, Weibull distribution plays an important role. However to augment its flexibility in modeling any type of data, researchers have developed Weibull- X family of distributions, where X is any absolutely continuous random variable. Similarly, the construction of the Burr XII negative binomial random variable (Ramos et al., 2015) was motivated by the fact that this new model subsumes as special cases, some important lifetime distributions, such as log-logistic, Weibull, Pareto (type II) and Burr XII distributions. With the advent of such an armory of new families of distributions, by compounding a distribution with a baseline lifetime distributions, it becomes necessary to develop new ways of characterizing such flexible families of distributions. In this paper, two such flexible families of distributions, namely the Weibull-X family and the Burr XII negative binomial distribution have been considered. It is our sincere hope that these characterization results presented in this paper will motivate researchers to find new avenues for characterization and construction of such flexible families in more than one dimension (bivariate and multivariate models).

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