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On Some Properties of the Reversed Variance Residual Lifetime

Bander Al-Zahrani[1] & Mashail Al-Sobhi[2]

[1] Department of Statistics, King Abdulaziz University, Jeddah, Saudi Arabia

[2] Department of Mathematics, Umm Al Qura University, Makkah, Saudi Arabia

Correspondence: Bander Al-Zahrani, Department of Statistics, King Abdulaziz University, Jeddah, Saudi Arabia. Tel: 966-2-965-2295. E-mail: bmalzahrani@kau.edu.sa

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Abstract

In this paper, we give an overview of recent results in the concept of reversed residual lifetime. We focus on properties of the reversed variance residual lifetime (RVR) and study the interrelations among reversed residual lifetime classes. We mention the most important results in the literature that are related to the RFR function for both continuous and discrete life distributions. We give properties of the reversed mean residual lifetime (RMR) and the RVR functions. Reversed entropy is briefly discussed. We study the relationships among the reversed classes.

Keywords: reversed residual lifetime, reversed variance residual lifetime, reversed entropy

1. Introduction

Properties of the so-called *reversed* residual lifetime have gained the interest of many researchers who have studied, for example, properties of the reversed failure rate (RFR). This function was introduced as the dual function of the original failure rate function. Andersen et al. (1993), Chandra and Roy (2001) and Block et al. (1998) showed that the RFR function plays the same role in the analysis of left censored data as the failure rate function plays in the analysis of right censored data.

Suppose that X is a random variable describing the lifetime of a component with distribution function F, density f, and let $X_t = [X - t|X \geq t]$ denote the residual lifetime of the component, which has survived up to time t. Then the corresponding distribution function of X_t is

$$F_t(x) = \mathbf{P}\{X - t \leq x|\, X \geq t\} = \frac{F(t + x) - F(t)}{\bar{F}(t)}, \quad x \geq 0,$$

where $\bar{F} = 1 - F$.

Now suppose that the component has failed at some time before t, for some $t > 0$, and we are interested in the past time from failure up to now. For this reason we let the random variable $\tilde{X}_t = [t - X|\, X \leq t]$ denote the time elapsed since failure up to time t, given that the component has already failed by time t. The values of $\tilde{X}_t$ are in the interval $(0, t]$. The corresponding distribution function of $\tilde{X}_t$ is denoted by $\tilde{F}_t(x)$, where

$$\tilde{F}_t(x) = \mathbf{P}\{t - X \leq x|\, X \leq t\} = \frac{F(t) - F(t - x)}{F(t)}, \text{ for } 0 < x \leq t. \tag{1.1}$$

The reversed failure rate function, denoted by $\tilde{r}(x)$, is the ratio of probability density function and the corresponding distribution function. Analytically,

$$\begin{aligned} \tilde{r}(x) &= \lim_{\Delta x \to 0} \frac{\mathbf{P}\{x - \Delta x < X \leq x|\, X \leq x\}}{\Delta x}. \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \frac{\mathbf{P}\{X \leq x\} - \mathbf{P}\{X \leq x - \Delta x\}}{Pr\{X \leq x\}}. \\ &= f(x)/F(x). \end{aligned} \tag{1.2}$$

The density of $\tilde{X}_t$ is $\tilde{f}_t(x) = f(t-x)/F(t)$, which implies that

$$\tilde{r}(t) = \tilde{f}_t(0), \quad \text{for any fixed } \ t > 0. \tag{1.3}$$

Relation (1.3) is similar to that between the failure rate function $r(x) = f(t)/\bar{F}(t)$, $x \geq 0$ and the density function $f(x)$, i.e., $r(0) = f(0)$.

Likewise the function $r(x)$, the function $\tilde{r}(x)$, $x \geq 0$ uniquely determines the distribution function through the equality

$$F(x) = \exp\left\{-\int_x^\infty \tilde{r}(t)\,\mathrm{d}t\right\}, \quad x \geq 0. \tag{1.4}$$

Note that we can write $F(x) = \exp[-\tilde{\Lambda}(x)]$, where $\tilde{\Lambda}$ is the cumulative reversed life function and is given by

$$\tilde{\Lambda}(x) = \int_x^\infty \tilde{r}(t)\,\mathrm{d}t, \quad x \geq 0. \tag{1.5}$$

The reversed failure rate function has many properties that are similar to those of the usual failure rate function.

Next we will give definitions for the decreasing reversed failure rate (DRFR), increasing reversed failure rate (IRFR) and decreasing reversed failure rate in average (DRFRA). We will also study the relation between the functions $\tilde{r}(x)$ and $r(x)$, $x \geq 0$. Shaked and Shanthikumar (1994) gave the following definitions for the DRFR and IRFR classes:

Definition 1. *A nonnegative random variable X, having distribution F, is said to have DRFR, if the function $\tilde{r}(t)$, $t > 0$ is decreasing.*

A wide range of distributions belong to the DRFR class. For example, the exponential distribution is a DRFR; Weibull and gamma with shape parameters less than 1 are also DRFR. Not only this, but also increasing failure rate IFR distributions like Weibull, gamma, and log normal distributions were found to be DRFR distributions. For details, we refer to Block et al. (1998).

Remark 1. *It has been proved mathematically that a decreasing failure rate (DFR) distribution is necessarily DRFR distribution.*

Definition 2. *A nonnegative random variable X, having distribution F, is said to have IRFR, if the function $\tilde{r}(t)$, $t > 0$ is increasing.*

Block et al. (1998) have noted the interesting fact that there is no life distribution which belongs to the class IRFR over the domain $[0, \infty)$.

Definition 3. *A nonnegative random variable X, having distribution F, is said to have DRFRA, if the following averaged function is increasing:*

$$\frac{1}{x}\int_0^x \tilde{r}(u)\,dt, \quad x > 0.$$

Proposition 1. *Suppose that F is a life distribution. Then the following two statements are equivalent:*

(a) F has a decreasing reversed failure rate: $F \in$ DRFR.

(b) F is log-concave.

Note that the equivalence of (a) and (b) is easily obtained, due to the fact that $\tilde{r}(x) = (\log F(x))'$, $x > 0$.

Making a simple transformation one can arrive at an important relation between the failure rate functions $r(x)$ and $\tilde{r}(x)$:

$$\begin{aligned} r(x) &= \frac{f(x)}{\bar{F}(x)} = \frac{\tilde{r}(x)\,F(x)}{1-F(x)} \\ &= \tilde{r}(x)\left(\exp\left\{\int_x^\infty \tilde{r}(u)\mathrm{d}u\right\} - 1\right)^{-1}, \quad x \geq 0 \end{aligned} \tag{1.6}$$

Therefore, if $r(x)$ is decreasing, then $\tilde{r}(x)$ is also decreasing.

Remark 2. *The reversed failure rate function* $\tilde{r}(x)$ *is equivalent to the density function* $f(x)$ *as* x *tends to infinity. Therefore, all distributions with monotonically decreasing densities on* $[0, \infty)$ *are characterized by DRFR.*

Now we deal with a discrete life distribution $\mathcal{P} = \{p_k = \mathbf{P}\{X = k\},\ k \in \mathbb{N}\}$, where $\mathbb{N}$ is the set of all nonnegative integers. With $A_k = \mathbf{P}\{X \le k\} = p_0 + p_1 + \ldots, p_k$, we define the discrete reversed failure rate (D-RFR) as follows:

$$\tilde{r}_k = \frac{p_k}{A_k}, \quad k \in \mathbb{N}. \tag{1.7}$$

Definition 4. *A discrete lifetime* X *with a distribution* $\mathcal{P}$ *is said to be discrete decreasing reversed failure rate (D-DRFR) if the function* $\tilde{r}_k$, $k \in \mathbb{N}$ *is decreasing.*

Examples for D-DRFR distributions are: binomial, Poisson, geometric, hypergeometric, negative binomial, Zeta and Yule distributions.

We state without proof the following characterizations. Details can be found in Nanda and Sengupta (2005) and the references therein.

Proposition 2. *Let* X *be a discrete random variable having the distribution* P_k *and the probability mass function* $\mathcal{P} = \{p_k,\ k \in \mathbb{N}\}$. *Then, with* A_k *as above:*

(a) $X \in$ *D-DRFR if and only if* A_{n+k}/A_k *is decreasing for all* $n, k \in \mathbb{N}$.

(b) If the sequence $\{p_k\}$, $k \in \mathbb{N}$ *is decreasing, then* $X \in$ *D-DRFR.*

(c) If $X \in$ *D-DRFR, then* $X \in$ *discrete decreasing failure rate (D-DFR).*

Here we mention, without proof, closure properties of D-DRFR under convolution and mixing operations.

Proposition 3. *Let* X *and* Y *be two independent D-DRFR random variables. Then their sum* $X+Y$ *is also D-DRFR.*

Proposition 4. *Mixture of D-DRFR distributions is not necessarily D-DRFR.*

We end this section by mentioning an observation by Nanda and Sengupta (2005) that there is no discrete life distribution with domain in $\mathbb{N}$, with an increasing reversed failure rate (D-IRFR).

2. Reversed Mean Residual Lifetime

The reversed mean residual lifetime (RMR), also known as mean inactivity time or mean waiting time, has been studied by many researchers. Li and Zuo (2004), Nanda et al. (2003), Finkelstein (2002) and Chandra and Roy (2001) have presented some results and characterizations based on the monotonicity of RFR and RMR.

We assume that a lifetime $X \sim F$ has a finite first moment and define the RMR:

$$\tilde{\mu}(t) = \mathbf{E}[t - X | X \le t] = \frac{1}{F(t)} \int_0^t F(u)\mathrm{d}u, \quad t > 0. \tag{2.1}$$

The function $\tilde{\mu}(t)$, $t > 0$ uniquely determines the distribution F by the following formula

$$F(x) = \exp\left[-\int_x^\infty \frac{1 - (\mathrm{d}/\mathrm{d}t)\tilde{\mu}(t)}{\tilde{\mu}(t)}\,\mathrm{d}t\right], \tag{2.2}$$

where, in view of (2.1),

$$\frac{\mathrm{d}}{\mathrm{d}x}\tilde{\mu}(x) = 1 - \tilde{r}(x)\tilde{\mu}(x), \quad x > 0. \tag{2.3}$$

Definition 5. *A random variable* $X \sim F$, *is said to have increasing reversed mean residual lifetime (IRMR) if, the function* $\tilde{\mu}(t)$, $t > 0$ *is increasing.*

Chandra and Roy (2001) noted the interesting fact that there is no life distribution with decreasing reversed mean life on the domain $(0, \infty)$. Details can be found in Nanda et al. (2003).

3. Reversed Variance Residual Lifetime

We assume that $X \sim F$ has a finite second moment, so the mean and the variance are well defined. The reversed variance residual lifetime is

$$\tilde{\sigma}^2(x) = \mathbf{Var}[x - X | X \leq x] = \frac{2}{F(x)} \int_0^x \int_0^y F(u)\mathrm{d}u\mathrm{d}y - \tilde{\mu}^2(x), \quad x > 0. \tag{3.1}$$

The three main characteristics $\tilde{r}(x)$, $\tilde{\mu}(x)$ and $\tilde{\sigma}^2(x)$ are linked to each other by the following simple relation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\tilde{\sigma}^2(x) = \tilde{r}(x)(\tilde{\mu}^2(x) - \tilde{\sigma}^2(x)), \quad x > 0. \tag{3.2}$$

Next we define the increasing reversed variance residual lifetime (IRVR).

Definition 6. *Suppose that F is a life distribution with finite second moment. Then F is said to have IRVR if the function $\tilde{\sigma}^2(x)$, $x > 0$ is increasing.*

Looking at relation (3.2), one can easily formulate the following:

Lemma 1. *A life distribution F is DRVR if and only if the following inequality holds for all $x > 0$.*

$$\tilde{\sigma}^2(x) \leq \tilde{\mu}^2(x), \tag{3.3}$$

The dual class, decreasing reversed variance lifetime (DRVR), can be obtained by reversing the inequality in (3.3). However, the following theorem is crucial for this kind of class:

Theorem 1. *There is no nonnegative random variable X for which the function $\tilde{\sigma}^2(x)$, $x \in [0, \infty)$ is decreasing.*

Proof. Define the integral $\tilde{V}(x) = \int_0^x \tilde{v}(u)\mathrm{d}u$, where $\tilde{v}(z) = \int_0^z F(u)\mathrm{d}u$. Integrating $\tilde{V}(x)$ by parts yields

$$\tilde{V}(x) = x\,\tilde{v}(x) - \int_0^x u\,F(u)\mathrm{d}u, \quad x > 0. \tag{3.4}$$

Thus, we can rewrite the RVR function $\tilde{\sigma}^2(x)$ as defined (3.1) in the following equivalent form:

$$\tilde{\sigma}^2(x) = 2x\tilde{\mu}(x) - \tilde{\mu}^2(x) - \frac{2}{F(x)} \int_0^x uF(u)\mathrm{d}u, \quad x > 0. \tag{3.5}$$

The next step is to take the limit for the function $\tilde{\sigma}^2(x)$, which is found to be zero, i.e. $\lim_{x\to 0} \tilde{\sigma}^2(x) = 0$. Now, if $\tilde{\sigma}^2(x)$ is decreasing in $x \geq 0$, then $\tilde{\sigma}^2(x) \leq \tilde{\sigma}^2(0) = 0$, for all $x \geq 0$. But $\tilde{\sigma}^2(x)$ cannot be negative and it is also not possible for it to be zero for all x. The result follows. □

4. Reversed Entropy

Let X be a continuous nonnegative random variable describing the random lifetime of a component. As usual, we denote by F, f and $\bar{F}$ its distribution function, density and the reliability function. We can introduce a number, called entropy, as the measure of uncertainty of the lifetime X.

We assume that f is positive on $(0, \infty)$. Then the entropy of X is denoted by $H(x)$ and is defined by

$$H(X) = -\mathbf{E}[\log f(X)] = -\int_0^\infty f(x) \log f(x)\mathrm{d}x. \tag{4.1}$$

Ebrahimi (1996) considers the residual entropy of X as follows:

$$\begin{aligned} H[X - t | X \geq t] &= -\int_t^\infty \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)}{\bar{F}(t)}\,\mathrm{d}x. \\ &= \log \bar{F}(t) - \frac{1}{\bar{F}(t)} \int_t^\infty f(x) \log f(x)\,\mathrm{d}x. \\ &= 1 - \frac{1}{\bar{F}(t)} \int_t^\infty f(x) \log r(x)\,\mathrm{d}x. \end{aligned} \tag{4.2}$$

The residual entropy has been used to measure the aging and also to characterize the life distributions. It has been shown that if $X \sim F$, then the entropy H uniquely determines the distribution F, see e.g. Ebrahimi (1996) and Belzunce et at. (2004).

In many realistic situations uncertainty is not only related to future time but can also refer to past time. For example, consider a system observed only at certain preassigned inspection times. If at time t the system is inspected for the first time and it is found to have failed, then the uncertainty depends on the past, i.e., on events happening on the interval $(0, t)$. Di Crescenzo and Longobardi (2002) consider the entropy $\tilde{H}$ of the random variable $\tilde{X} = [t-X|X \le t]$ is the random variable of the reversed life. Here H is a measure of the uncertainty of the past life of a failed component given that the component has failed on or before time t. They called it "the past entropy at time t of X":

$$\tilde{H}(t) = H[t - X|\, X \le t] \quad = \quad 1 - \frac{1}{F(t)} \int_0^t f(x)\, \log \tilde{r}(x)\, \mathrm{d}x. \tag{4.3}$$

The role of the function $\tilde{r}(x)$ in the analysis of the reversed entropy is analogous to that of the function $r(x)$ in the analysis of the residual entropy as performed by Ebrahimi (1996).

Based on relation (4.3), it is not difficult to establish the following:

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{H}(t) \quad = \quad \tilde{r}(t)[1 - \tilde{H}(t) - \log \tilde{r}(t)], \quad t > 0. \tag{4.4}$$

Proposition 5. *The reversed entropy of a random variable X is decreasing in the domain* $(0, \infty)$ *if and only if the following inequality holds:*

$$\tilde{H}(t) \ge 1 - \log \tilde{r}(t), \quad \text{for all } \ t > 0. \tag{4.5}$$

Definition 7. *A life distribution F is said to have reversed decreasing uncertainty of residual lifetime (RDUR) if its entropy function* $\tilde{H}(x), x > 0$ *is decreasing.*

Note that the dual class of RDUR is the reversed increasing uncertainty of residual lifetime, denoted by RIUR. It can be obtained by reversing the inequality sign in relation (4.5).

5. Relationships between Reversed Classes

It is of general interest to consider the classes of life distributions defined in terms of the reversed mean residual lifetime, and classes defined in terms of the reversed variance residual lifetime, and establish relationships between them. One of our goals is to construct specific counterexamples in order to show that some implications among these classes are impossible.

The following proposition is due to Nanda et al. (2003). It gives a relationship between DRFR and IRMR life distributions.

Proposition 6. *Let F be a life distribution with a finite second moment. Then: If the reversed failure rate* $\tilde{r}(x)$, $x > 0$ *is decreasing, then the reversed mean residual lifetime* $\tilde{\mu}(x)$, $x > 0$ *is increasing. Symbolically:*

$$F \in DRFR \Longrightarrow F \in IRMR.$$

The converse of this implication is not always true, i.e. there is a life distribution $F \in$ IRMR, such that $F \notin$ DRFR. For counterexamples, see Nanda et al. (2003).

Our next theorem deals with the relationship between RMR and RVR.

Theorem 2. *Suppose that F is a life distribution. Then, the following two statements hold:*

(a) If the reversed mean residual lifetime $\tilde{\mu}(x)$, $x > 0$ *is increasing, so is the reversed variance residual lifetime* $\tilde{\sigma}^2(x)$, $x > 0$. *Symbolically:*

$$F \in IRMR \Longrightarrow F \in IRVR.$$

(b) The converse implication to that in (a) is not generally true: there is a life distribution $F \in IRVR$, *such that* $F \notin IRMR$.

Proof. We start by giving a **proof for claim (a):** Given that the function $\tilde{\mu}(x)$, $x \geq 0$ is increasing, we want to show that the function $\tilde{\sigma}^2(x)$, $x \geq 0$ is also increasing. It is sufficient to establish that the inequality $\tilde{\sigma}^2(x) \leq \tilde{\mu}^2(x)$ is satisfied. Since $\tilde{\mu}(x)$ is an increasing function, then for any $x, y \geq 0$ we have:

$$\begin{aligned} \tilde{\mu}(x) &\leq \tilde{\mu}(y), \quad \text{for } x \leq y, \\ \tilde{\mu}(x) - \tilde{\mu}(y) &\leq 0. \\ \frac{2}{F(x)} \int_0^x F(u)[\tilde{\mu}(u) - \tilde{\mu}(y)]\mathrm{d}u &\leq 0. \end{aligned} \tag{5.1}$$

Making simple calculations on the left-hand-side of (5.1) gives

$$\begin{aligned} \frac{2}{F(x)} \int_0^x F(u)[\tilde{\mu}(u) - \tilde{\mu}(y)]\mathrm{d}u &\leq \frac{2}{F(y)} \int_0^y F(u)\tilde{\mu}(u)\mathrm{d}u - \frac{2\tilde{\mu}(y)}{F(y)} \int_0^y F(u)\mathrm{d}u. \\ &= \tilde{\sigma}^2(y) - \tilde{\mu}^2(y). \end{aligned} \tag{5.2}$$

Hence claim (a) immediately follows from (5.1) and (5.2).
Proof of claim (b): Suppose X is a random variable having the following distribution:

$$F(x) = \begin{cases} \mathrm{e}^{-(1+1/x)}, & \text{if } 0 < x < 1, \\ \mathrm{e}^{(x^2-5)/2}, & \text{if } 1 \leq x < 2, \\ \mathrm{e}^{-1/x}, & \text{if } x \geq 2. \end{cases}$$

This example has already been considered by Block et al., (1998) in order to show some results on the RFR function. Using Maple, or any other mathematical software, one can find $\tilde{\mu}(x)$ and $\tilde{\sigma}^2(x)$ and easily note that the function $\tilde{\sigma}^2(x)$ is increasing for all $x > 0$, i.e., $F \in$ IRVR, while the function $\tilde{\mu}(x)$ is not a monotonic increasing function, i.e., $F \notin$ IFMR. This is shown in Figure 5. □

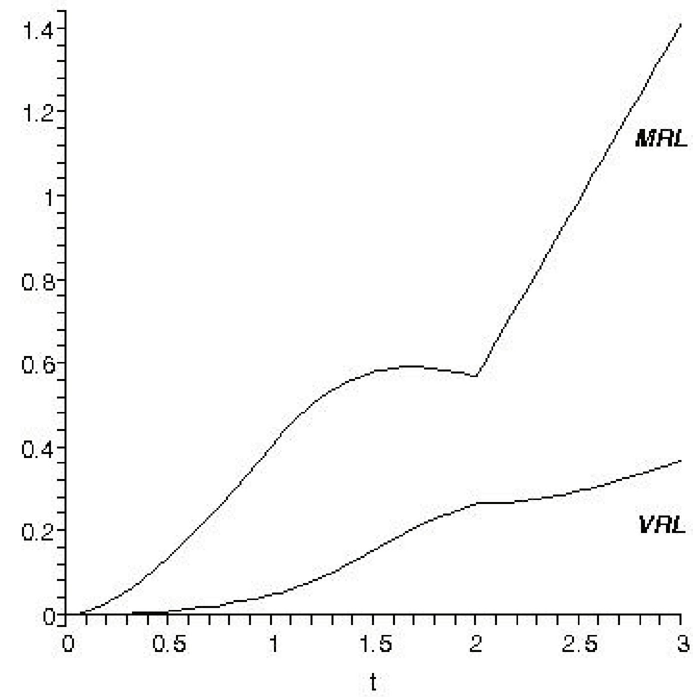

Figure 1. Plot of the reversed mean residual lifetime (MRL) and reversed variance residual lifetime (VRL) for $x \in (0, 3]$

6. Stochastic Ordering and Closure Properties

Useful properties of the stochastic orders are their closure with respect to typical reliability operations like convolution or mixture. We start by giving definitions for stochastic orders in terms of the reversed aging classes.

Definition 8. *Let X and Y be two nonnegative random variables, with continuous distribution functions F_X and F_Y and densities f_X and f_Y, respectively. Then:*

(i) X is said to be smaller than Y in the sense of the reversed failure rate (written as $X \leq_{RFR} Y$) if $\tilde{r}_X(t) \leq \tilde{r}_Y(t)$ for all $t \geq 0$.

(ii) X is said to be smaller than Y in the sense of the reversed mean residual lifetime (written as $X \leq_{RMR} Y$) if $\tilde{\mu}_X(t) \geq \tilde{\mu}_Y(t)$ for all $t \geq 0$. Equivalently, $\int_0^t F_X(u)du / \int_0^t F_Y(u)du$ is decreasing in t.

(iii) X is said to be smaller than Y in the sense of the reversed variance residual lifetime (written as $X \leq_{RVR} Y$) if $\tilde{\sigma}^2_X(t) \geq \tilde{\sigma}^2_Y(t)$, for all $t \geq 0$. Equivalently, $\int_0^t v_X(u)du / \int_0^t v_Y(u)du$ is decreasing, where $v(x) = \int_0^x F(u)du$.

One of our results is in the next theorem giving the relationship between the orderings $\leq_{RFR}$ and $\leq_{RMR}$.

Theorem 3. *The reversed mean residual lifetime ordering implies the reversed variance residual lifetime ordering. Formally:*

$$X \leq_{RMR} Y \Longrightarrow X \leq_{RVR} Y.$$

Proof. The idea of the proof is similar to one used for a different purpose by Nanda et al. (2003). Suppose that $X \leq_{RMR} Y$, then from (ii) in the above definition it follows that the function

$$\frac{\int_0^s F_X(u)\mathrm{d}u}{\int_0^s F_Y(u)\mathrm{d}u} \text{ is decreasing, } \quad s > 0.$$

Let $\tilde{v}_X(s) = \int_0^s F_X(u)\mathrm{d}u$ and take λ to be fixed positive number. Then:

$$\Delta(\tilde{v}_X(s), \tilde{v}_Y(s)) = \int_0^s F_X(u)\mathrm{d}u - \lambda \int_0^s F_Y(u)\mathrm{d}u$$

will have at most one change of sign and if one such change does occur, it occurs from + to −. This implies that, for each λ, the function

$$\int_0^t \Delta(\tilde{v}_X(s), \tilde{v}_Y(s))\mathrm{d}s = \int_0^t \left\{ \int_0^s F_X(u)\mathrm{d}u - \lambda \int_0^s F_Y(u)\mathrm{d}u \right\} \mathrm{d}s$$

will also have at most one change of sign and if one such change does occur, it occurs from + to −. This establishes the following:

$$\frac{\int_0^s \int_0^t F_X(u)\mathrm{d}u\mathrm{d}t}{\int_0^s \int_0^t F_Y(u)\mathrm{d}u\mathrm{d}t} \text{ is decreasing, } \quad s > 0.$$

The result follows. □

Based on the above discussion, we arrive at the following implications among the reversed aging classes:

$$X \leq_{RFR} Y \Longrightarrow X \leq_{RMR} Y \Longrightarrow X \leq_{RVR} Y.$$

It would be interesting to clarify whether or not the inverse implications hold.

As an important reliability operation, convolutions of a certain stochastic order are often paid much attention. In the following theorem we establish the closure property of $\leq_{RVR}$ ordering under the convolution operation when appropriate assumptions are satisfied.

Proposition 7. *Suppose that X_1 and X_2 are independent lifetimes and Y_1 and Y_2 are also another pair of independent lifetimes. Suppose further that $X_1 \leq_{RVR} Y_1$ and $X_2 \leq_{RVR} Y_2$. Then it is not necessarily true that $X_1 + X_2 \leq_{RVR} Y_1 + Y_2$.*

The next theorem is one of our results.

Theorem 4. *Let X_1, X_2 and Y be three life times, or nonnegative random variables, where Y is independent of both X_1 and X_2 , and let Y have density g. Suppose that $X_1 \leq_{RVR} X_2$ and that g is log-concave. Then $X_1 + Y \leq_{RVR} X_2 + Y$.*

Before giving the proof, we need to recall the definition of the total positivity of order 2, denoted by ?, also known as a Polya function.

Definition 9. *Let S_1 and S_2 be two subsets of the real line. A nonnegative function $h(x, y)$, defined for $x, y \in S_1 \times S_2$, is said to be ?, if*

$$h(x_1, y_1)\, h(x_2, y_1) \geq h(x_1, y_2)\, h(x_2, y_2),$$

whenever $x_1 \leq x_2$, $y_1 \leq y_2$ and $x_1, x_2 \in S_1, y_1, y_2 \in S_2$.

We also need the following lemma, see e.g. Joag-Dev et al., (1995), Lynch et al. (1987), Fagiuoli and Pellerey (1994), and Hu et al., (2003).

Lemma 2. *Let $\psi(\theta, x)$ be a ? function in $\theta \in S_1$ and $x \in S_2$. We are given also two distribution functions, $F_1 = \{F_1(\theta), \theta \in S_1\}$ and $F_2 = \{F_2(\theta), \theta \in S_1\}$. Define:*

$$H_1(x) = \int_{S_1} \psi(\theta, x) dF_1(\theta) \quad and \quad H_2(x) = \int_{S_1} \psi(\theta, x) dF_2(\theta).$$

Suppose $F_i(\theta)$, as a function of the two arguments, $(i, \theta) \in \{1, 2\} \times S_1$, is a TP_2 function. Suppose also that for any fixed $x \in S_2$, the function $\psi(\theta, x)$ is decreasing in $\theta \in S_1$. Then $H_i(x)$, $(i, x) \in \{1, 2\} \times S_2$, is a TP_2 function.

Now we are in a good position to prove Theorem 4.

Proof. Let X_1, X_2 and Y be random variables as given above. Define two functions:

$$v(i, s) = \int_0^s F_{X_i}(z) \mathrm{d}z, \quad \text{and} \quad V(i, s) = \int_0^s \int_0^t F_{X_i}(z) \mathrm{d}z \mathrm{d}t.$$

By definition, since $X \leq_{RVR} Y$, $V(i, p)$ is TP_2. We also define the function:

$$\begin{aligned} U(i, s) &= \int_0^s \int_0^t F_{X_i + Y}(z) \mathrm{d}z \mathrm{d}t \\ &= \int_0^s \int_0^t \left\{ \int_0^\infty F_{X_i}(z) f_Y(z - s) \mathrm{d}s \right\} \mathrm{d}z \mathrm{d}t \\ &= \int_0^\infty V(i, s)\, f_Y(z - s)\, \mathrm{d}s. \end{aligned}$$

Since Y has log-concave density, $f_Y(z - s)$, $(z, s) \in S_1 \times S_2$ is TP_2. Therefore, by composition, it follows that $U(i, s)$, $(i, s) \in \{1, 2\} \times S_i$. This completes the proof. □

7. Concluding Remarks

The properties of the reversed failure rate have become a topic of interest not only in reliability, but also in other fields. In this paper we have focused on properties of the reversed variance residual lifetime and the interrelations between reversed aging classes. Also, the increasing reversed variance residual lifetime is introduced. We have established the closure property of the reversed variance residual ordering under the convolution operation when appropriate assumptions are satisfied.

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