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Independence Distribution-preserving Covariance Structures for the Likelihood Ratio Test for $\mathbf{LX}\boldsymbol{\beta} = \mathbf{0}$ in the General Linear Model

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Abstract

Research has been ongoing for over fifty years with respect to violating the assumption of i.i.d. observations in the error covariance matrix. However, there exist test statistics and dependency structures for which the sampling distribution of the test statistic is identical to the test statistic's distribution under the assumption of i.i.d. observations. We derive an explicit representation of the general non-i.i.d. error covariance matrix of the general linear model error vector such that the likelihood ratio test statistic for testing certain linear restrictions on the parameter vector is robust against certain forms of dependency and heteroscedasticity. In doing so, we correct two proposed explicit covariance matrix characterizations given in Khatri (1981).

Keywords: covariance-matrix robustness, orthogonal projection matrix, column space, row space, matrix equation

1. Introduction

1.1 The Problem

Consider the general linear model

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}, \tag{1}$$

where $\mathbf{y}$ is an $n \times 1$ vector of observations, $\mathbf{X}$ is an $n \times p$ known fixed non-null model (design) matrix with $rank(\mathbf{X}) = r \leq p$, $\boldsymbol{\beta}$ is a $p \times 1$ vector of unknown model parameters, and $\boldsymbol{\epsilon}$ is an $n \times 1$ vector of random perturbations such that $\boldsymbol{\epsilon} \sim N\left(\mathbf{0}, \sigma^2\boldsymbol{\Sigma}\right)$, where $\boldsymbol{\Sigma}$ is symmetric nonnegative definite (s.n.d.) and $\sigma^2 > 0$. We assume that (1) is consistent, i.e., $\mathbf{y} \in C(\boldsymbol{\Sigma} : \mathbf{X})$, where $C(\mathbf{A})$ represents the column space of a matrix $\mathbf{A}$.

The usual assumptions of the general linear model include the restriction $\boldsymbol{\Sigma} = \mathbf{I}$ on the error covariance structure for (1). Ignoring this assumption can result in poor statistical references. However, for certain statistics, conditions on the error covariance matrix exist that allow particular types of perturbations of the usual i.i.d. covariance $Var(\mathbf{y}) = \sigma^2\mathbf{I}$ for sampled observations without affecting the characteristics of the sampling distribution of the statistic. That is, non-i.i.d. covariance structures for the data exist for which the sampling distribution of a statistic is identical to the sampling distribution under the assumption of i.i.d. observations. We refer to such covariance structures as *independence distribution-preserving* (IDP) covariance matrices.

The form of IDP dependency structures yields insight into model-misspecification and error-term dependence robustness for a statistic of interest. That is, one can see the form of covariance matrices for which the statistic of interest holds in addition to the often-assumed i.i.d. covariance structure. The existence of IDP covariance structures for a statistic implies that normally distributed observations need not be independent nor must the marginal variances be equal for the usual i.i.d.-induced properties of the statistic to hold. Hence, for certain statistics, some degree of robustness against dependent observations and heteroscedasticity with non-i.i.d. covariance matrices exists.

Khatri (1981) has proposed one implicit and two explicit expressions for the set of s.n.d. error covariance structures such that the distributional properties of the likelihood ratio (LR) statistic under $\boldsymbol{\Sigma} = \mathbf{I}$ for testing

$$H_0 : \mathbf{LX}\boldsymbol{\beta} = \mathbf{0} \text{ against } H_1 : \mathbf{LX}\boldsymbol{\beta} \neq \mathbf{0}, \tag{2}$$

are preserved. Unfortunately, his two proposed explicit IDP covariance matrix characterizations are not general IDP covariance matrices. Here, we explicitly characterize the set of s.n.d. IDP dependency structures $\boldsymbol{\Sigma}^{\geq}_{IDP}$ and symmetric positive definite (s.p.d). dependency structures $\boldsymbol{\Sigma}^{>}_{IDP}$ such that the distribution of the LR statistic for testing (2) is identical to the distribution of the LR statistic under the i.i.d. covariance structure $\boldsymbol{\Sigma} = \mathbf{I}$ for all nonzero $\mathbf{y} \in \mathcal{C}[\boldsymbol{\Sigma} : \mathbf{X}]$.

1.2 Background

Research on IDP covariance structures for univariate statistics has been ongoing for at least fifty years. In one of the first papers published in this area, Walsh (1947) studied the effect of equicorrelated data on certain two-group hypothesis test statistics. Other authors, such as Baldessari (1965), Baldessari and Gallo (1981), Stadje (1984), and Jensen (1989), have considered the derivation of the general s.p.d. IDP covariance structures for the univariate sample variance. In addition, Bhat (1962), Rogers (1980), Stadje (1984), and Young, Thompson, and Turner (2003) have derived the general s.p.d. IDP covariance structures for the univariate sample variance such that the sample variance is a multiple of a chi-squared random variable and is independent of the sample mean.

Recently, Young, Thompson, and Turner (2003) have characterized the general s.n.d. IDP covariance matrix for the multivariate sample variance and Young and Turner (2001) have characterized the general s.n.d. IDP joint covariance structure for the multivariate two-group problem. Also, Young et al. (2003) have characterized the s.n.d. covariance structures such that the sample covariance matrix is distributed as a central Wishart random matrix with $n - 1$ degrees of freedom and is independent of the sample mean vector.

In addition to Khatri (1981), research has been published on test statistics for the univariate dependent-variable regression case. For instance, Halperin (1951) has described an IDP covariance structure for a statistic used to test model adequacy, and Jeyaratnam (1982) has provided a sufficient IDP covariance structure for a statistic in testing linear hypotheses of the form (2) where $\mathbf{L} \in \mathbb{R}_{s\times n}$ and $rank(\mathbf{LX}) = s$, and $\mathbf{X}$ is the design matrix so that $\mathbf{LX}\boldsymbol{\beta}$ is a set of estimable functions. Tranquilli and Baldessari (1988) have devised an IDP structure for a test of model adequacy in multiple regression analysis, and Ghosh and Sinha (1980) have derived conditions on the design matrix $\mathbf{X}$ to yield the usual F-statistic for testing (2).

While these contributions are significant, we note that our primary focus is on the solutions obtained in Khatri (1981). We improve upon Khatri's (1981) results in three ways. First, we provide a correct general representation of the general IDP covariance structure. Second, our IDP characterization has no indeterminate matrices. In other words, we rigorously define all matrices in our general representation of the IDP covariance structure so that the structure is s.n.d. Third, our representation is not restricted such that $(\boldsymbol{\Sigma} - \theta\mathbf{I})$ is s.n.d., where $\theta > 0$.

1.3 Outline

We have organized the remainder of the paper as follows. In Section 2, we define notation and give two LR statistics of interest for testing (2). In Section 3, we present Khatri's (1981) proposed explicit IDP general covariance structure characterization along with six lemmas needed for the derivation of our main results. We derive our IDP characterization of the general s.n.d. IDP dependency structure for the LR statistic for testing (2) in Section 4. Finally, we provide some brief concluding remarks in Section 5.

2. Notation and Previous IDP Results for a Maximum Likelihood-Ratio Statistic

2.1 General Mathematical Notation

We use the following notation throughout the remainder of the paper. The symbol $\mathbb{C}_{m\times n}$ $(\mathbb{R}_{m\times n})$ represents the vector space of all $m \times n$ matrices over the complex (real) field $\mathbb{C}$ $(\mathbb{R})$. The notation $\mathbb{R}^{S}_{n}$ represents the set of symmetric matrices in $\mathbb{R}_{n\times n}$. The symbol $\mathbb{C}^{\geq}_{n}$ $(\mathbb{R}^{\geq}_{n})$ denotes the cone of all Hermitian s.n.d. matrices in $\mathbb{C}_{n\times n}$ $(\mathbb{R}_{n\times n})$, and $\mathbb{C}^{>}_{n}$ $(\mathbb{R}^{>}_{n})$ represents the interior cone composed of the set of Hermitian s.p.d. matrices in $\mathbb{C}_{n\times n}$ $(\mathbb{R}_{n\times n})$. The notation $\mathbf{A}^{*}$ represents the conjugate transpose of $\mathbf{A} \in \mathbb{C}_{n\times m}$, and the symbol $\mathbf{A}'$ denotes the transpose of $\mathbf{A} \in \mathbb{R}_{n\times m}$. Also, $\mathbf{B}^{-} \in \mathbb{R}_{n\times m}$ represents a generalized inverse of $\mathbf{B} \in \mathbb{R}_{m\times n}$, and $\mathbf{B}^{+} \in \mathbb{R}_{n\times m}$ denotes the Moore-Penrose pseudoinverse of $\mathbf{B} \in \mathbb{R}_{m\times n}$. Additionally, $\mathcal{C}(\mathbf{A})$ and $\mathcal{R}(\mathbf{A})$ denote the column space and row space, respectively, of $\mathbf{A}$, and $\mathcal{C}(\mathbf{A})^{\perp}$ and $\mathcal{R}(\mathbf{A})^{\perp}$ represent the orthogonal complement of $\mathcal{C}(\mathbf{A})$ and $\mathcal{R}(\mathbf{A})$, respectively. Also, $\mathbf{P_A}$ denotes the orthogonal projection matrix onto $\mathcal{C}(\mathbf{A})$, $\mathbf{R_A}$ denotes the orthogonal projection onto $\mathcal{R}(\mathbf{A})$, and $\mathbf{P}^{\perp}_{\mathbf{A}}$ and $\mathbf{R}^{\perp}_{\mathbf{A}}$ represent the orthogonal projection matrix onto $\mathcal{C}(\mathbf{A})^{\perp}$ and $\mathcal{R}(\mathbf{A})^{\perp}$, respectively.

2.2 IDP Notation

We need the following notation and terms to specifically define the IDP problem we address. Let $\mathbf{L} \in \mathbb{R}_{s\times n}$ such that $rank(\mathbf{LX}) = s$, and let

$$\begin{bmatrix} \mathbf{V}_{11} & \mathbf{V}_{12} \\ \mathbf{V}_{21} & \mathbf{V}_{22} \end{bmatrix} \equiv \begin{bmatrix} \mathbf{P_X} \\ \mathbf{P_X^\perp} \end{bmatrix} \mathbf{\Sigma} \begin{bmatrix} \mathbf{P_X} : & \mathbf{P_X^\perp} \end{bmatrix}$$

with $rank(\mathbf{V}_{22}) = n - r_1$ and $rank\left[\mathbf{LP_X}\left(\mathbf{V}_{11} - \mathbf{V}_{12}\mathbf{V}_{22}^{-}\mathbf{V}_{12}'\right)\mathbf{P_X L}'\right] = s_1$.

Assuming the Gauss-Markov model (1), Khatri (1981) has derived the LR statistic for testing the hypothesis in (2), which is

$$F_{\mathbf{\Sigma}} = \frac{(n-r_1)\left(\mathbf{P_X y} - \mathbf{V}_{12}\mathbf{V}_{22}^{-}\mathbf{P_X^\perp y}\right)' \mathbf{P_X L}'\left[\mathbf{LP_X}\left(\mathbf{V}_{11} - \mathbf{V}_{12}\mathbf{V}_{22}^{-}\mathbf{V}_{12}'\right)\mathbf{P_X L}'\right]^{-} \mathbf{LP_X}\left(\mathbf{P_X y} - \mathbf{V}_{12}\mathbf{V}_{22}^{-}\mathbf{P_X^\perp y}\right)}{s_1\left(\mathbf{y}'\mathbf{P_X^\perp}\mathbf{V}_{22}^{-}\mathbf{P_X^\perp y}\right)}, \tag{3}$$

where $F_{\mathbf{\Sigma}} \sim F_{s_1, n-r_1}$ if H_0 holds and $\mathbf{\Sigma}$ is known. If $\mathbf{\Sigma} = \mathbf{I}$, then because $\mathcal{C}(\mathbf{L}') \subset \mathcal{C}(\mathbf{X})$, $s = s_1$, and $r = r_1$, the test statistic (3) becomes

$$F_{\mathbf{I}} = \frac{(n-r)\,\mathbf{y}'\mathbf{L}'\left[\mathbf{LL}'\right]^{-}\mathbf{Ly}}{s\left(\mathbf{y}'\mathbf{P_X^\perp y}\right)} \tag{4}$$

where $F_{\mathbf{I}} \sim F_{s,n-r}$.

3. Mathematical Preliminaries

3.1 Khatri's (1981) IDP Solutions

Khatri (1981) has implicitly characterized the set of s.n.d. IDP covariance structures $\mathbf{\Sigma} \in \mathbb{R}_n^{\geq}$ such that $F_{\mathbf{\Sigma}} = F_{\mathbf{I}}$ as

$$\mathbf{\Sigma}_{IDP}^{K_{IM}} \equiv \left\{\mathbf{\Sigma} \in \mathbb{R}_n^{\geq} : \begin{bmatrix} \mathbf{L} \\ \mathbf{P}_X^\perp \end{bmatrix}\left[\mathbf{\Sigma} - \theta\mathbf{I}\right]\begin{bmatrix} \mathbf{L}' : & \mathbf{P}_X^\perp \end{bmatrix} = \mathbf{0}\right\} \tag{5}$$

given $\theta > 0$ is fixed. Khatri (1981) has also proposed two explicit characterizations for the set of IDP covariance structures such that $F_{\mathbf{\Sigma}} = F_{\mathbf{I}}$. Khatri's (1981) first IDP characterization, which he found by deriving the general solution for $[\mathbf{\Sigma} - \theta\mathbf{I}]$ to the matrix equation (5), is

$$\mathbf{\Sigma}_{IDP}^{K_1} \equiv \left\{\mathbf{\Sigma} \in \mathbb{R}_n^{\geq} : \mathbf{\Sigma} = \theta\mathbf{I} + \mathbf{W} - (\mathbf{I} - \mathbf{P_X G P_X})\,\mathbf{W}\,(\mathbf{I} - \mathbf{P_X G P_X})\right\}, \tag{6}$$

where $\mathbf{G} = \mathbf{I}_n - \mathbf{L}'\left[\mathbf{LP_X L}'\right]^{-}\mathbf{L}$, $\mathbf{W} \in \mathbb{R}_{n\times n}$ is an arbitrary symmetric matrix such that $\mathbf{\Sigma} \in \mathbb{R}_n^{\geq}$ and $\theta > 0$. Unfortunately, Khatri's (1981) IDP general covariance matrix (6) is not an explicit characterization of the set of IDP covariance structures such that $F_{\mathbf{\Sigma}} = F_{\mathbf{I}}$ (see appendix).

Khatri's (1981) second proposed general IDP covariance structure is purportedly a sufficient IDP covariance structure because it is restricted to the case where $(\mathbf{\Sigma} - \theta\mathbf{I}) \in \mathbb{R}_n^{\geq}$. This explicit general IDP covariance matrix is

$$\mathbf{\Sigma}_{IDP}^{K_2} \equiv \left\{\mathbf{\Sigma} \in \mathbb{R}_n^{\geq} : \mathbf{\Sigma} = \theta\mathbf{I} + \mathbf{P_X G P_X W P_X G P_X}\right\}, \tag{7}$$

where $\theta > 0$ and $\mathbf{W} \in \mathbb{R}_n^{\geq}$ is arbitrary. We remark that (7) has the unnecessary restriction that $(\mathbf{\Sigma} - \theta\mathbf{I}) \in \mathbb{R}_n^{\geq}$. Because (7) follows from (6), neither the covariance matrices given in (6) nor (7) represents an explicit characterization of the set of s.n.d. IDP covariance matrices for $F_{\mathbf{\Sigma}} = F_{\mathbf{I}}$.

3.2 Lemmas

We now state six lemmas that we use in the proof of our IDP covariance-matrix characterization result. The first lemma gives a representation of the general s.n.d. solution to the matrix equation $\mathbf{AXA}^* = \mathbf{B}$. We remark that Groß (2000) gives another form of a general n.n.d. solution to $\mathbf{AXA}^* = \mathbf{B}$.

Lemma 3.1. *(Baksalary 1984, Theorem 3) Let* $\mathbf{A} \in \mathbb{C}_{m\times n}$ *and* $\mathbf{B} \in \mathbb{C}_m^{\geq}$ *such that* $\mathcal{C}(\mathbf{B}) = \mathcal{C}(\mathbf{A})$. *Then, a represnta-tion of the general Hermitian n.d. solution to* $\mathbf{AXA}^* = \mathbf{B}$ *is*

$$\mathbf{X} = \left[\mathbf{A}^{-}\mathbf{D} + (\mathbf{I}_n - \mathbf{A}^{-}\mathbf{A})\,\mathbf{Z}\right]\left[\mathbf{A}^{-}\mathbf{D} + (\mathbf{I}_n - \mathbf{A}^{-}\mathbf{A})\,\mathbf{Z}\right]^*,$$

where $\mathbf{D} \in \mathbb{C}_{m\times n}$ *is an arbitrary but fixed matrix such that* $\mathbf{B} = \mathbf{DD}^*$, *and* $\mathbf{Z} \in \mathbb{C}_{n\times n}$ *is free to vary.*

The following lemma gives a representation of a generalized inverse of a vertically partitioned matrix.

Lemma 3.2. *(Harville 1999, Exercise 17.3.11) If* $\mathbf{A} \in \mathbb{R}_{m\times p}$ *and* $\mathbf{B} \in \mathbb{R}_{n\times p}$*, then*

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix}^{-} = \begin{bmatrix} \mathbf{A}^{-} - \mathbf{F_A}\mathbf{Y}^{-}\mathbf{B}\mathbf{A}^{-} : & \mathbf{F_A}\mathbf{Y}^{-} \end{bmatrix},$$

where $\mathbf{F_A} = \mathbf{I}_p - \mathbf{A}^{-}\mathbf{A}$ *and* $\mathbf{Y} = \mathbf{B}\mathbf{F_A}$.

The next lemma provides conditions for two orthogonal projection matrices to commute.

Lemma 3.3 *(Werner 2004, Solution to 31.75) Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{R}_n$ *be orthogonal projectors of the same order. Then,* $(\mathbf{PQ})^{+} = \mathbf{PQ}$ *if and only if* $\mathbf{QP} = \mathbf{PQ}$.

We use the following three lemmas in the proof of our main result in Section 4.

Lemma 3.4. *Let* $\mathbf{\Sigma}$ *denote the general covariance matrix given in* (8)*, and let* $\mathbf{V}_{12} \equiv \mathbf{P_X\Sigma P_X^{\perp}}$*. Then,* $\mathbf{V}_{12} = \theta\,(\mathbf{P_X} - \mathbf{R_L})\,\mathbf{Z}\mathbf{P_X^{\perp}}$.

Proof. Using the fact that $\mathbf{P_X R_L} = \mathbf{R_L P_X}$ and that $\mathbf{P_X^{\perp} R_L^{\perp}} = \mathbf{R_L^{\perp} P_X^{\perp}}$, which follow from Lemma 3.3, we have that

$$\begin{aligned}
\mathbf{P_X\Sigma P_X^{\perp}} &= \theta\mathbf{P_X}\left[\left(\mathbf{R_L} + \mathbf{P_X^{\perp}}\right) + \left(\mathbf{R_L^{\perp}P_X}\right)\mathbf{H} + \mathbf{H}'\left(\mathbf{P_X R_L^{\perp}}\right)\right]\mathbf{P_X^{\perp}} \\
&= \theta\mathbf{P_X}\left[\left(\mathbf{R_L} + \mathbf{P_X^{\perp}}\right) + \left(\mathbf{R_L^{\perp}P_X}\right)\mathbf{Z}\right]\left[\left(\mathbf{R_L} + \mathbf{P_X^{\perp}}\right) + \left(\mathbf{R_L^{\perp}P_X}\right)\mathbf{Z}\right]'\mathbf{P_X^{\perp}} \\
&= \theta\left[\mathbf{P_X R_L} + \mathbf{P_X P_X^{\perp}} + \mathbf{P_X R_L^{\perp} P_X Z}\right]\mathbf{P_X^{\perp}} \\
&= \theta\left[\mathbf{R_L} + (\mathbf{P_X} - \mathbf{R_L})\,\mathbf{Z}\right]\mathbf{P_X^{\perp}} \\
&= \theta\,(\mathbf{P_X} - \mathbf{R_L})\,\mathbf{Z}\mathbf{P_X^{\perp}}.
\end{aligned}$$

Lemma 3.5. *Let* $\mathbf{\Sigma}$ *denote the general covariance matrix given in* (8)*, and let* $\mathbf{V}_{11} \equiv \mathbf{P_X\Sigma P_X}$*. Then,*

$$\mathbf{V}_{11} = \theta\left[\mathbf{R_L} + \mathbf{R_L Z}'\,(\mathbf{P_X} - \mathbf{R_L'}) + (\mathbf{P_X'} - \mathbf{R_L})\,\mathbf{Z R_L} + \mathbf{R_L^{\perp} P_X Z Z' P_X R_L^{\perp\prime}}\right].$$

Proof. Again, using the identities $\mathbf{P_X R_L} = \mathbf{R_L P_X}$ and $\mathbf{P_X^{\perp} R_L^{\perp}} = \mathbf{R_L^{\perp} P_X^{\perp}}$, we have

$$\begin{aligned}
\mathbf{P_X\Sigma P_X} &= \theta\mathbf{P_X}\left[\left(\mathbf{R_L} + \mathbf{P_X^{\perp}}\right) + \left(\mathbf{R_L^{\perp}P_X^{\perp}}\right)\mathbf{H} + \mathbf{H}'\left(\mathbf{P_X R_L^{\perp}}\right)\right]\mathbf{P_X} \\
&= \theta\mathbf{P_X}\left[\left(\mathbf{R_L} + \mathbf{P_X^{\perp}}\right) + \left(\mathbf{R_L^{\perp}P_X}\right)\mathbf{Z}\right]\left[\left(\mathbf{R_L} + \mathbf{P_X^{\perp}}\right) + \left(\mathbf{R_L^{\perp}P_X}\right)\mathbf{Z}\right]'\mathbf{P_X} \\
&= \theta\left[\mathbf{P_X R_L} + \mathbf{P_X P_X^{\perp}} + \mathbf{P_X R_L^{\perp} P_X Z}\right]\left[\mathbf{R_L' P_X} + \mathbf{P_X^{\perp\prime} P_X} + \mathbf{Z' P_X' R_L^{\perp\prime} P_X}\right] \\
&= \theta\left(\mathbf{R_L} + \mathbf{P_X R_L^{\perp} Z}\right)\left(\mathbf{R_L'} + \mathbf{Z' P_X' R_L^{\perp\prime}}\right)\mathbf{P_X} \\
&= \theta\left[\mathbf{R_L} + \mathbf{R_L Z}'\,(\mathbf{P_X} - \mathbf{R_L'}) + (\mathbf{P_X'} - \mathbf{R_L})\,\mathbf{Z R_L} + \mathbf{R_L^{\perp} P_X Z Z' P_X R_L^{\perp\prime}}\right].
\end{aligned}$$

Lemma 3.6. *Let* $\mathbf{\Sigma}$ *denote the general covariance matrix given in* (8)*, and let* $\mathbf{V}_{22} \equiv \mathbf{P_X^{\perp}\Sigma P_X^{\perp}}$*. Then,* $\mathbf{V}_{22} = \theta\mathbf{P_X^{\perp}}$.

Proof. Using the fact that $\mathbf{P_X^{\perp} R_L^{\perp}} = \mathbf{R_L^{\perp} P_X^{\perp}}$, we have

$$\begin{aligned}
\mathbf{P_X^{\perp}\Sigma P_X^{\perp}} &= \theta\mathbf{P_X^{\perp}}\left[\left(\mathbf{R_L} + \mathbf{P_X^{\perp}}\right) + \left(\mathbf{R_L^{\perp}P_X^{\perp}}\right)\mathbf{H} + \mathbf{H}'\left(\mathbf{P_X R_L^{\perp}}\right)\right]\mathbf{P_X^{\perp}} \\
&= \theta\mathbf{P_X^{\perp}}\left[\left(\mathbf{R_L} + \mathbf{P_X^{\perp}}\right) + \left(\mathbf{R^{\perp}P_X}\right)\mathbf{Z}\right]\left[\left(\mathbf{R_L} + \mathbf{P_X^{\perp}}\right) + \left(\mathbf{R^{\perp}P_X}\right)\mathbf{Z}\right]'\mathbf{P_X^{\perp}} \\
&= \theta\left[\mathbf{P_X^{\perp} R_L} + \mathbf{P_X^{\perp} P_X^{\perp}} + \mathbf{P_X^{\perp} R_L^{\perp} P_X Z}\right]\left[\mathbf{R_L}'\mathbf{P_X^{\perp\prime}} + \mathbf{P_X^{\perp} P_X^{\perp\prime}} + \mathbf{Z' P_X' R_L^{\perp\prime} P_X^{\perp}}\right] \\
&= \theta\left[\mathbf{P_X^{\perp} R_L} + \mathbf{P_X^{\perp} P_X^{\perp}} + \mathbf{R_L^{\perp}}\left(\mathbf{P_X^{\perp} P_X}\right)\mathbf{Z}\right]\left[\mathbf{R_L}'\mathbf{P_X^{\perp\prime}} + \mathbf{P_X^{\perp} P_X^{\perp\prime}} + \mathbf{Z' P_X' P_X^{\perp} R_L^{\perp\prime}}\right] \\
&= \theta\mathbf{P_X^{\perp}}.
\end{aligned}$$

4. The Main Result

We next derive our new general s.n.d. IDP covariance structure such that $F_{\mathbf{\Sigma}} = F_{\mathbf{I}}$.

Theorem. *Consider the Gauss-Markov model defined in* (1)*, where* $\mathbf{\Sigma} \in \mathbb{R}_n^{\geq}$*. Then, the LR statistic* $F_{\mathbf{\Sigma}}$*, given in* (3) *for testing* (2)*, is the LR statistic* $F_{\mathbf{I}}$ *defined in* (4) *if and only if* $\mathbf{\Sigma} \in \mathbf{\Sigma}_{IDP}^{\geq}$ *such that*

$$\mathbf{\Sigma}_{IDP}^{\geq} \equiv \left\{\mathbf{\Sigma} \in \mathbb{R}_n^{\geq} : \mathbf{\Sigma} = \theta\left(\mathbf{R_L} + \mathbf{P_X^{\perp}}\right) + \left(\mathbf{R_L^{\perp}P_X}\right)\mathbf{H} + \mathbf{H}'\left(\mathbf{P_X R_L^{\perp}}\right)\right\}, \tag{8}$$

where

$$\mathbf{H} \equiv \theta\left[\mathbf{Z}\left(\mathbf{R_L} + \mathbf{P_X^\perp}\right) + \frac{1}{2}\left(\mathbf{R_L^\perp P_X}\right)\right] \tag{9}$$

such that $\mathbf{Z} \in \mathbb{R}_{n\times n}$ *is free to vary and* $\theta > 0$.

Proof. First, using the fact that $\mathbf{P_X^\perp R_L^\perp} = \mathbf{R_L^\perp P_X^\perp}$, we have

$$\begin{aligned}\begin{bmatrix}\mathbf{L}\\ \mathbf{P_X^\perp}\end{bmatrix}^+ &= \begin{bmatrix}\mathbf{L}^+ - \mathbf{R_L}\left(\mathbf{P_X^\perp R_L^\perp}\right)^+\mathbf{P_X^\perp L}^+ : & \mathbf{R_L^\perp}\left(\mathbf{P_X^\perp R_L^\perp}\right)^+\mathbf{P_X^\perp}\end{bmatrix}\\ &= \begin{bmatrix}\mathbf{L}^+ : & \mathbf{R_L^\perp P_X^\perp}\end{bmatrix}.\end{aligned} \tag{10}$$

Next, assume $F_{\mathbf{\Sigma}} = F_{\mathbf{I}}$. Then, from (5) and from Lemma 3.1, a s.n.d. solution to

$$\begin{bmatrix}\mathbf{L}\\ \mathbf{P_X^\perp}\end{bmatrix}\mathbf{\Sigma}\begin{bmatrix}\mathbf{L}' : & \mathbf{P_X^\perp}\end{bmatrix} = \theta\begin{bmatrix}\mathbf{LL}' & \mathbf{0}\\ \mathbf{0} & \mathbf{P_X^\perp}\end{bmatrix} \tag{11}$$

exists because $C\left(\begin{bmatrix}\mathbf{L}\\ \mathbf{P_X^\perp}\end{bmatrix}\right) = C\left(\begin{bmatrix}\mathbf{LL}' & \mathbf{0}\\ \mathbf{0} & \mathbf{P_X^\perp}\end{bmatrix}\right)$. Hence, from Lemma 3.1 and (10), we have

$$\begin{aligned}\mathbf{\Sigma} &= \theta\left[\begin{bmatrix}\mathbf{L}\\ \mathbf{P_X^\perp}\end{bmatrix}^+\begin{bmatrix}\mathbf{L}\\ \mathbf{P_X^\perp}\end{bmatrix} + \left(\mathbf{I} - \begin{bmatrix}\mathbf{L}\\ \mathbf{P_X^\perp}\end{bmatrix}^+\begin{bmatrix}\mathbf{L}\\ \mathbf{P_X^\perp}\end{bmatrix}\right)\mathbf{Z}\right]\left[\begin{bmatrix}\mathbf{L}\\ \mathbf{P_X^\perp}\end{bmatrix}^+\begin{bmatrix}\mathbf{L}\\ \mathbf{P_X^\perp}\end{bmatrix} + \left(\mathbf{I} - \begin{bmatrix}\mathbf{L}\\ \mathbf{P_X^\perp}\end{bmatrix}^+\begin{bmatrix}\mathbf{L}\\ \mathbf{P_X^\perp}\end{bmatrix}\right)\mathbf{Z}\right]'\\ &= \theta\left[\left(\mathbf{L}^+\mathbf{L} + \mathbf{R_L^\perp P_X^\perp}\right) + \left(\mathbf{I} - \left(\mathbf{L}^+\mathbf{L} + \mathbf{R_L^\perp P_X^\perp}\right)\right)\mathbf{Z}\right]\left[\left(\mathbf{L}^+\mathbf{L} + \mathbf{R_L^\perp P_X^\perp}\right) + \left(\mathbf{I} - \left(\mathbf{L}^+\mathbf{L} + \mathbf{R_L^\perp P_X^\perp}\right)\right)\mathbf{Z}\right]'\\ &= \theta\left[\left(\mathbf{R_L} + \mathbf{P_X^\perp}\right) + \left(\mathbf{R_L^\perp P_X}\right)\mathbf{Z}\right]\left[\left(\mathbf{R_L} + \mathbf{P_X^\perp}\right) + \left(\mathbf{R_L^\perp P_X}\right)\mathbf{Z}\right]'\\ &= \theta\left(\mathbf{R_L} + \mathbf{P_X^\perp}\right) + \left(\mathbf{R_L^\perp P_X^\perp}\right)\mathbf{H} + \mathbf{H}'\left(\mathbf{P_X R_L^\perp}\right),\end{aligned}$$

where $\mathbf{H}$ is given in (9).

Now let the Gauss-Markov model (1) hold, and suppose we wish to test the hypothesis given in (2). We remark that Khatri (1981) has shown that the LR statistic for testing the hypothesis in (2) is (3) for $\mathbf{V} \neq \mathbf{I}$ and (4) for $\mathbf{\Sigma} = \mathbf{I}$. Also, let $\mathbf{\Sigma} \in \mathbf{\Sigma}_{IDP}^{\geq}$, where $\mathbf{\Sigma}_{IDP}^{\geq}$ is defined in (8), and let $\mathbf{V}_S \equiv \mathbf{V}_{11} - \mathbf{V}_{12}\mathbf{V}_{22}^-\mathbf{V}_{12}'$. Then, using Lemmas 3.3 - 3.6, we have that

$$\begin{aligned}F_{\mathbf{\Sigma}} &= \frac{(n-r_1)\left(\mathbf{P_X y} - \mathbf{V}_{12}\mathbf{V}_{22}^-\mathbf{P_X^\perp y}\right)'\mathbf{P_X L}'\left[\mathbf{LP_X V}_S\mathbf{P_X L}'\right]^-\mathbf{LP_X'}\left(\mathbf{P_X y} - \mathbf{V}_{12}\mathbf{V}_{22}^-\mathbf{P_X^\perp y}\right)}{s_1\left(\mathbf{y}'\mathbf{P_X^\perp V}_{22}^-\mathbf{P_X^\perp y}\right)}\\ &= \frac{(n-r_1)\left(\mathbf{P_X y} - \left(\mathbf{P_X}' - \mathbf{R_L}\right)\mathbf{ZP_X^\perp y}\right)'\mathbf{P_X L}'\left[\mathbf{LP_X V}_S\mathbf{P_X L}'\right]^-\mathbf{LP_X'}\left(\mathbf{P_X y} - \left(\mathbf{P_X}' - \mathbf{R_L}\right)\mathbf{ZP_X^\perp y}\right)}{s_1\left(\mathbf{y}'\mathbf{P_X^\perp V}_{22}^-\mathbf{P_X^\perp y}\right)}\\ &= \frac{(n-r_1)\left(\mathbf{y}'\mathbf{P_X}\right)\mathbf{P_X L}'\left[\mathbf{LV}_{11}\mathbf{L}'\right]^-\mathbf{LP_X}\left(\mathbf{P_X y}\right)}{s_1\left(\mathbf{y}'\mathbf{P_X^\perp V}_{22}^-\mathbf{P_X^\perp y}\right)}\\ &= \frac{(n-r_1)\left(\mathbf{y}'\mathbf{P_X}\right)\mathbf{P_X L}'\left\{\theta\mathbf{L}\left[\mathbf{R_L} + \mathbf{R_L Z}'\left(\mathbf{P_X} - \mathbf{R_L}'\right) + \left(\mathbf{P_X}' - \mathbf{R_L}\right)\mathbf{ZR_L} + \mathbf{R_L^\perp P_X ZZ'P_X R_L^{\perp\prime}}\right]\mathbf{L}'\right\}^-\mathbf{LP_X}\left(\mathbf{P_X y}\right)}{s_1\left(\mathbf{y}'\mathbf{P_X^\perp V}_{22}^-\mathbf{P_X^\perp y}\right)}\\ &= \frac{\theta(n-r_1)\mathbf{y}'\mathbf{L}'\left[\mathbf{LL}'\right]^-\mathbf{Ly}}{\theta s_1\left(\mathbf{y}'\mathbf{P_X^\perp V}_{22}^-\mathbf{P_X^\perp y}\right)}\\ &= \frac{(n-r_1)\mathbf{y}'\mathbf{L}'\left[\mathbf{LL}'\right]^-\mathbf{Ly}}{s_1\mathbf{y}'\mathbf{P_X^\perp y}}\\ &= F_{\mathbf{I}}.\end{aligned}$$

The following corollary provides a characterization of the general s.p.d. IDP covariance structures such that $F_{\mathbf{\Sigma}} = F_{\mathbf{I}}$.

Corollary. *Consider the Gauss-Markov model defined in* (1) *with* $\mathbf{\Sigma} \in \mathbb{R}_n^{>}$. *Then, the LR test statistic* $F_{\mathbf{\Sigma}}$ *given in* (3) *for testing* (2) *is identical to the test statistic* $F_{\mathbf{I}}$ *given in* (4) *if and only if* $\mathbf{\Sigma} \in \mathbf{\Sigma}_{IDP}^{>}$, *where*

$$\mathbf{\Sigma}_{IDP}^{>} \equiv \left\{\mathbf{\Sigma} \in \mathbb{R}_n^{\geq} : \mathbf{\Sigma} = \theta\left(\mathbf{R_L} + \mathbf{P_X^\perp}\right) + \left(\mathbf{R_L^\perp P_X}\right)\mathbf{H} + \mathbf{H}'\left(\mathbf{P_X R_L^\perp}\right)\right\}, \tag{12}$$

where $\mathbf{H}$ *is given in* (9) *and* $\mathbf{Z} \in \mathbf{S_Z}$ *with*

$$\mathbf{S_Z} \equiv \left\{ \mathbf{Z} \in \mathbb{R}_n : \mathcal{N}(\mathbf{Z}') \cap \mathcal{N}\left(\begin{bmatrix} \mathbf{L} \\ \mathbf{P}_\mathbf{X}^\perp \end{bmatrix} \right) = \{\mathbf{0}\}, \mathcal{C}\left(\begin{bmatrix} \mathbf{L} \\ \mathbf{P}_\mathbf{X}^\perp \end{bmatrix}' \right) \cap \mathcal{C}\left[\mathbf{Z}'\left(\mathbf{R}_\mathbf{L}^\perp - \mathbf{P}_\mathbf{X}^\perp \right) \right] = \{\mathbf{0}\} \right\}. \tag{13}$$

Proof. The proof of the corollary follows from our Theorem and from the theorem in Marsaglia and Styan (1972) to obtain the conditions given in (13) so that the matrices in (12) are s.p.d.

5. Concluding Remarks

In this paper, we have characterized the general s.n.d. explicit IDP covariance structure for the LR statistic (3) for testing (2). Thus, in (7) one can see the general form of the non-i.i.d. covariance structures such that (3) yields the same F-statistic as in the i.i.d. covariance structure case given in (4). That is, we have rigorously characterized the covariance structures $\mathbf{\Sigma}_{IDP}^{\geq}$ such that one can test (2) using the simplified test statistic (4) without loss of power. In summary, we have completely described the degrees of observation-dependence and heteroscedastic robustness of the test statistic (4) with respect to the covariance structure $\mathbf{\Sigma}$ in (1).

Appendix.

Khatri (1981) has derived his IDP covariance structure characterization by solving the matrix equation

$$\begin{bmatrix} \mathbf{L} \\ \mathbf{P}_\mathbf{X}^\perp \end{bmatrix} [\mathbf{\Sigma} - \theta\mathbf{I}] \begin{bmatrix} \mathbf{L}' : & \mathbf{P}_\mathbf{X}^\perp \end{bmatrix} = \mathbf{0} \tag{14}$$

for $(\mathbf{\Sigma} - \theta\mathbf{I})$. His general s.n.d. covariance structure is given in (6). One can easily show that $\theta\left(\mathbf{R_L} + \mathbf{P}_\mathbf{X}^\perp\right)$ is an IDP covariance matrix so that $F_\mathbf{\Sigma} = F_\mathbf{I}$. Thus, $\theta\left(\mathbf{R_L} + \mathbf{P}_\mathbf{X}^\perp\right) - \theta\mathbf{I} = \theta\left(\mathbf{R_L} + \mathbf{P_X}\right)$ should satisfy (14). Therefore, from (6), we should have that

$$\theta\mathbf{R_L} + \theta\mathbf{P}_\mathbf{X}^\perp = \theta\mathbf{I} + \mathbf{W} - (\mathbf{I} - \mathbf{P_X}\mathbf{G}\mathbf{P_X})\,\mathbf{W}\,(\mathbf{I} - \mathbf{P_X}\mathbf{G}\mathbf{P_X}) \tag{15}$$

for some $\mathbf{W} \in \mathbb{R}_n^S$ and $\mathbf{G} = \mathbf{I}_n - \mathbf{L}'\left[\mathbf{L}\mathbf{P_X}\mathbf{L}'\right]^- \mathbf{L}$. Hence,

$$\theta\left[(\mathbf{R_L} + \mathbf{P_X}) - \mathbf{I}\right] \in \{\mathbf{T} : \mathbf{T} = \mathbf{W} + (\mathbf{I} - \mathbf{P_X}\mathbf{G}\mathbf{P_X})\,\mathbf{W}\,(\mathbf{I} - \mathbf{P_X}\mathbf{G}\mathbf{P_X})\},$$

where $\mathbf{T}$ is a solution to (14). However, from (14),

$$\theta \begin{bmatrix} \mathbf{L} \\ \mathbf{P}_\mathbf{X}^\perp \end{bmatrix} [\mathbf{R_L} + \mathbf{P_X} - \mathbf{I}] \begin{bmatrix} \mathbf{L}' : & \mathbf{P}_\mathbf{X}^\perp \end{bmatrix} = \begin{bmatrix} 2\theta\mathbf{L}\mathbf{L}' & \mathbf{L}\mathbf{P}_\mathbf{X}^\perp \\ \mathbf{P}_\mathbf{X}^\perp\mathbf{L} & \mathbf{P}_\mathbf{X}^\perp \end{bmatrix} \neq \mathbf{0}.$$

Therefore, Khatri's (1981) proposed general s.n.d. solution (6) is not a general s.n.d. solution for $\mathbf{\Sigma}$ such that $F_\mathbf{\Sigma} = F_\mathbf{I}$.

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