International Journal of Statistics and Probability; Vol. 3, No. 4; 2014
ISSN 1927-7032 E-ISSN 1927-7040
Published by Canadian Center of Science and Education

Testing Bivariate Normality Based on Nonlinear Canonical Analysis

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Received: May 18, 2014 Accepted: September 11, 2014 Online Published: September 23, 2014

doi:10.5539/ijsp.v3n4p35 URL: http://dx.doi.org/10.5539/ijsp.v3n4p35

The research is financed by Agence Universitaire de la Francophonie (AUF)

Abstract

Using a test statistic constructed on wavelets-based estimation of canonical coefficients of nonlinear canonical analysis, we introduce a new class for bivariate normality test. The limit distribution of the new test statistic is established. We also give some critical values of the distribution. The finite sample performance of the proposed test, with comparison to that of an existing method, is evaluated through Monte Carlo power study.

Keywords: nonlinear canonical analysis, estimation, wavelets, test for bivariate normality

MSC 2000 mathematics subject classification: 62F05, 62H15

1. Introduction

Multivariate statistical methods for data analysis often require the assumption of normality of the underlying population. A severe departure from normality could result in unreliable statistical conclusions for models based on the normality assumption. In the univariate case, a departure from normality can usually be attributed to the skewness or kurtosis of the data being analyzed. In the multivariate case, the situation becomes much more complicated : a departure from multivariate normality could come from any direction in a multidimensional Euclidean space. Because of this fact, an existing statistic for testing multinormality can only provide partial information on the assumption. No statistic can outperform others in all aspects. Therefore, to get a relatively complete understanding of the nature and extent of violations of normality assumption with multivariate data, it is wise to use several testing statistics at the same time. This is why interest in developing normal test statistics has been continuing (see, e.g., Mardia, 1980; Csörgo, 1986; Zhu et al., 1995; Yang et al., 1996; Henze & Wagner, 1997; Liang et al., 2000; Kim & Bickel, 2003; Von Eye & Bogat, 2004; Székely & Rizzo, 2005). In this paper, we propose a new class of test for bivariate normality based on wavelets estimation of canonical coefficients of nonlinear canonical analysis (NLCA) of two random variables. The paper is organized as follows. The next section sets up the notation and some preliminary results. Section 3 is devoted to the construction of the new test statistic. In Section 4, we give the limit distribution of the test statistic defined in Section 3. Section 5 presents some power comparisons to others statistics.

2. Notations and Preliminary Results

We consider a probability space $(\Omega, \mathcal{A}, P)$ and denote by $L^2(P)$ the Hilbert space of random variables with finite second-order moment. Let X and Y be random variables defined on $(\Omega, \mathcal{A}, P)$, with values in measurable spaces $(E_X, \mathcal{T}_X)$ et $(E_Y, \mathcal{T}_Y)$ respectively, and with probability distribution measures denoted by P_X and P_Y. We denote by $L^2(P_X)$ the space of measurable real functions φ defined on E_X and such that $\mathbb{E}(\varphi^2(X)) < +\infty$, and by $L^2(P_Y)$ the analogous of $L^2(P_X)$ with respect to Y. Nonlinear canonical analysis (NLCA) of X and Y is defined by Dauxois and Pousse (1975) as the search of orthonormal bases $(\varphi_i)_{i\geq 1}$ and $(\psi_i)_{i\geq 1}$ of $L^2(P_X)$ and $L^2(P_Y)$ respectively, satisfying:

$$\mathbb{E}\left(\varphi_i(X)\psi_i(Y)\right) = \max_{(\varphi,\psi)\in W_X^{(i)}\times W_Y^{(i)}} \mathbb{E}\left(\varphi(X)\psi(Y)\right) \tag{1.1}$$

where $\mathbb{E}$ denotes the mathematical expectation, and

$$W_X^{(1)} = \{\varphi \in L^2(P_X)\,/\mathbb{E}\left(\varphi^2(X)\right) = 1\},\quad W_Y^{(1)} = \{\psi \in L^2(P_Y)\,/\mathbb{E}\left(\psi^2(Y)\right) = 1\} \tag{1.2}$$

and for $i \geq 2$:

$$W_X^{(i)} = \{\varphi \in W_X^{(1)}/\mathbb{E}\left(\varphi(X)\varphi_1(X)\right) = \cdots = \mathbb{E}\left(\varphi(X)\varphi_{i-1}(X)\right) = 0\}, \tag{1.3}$$

$$W_Y^{(i)} = \{\psi \in W_Y^{(1)}/\mathbb{E}\left(\psi(Y)\psi_1(Y)\right) = \cdots = \mathbb{E}\left(\psi(Y)\psi_{i-1}(Y)\right) = 0\}. \tag{1.4}$$

2.1 Nonlinear Canonical Analysis (NLCA) of Random Variables

Considering the subspaces

$$H_X = \{\varphi(X)\ ,\ \varphi \in L^2(P_X)\} \quad \text{and} \quad H_Y = \{\psi(Y)\ ,\ \psi \in L^2(P_Y)\}$$

of $L^2(P)$, it is known (see Dauxois & Pousse, 1975) that the solution for the NLCA problem is obtained from the spectral analysis of the self-adjoint operator $T = \mathbb{E}^X\mathbb{E}^Y_{|_{H_X}}$, that is the restriction of $T = \mathbb{E}^X\mathbb{E}^Y$ at H_X, where $\mathbb{E}^X$ and $\mathbb{E}^Y$ are the conditional expectations relative to X and Y, respectively. If T is a compact operator, NLCA exists and is characterized by a triple:

$$\{(\rho_i)_{i=0,\cdots,N},\ (\varphi_i(X))_{i=0,\cdots,N_1},\ (\psi_i(Y))_{i=0,\cdots,N_2}\},$$

where N, N_1, N_2 are elements of $\mathbb{N}\cup\{+\infty\}$. In this triple, the ρ_i's, called canonical coefficients, are real numbers contained in $]0,1]$, the systems $(\varphi_i)_{i=0,\cdots,N_1}$ and $(\psi_i)_{i=0,\cdots,N_2}$ are orthonormal bases of $L^2(P_X)$ and $L^2(P_Y)$, respectively, satisfying

$$\forall i = 0,\cdots,N,\quad T(\varphi_i(X)) = \rho_i^2\varphi_i(X) \quad \text{and} \quad \psi_i(Y) = \rho_i^{-1}\mathbb{E}^Y(\varphi_i(X)).$$

The sequence of canonical coefficients is non increasing and unique and, when T is a compact operator, one has $\lim_{i\to+\infty}\rho_i = 0$. In this paper, we suppose that T is compact and that the aforementioned sequence is strictly decreasing, that is: $\rho_i > \rho_{i+1}$ for any $i \geq 1$. These hypotheses are satisfied when (X,Y) has a bivariate normal distribution (see Dauxois & Pousse, 1975) but also for other families of bivariate distributions (see Buja, 1990). When (X,Y) has the bivariate standard normal distribution with correlation ρ, then (see, e.g., Dauxois & Pousse, 1975) the NLCA of X and Y is given by

$$\rho_i = \rho^i, \qquad \varphi_i = \psi_i = \frac{H_i}{\sqrt{i!}}, \tag{2.1}$$

where H_i are the Hermite polynomials.

2.2 Estimation of Nonlinear Canonical Analysis

Let $\{(X_i,Y_i)\}_{1\leq i\leq n}$ be an i.i.d. sample of size n, where each pair (X_i,Y_i) has the same distribution as (X,Y). The aim of this section is to remind the principle of estimation by wavelets of NLCA. Let $\{V_j^{(1)}\}_{j\in\mathbb{Z}}$ and $\{V_j^{(2)}\}_{j\in\mathbb{Z}}$ be two multiresolution analysis(see, e.g., Meyer (1990) for a definition) with fathers wavelets respective ϕ_1 and ϕ_2. Given a nondecreasing sequence $(j_n)_{n\in\mathbb{N}}$ in $\mathbb{Z}$ such that $\lim\limits_{n\to+\infty} j_n = +\infty$, we consider the estimator $\hat{f}_n$ of f defined (in Vidakovic (1999) for example) by

$$\hat{f}_n(x,y) = \frac{1}{n}\sum_{i=1}^{n} K^{(j_n)}[(X_i,Y_i)\,;(x,y)], \tag{2.2}$$

where,

$$K^{(j)}[(x_1,x_2)\,;(t_1,t_2)] = K_{1\cdot j}(x_1,t_1)\,K_{2\cdot j}(x_2,t_2)\,,$$

with, for all $(\ell, j)\in\{1,2\}\times\mathbb{Z}$ and all $(x,y)\in\mathbb{R}^2$

$$K_{\ell\cdot j}(x,y) = 2^jK_\ell\left(2^jx, 2^jy\right) \quad \text{and} \quad K_\ell(x,y) = \sum_{k\in\mathbb{Z}}\phi_\ell(x-k)\,\phi_\ell(y-k)\,.$$

We consider the estimation by wavelets of NLCA defined in Niang et al. (2012) by the family $\left(\hat{\rho}_i^{(n)}, \hat{\varphi}_i^{(n)}, \hat{\psi}_i^{(n)}\right)_{i\geq 1}$ such that, putting

$$\langle \varphi, \psi \rangle_n := \int_{\mathbb{R}^2} \varphi(x)\psi(y)\hat{f}_n(x,y)\,dxdy,$$

$$\hat{f}_{n,X}(x) := \int_{\mathbb{R}} \hat{f}_n(x,y)\,dy, \quad \hat{f}_{n,Y}(y) := \int_{\mathbb{R}} \hat{f}_n(x,y)\,dx,$$

and

$$\langle \varphi, \psi \rangle_{n,X} := \int_{\mathbb{R}} \varphi(x)\psi(x)\hat{f}_{n,X}(x)\,dx, \ \langle \varphi, \psi \rangle_{n,Y} := \int_{\mathbb{R}} \varphi(y)\psi(y)\hat{f}_{n,Y}(y)\,dy,$$

one has $\langle \hat{\varphi}_i^{(n)}, \hat{\psi}_i^{(n)} \rangle_n = \max_{(\varphi,\psi)\in \hat{W}_{n,X}^{(i)} \times \hat{W}_{n,Y}^{(i)}} \langle \varphi, \psi \rangle_n = \hat{\rho}_i^{(n)}$, with

$$\hat{W}_{n,X}^{(1)} = \{\varphi / \langle \varphi, \varphi \rangle_{n,X} = 1\}, \ \hat{W}_{n,Y}^{(1)} = \{\psi / \langle \psi, \psi \rangle_{n,Y} = 1\}$$

and for $i \geq 2$:

$$\hat{W}_{n,X}^{(i)} = \{\varphi \in \hat{W}_{n,X}^{(1)} / \langle \varphi, \hat{\varphi}_1^{(n)} \rangle_{n,X} = \cdots = \langle \varphi, \hat{\varphi}_{i-1}^{(n)} \rangle_{n,X} = 0\}, \tag{1}$$

$$\hat{W}_{n,Y}^{(i)} = \{\psi \in \hat{W}_{n,Y}^{(1)} / \langle \psi, \hat{\psi}_1^{(n)} \rangle_{n,Y} = \cdots = \langle \psi, \hat{\psi}_{i-1}^{(n)} \rangle_{n,Y} = 0\}. \tag{2}$$

Remark 1 For practical computation of the above introduced estimators, see Niang et al. (2012). Under some conditions, asymptotic properties for these estimators are established by Niang et al. (2012). Let us word in the following lemma the result that allows to obtain a limiting distribution for $\hat{\rho}_i^{(n)}$. Note that we consider, without loss of generality, the squared canonical coefficients $\lambda_i = \rho_i^2$ and their estimators $\hat{\lambda}_i^{(n)} = \left(\hat{\rho}_i^{(n)}\right)^2$.

Lemma 1 *For all $i \geq 1$, we have the convergence in distribution, as $n \to +\infty$, of the random variable $\sqrt{n}\left(\hat{\lambda}_i^{(n)} - \lambda_i\right)$ to a random variable with normal distribution $N(0, \sigma_i^2)$, where*

$$\sigma_i^2 = Var\left(2\lambda_i \varphi_i(X)\psi_i(Y)\right).$$

We can find the proof of Lemma 1 in Niang et al. (2012). This lemma will be useful for establishing the asymptotic normality for our proposal test statistic.

3. Constructing the Test Statistic

When (X, Y) have a standard normal distribution with correlation ρ and a null expectation, then

$$\sum_{i=1}^{\infty} \lambda_i = \sum_{i=1}^{\infty} \rho^{2i} = \frac{\rho^2}{1-\rho^2}.$$

Thus, for all $m \in \mathbb{N}^*$, putting $\Phi^{(m)} = \sum_{i=1}^{m} \lambda_i$, one can test the fact that (X, Y) follows the normal distribution described above considering the null hypothesis test

$$\mathcal{H}_0 : \text{``}\Phi^{(m)} = \rho^2 \frac{1-\rho^{2m+2}}{1-\rho^2}\text{''}$$

versus the alternative hypothesis

$$\mathcal{H}_1 : \text{``}\Phi^{(m)} \neq \rho^2 \frac{1-\rho^{2m+2}}{1-\rho^2}\text{''}.$$

In order to do that, wa can take as test statistic the random variable $\widehat{\Phi}_n^{(m)} = \sum_{i=1}^{m} \hat{\lambda}_i^{(n)}$, where $\hat{\lambda}_i^{(n)}$ are the wavelets estimators of $\lambda_i = \rho^{2i}$ (see section 2.2). We are now going to describe the asymptotic properties of $\widehat{\Phi}_n^{(m)}$.

4. Limiting Distribution of the Test Statistic

Under $\mathcal{H}_0$, the limiting distribution of the previously defined test statistic, is given in the following theorem.

Theorem 1 *Under hypothesis $\mathcal{H}_0$, the random variable $\sqrt{n}\left(\widehat{\Phi}_n^{(m)} - \Phi^{(m)}\right)$ converges in distribution, as $n \to +\infty$, to a random variable with normal distribution $N(0, \sigma(m)^2)$, where $\sigma(m)^2 = Var(g_m(X,Y))$, with $g_m(x,y) = \sum_{i=1}^{m} \frac{2\rho^i}{i!} H_i(x) H_i(y)$.*

Proof. This result is a consequence of Lemma 1. In fact, we have

$$\sqrt{n}\left(\widehat{\Phi}_n^{(m)} - \Phi^{(m)}\right) = \sum_{i=1}^{m} \sqrt{n}(\hat{\lambda}_i^{(n)} - \lambda_i).$$

But (see Lemma 1),

$$\sqrt{n}(\hat{\lambda}_i^{(n)} - \lambda_i) = \sqrt{n} \int_{\mathbb{R}^2} g_i(x,y)(\hat{f}_n(x,y) - f(x,y))\, dx\, dy + R_{n,i},$$

with $g_i(x,y) = 2\lambda_i^{\frac{1}{2}} \varphi_i(x)\psi_i(y)$ and $R_{n,i} = o_p(1)$, for $i = 1, \cdots, m$ so,

$$\sqrt{n}\left(\widehat{\Phi}_n^{(m)} - \Phi^{(m)}\right) = \sqrt{n} \int_{\mathbb{R}^2} \sum_{i=1}^{m} g_i(x,y)(\hat{f}_n(x,y) - f(x,y))\, dx\, dy + \sum_{i=1}^{m} R_{n,i},$$

$$\sqrt{n}\left(\widehat{\Phi}_n^{(m)} - \Phi^{(m)}\right) = \sqrt{n} \int_{\mathbb{R}^2} g_m(x,y)(\hat{f}_n(x,y) - f(x,y))\, dx\, dy + R_n^{(m)},$$

where $g_m(x,y) = \sum_{i=1}^{m} g_i(x,y)$ and $R_n^{(m)} = \sum_{i=1}^{m} R_{n,i}$. The relation (2.1) allows to write

$$g_m(x,y) = \sum_{i=1}^{m} g_i(x,y) = \sum_{i=1}^{m} \frac{2\rho^i}{i!} H_i(x) H_i(y).$$

Since, for $i = 1, \cdots, m$, $R_{n,i} = o_p(1)$ so $R_n^{(m)} = o_p(1)$, as $n \to +\infty$. It comes from Niang et al. (2012) that the random variable

$$\sqrt{n} \int_{\mathbb{R}^2} g_m(x,y)(\hat{f}_n(x,y) - f(x,y)) dxdy$$

converges in distribution, as $n \to +\infty$, to a random variable with normal distribution $N(0, \sigma(m)^2)$, where $\sigma(m)^2 = Var(g_m(X,Y))$, this yields the proof. □

5. Simulations

In this section, we illustrate the previous procedure for testing bivariate normality by applying it to various data sets. In order to assess performance on finite samples, the procedure is applied to simulated data from bivariate random variables (X, Y) with known distributions. The objective is to estimate the powers of some tests of our class and to compare these powers to those of the below three affine invariant tests for bivariate normality.

5.1 Mardia's Multivariate Kurtosis and Skewness Test

Mardia (1980) proposed a test of multivariate normality based on skewness and kurtosis. The multivariate skewness test proposed by Mardia (MARD) is based on the sample skewness statistic defined

$$m_{1,d} = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\left(X_i - \bar{X}\right)^t \hat{\Sigma}^{-1} \left(X_j - \bar{X}\right)\right)^3, \tag{3}$$

where $\hat{\Sigma} = n^{-1} \sum_{j=1}^{n} \left(X_j - \bar{X}\right)\left(X_j - \bar{X}\right)^t$ denotes the maximum likelihood estimator of population covariance and A^t is the transpose of A. Normality is rejected for large values of $m_{1,d}$.

5.2 Test of Malkovich and Afifi

The test of normality proposed by Malkovich and Afifi (1973) is a generalization of an univariate Shapiro-Wilk's test. For comparison, we also put the power of Malkovich and Afifi's (MA) generalized Shapiro-Wilk's W statistic. The Shapiro-Wilk's W statistic for testing univariate normality is

$$W(Z_1, \cdots, Z_n) = \frac{[\sum a_j(Z_{(j)} - \bar{Z})]^2}{\sum (Z_{(j)} - \bar{Z})^2} \tag{4}$$

where $Z_{(j)}$'s are the univariate order statistics of $Z_1, \cdots, Z_n$, $\bar{Z} = n^{-1} \sum Z_j$, and a_j's are the coefficients tabulated in Shapiro and Wilk (1965). The test of Malkovich and Afifi accepts the hypothesis of multivariate normality if

$$\min_c W(c^t X_1, \cdots, c^t X_n) \geq K_\omega,$$

where k_ω is a constant.

5.3 Test of Székely and Rizzo

Recently Székely and Rizzo (2005) (SR) proposed a test of multivariate normality based on Euclidean distance between sample elements. Let $X_1, \cdots, X_n$ is a random sample from a d-variate population with distribution F, and $x_1, \cdots, x_n$ are the observed values of the random sample. The statistic test proposed by Székely and Rizzo for testing H_0: $F = F_0$ vs. H_1: $F \neq F_0$ is

$$\mathcal{E}_{n,d} = n\left(\frac{2}{n}\sum_{j=1}^{n}\mathbb{E}\|x_j - X\| - \mathbb{E}\|X - X'\| - \frac{1}{n^2}\sum_{j=1}^{n}\sum_{k=1}^{n}\|x_j - x_k\|\right),$$

where X and X' are independent and identically distributed with the distribution F_0. If the hypothesized distribution is d-variate normal with mean vector μ and nonsingular covariance matrix Σ, denoted $\mathcal{N}_d(\mu, \Sigma)$, consider the transformed sample $y_j = \Sigma^{\frac{-1}{2}}\left(x_j - \mu\right)$, $j = 1, \cdots, n$. The test statistic for d-variate normality is

$$\mathcal{E}_{n,d} = n\left(\frac{2}{n}\sum_{j=1}^{n}\mathbb{E}\|y_j - Z\| - \mathbb{E}\|Z - Z'\| - \frac{1}{n^2}\sum_{j=1}^{n}\sum_{k=1}^{n}\|y_j - y_k\|\right),$$

where Z and Z' denote iid $\mathcal{N}_d(0, I)$ random variables, and I is the $d \times d$ identity matrix. A test of the simple hypothesis d-variate normality, $d \geq 1$, rejects the null hypothesis for large values of $\mathcal{E}_{n,d}$.

In the following section, we consider the above three test statistics in the bivariate case.

5.4 Simulation Results

A Monte Carlo experiment was performed to study the power of the test based on $\widehat{\Phi}_n^{(m)}$. The critical values are given in Table 1 and Table 2 for sample sizes $n = 20, 30, 50, 100$, the usual significance levels $\alpha = 0.01, 0.05$ and 0.10 and for correlation $\rho = 0.5$, $\rho = 0.8$. Each empirical percentage is based on 1000 realizations of $\widehat{\Phi}_n^{(m)}$.

Table 1. Simulated critical values t_α of statistic $\widehat{\Phi}_n^{(m)}$: $P_{\mathcal{H}_0}(|\widehat{\Phi}_n^{(m)} - \Phi^{(m)}| \geq t_\alpha) = \alpha$ for $\rho = 0.5$

n	α=0.10	α=0.05	α=0.01
20	0.634	0.733	0.859
30	0.532	0.590	0.784
50	0.419	0.543	0.668
100	0.321	0.356	0.532

Table 2. Simulated critical values t_α of statistic $\widehat{\Phi}_n^{(m)}$: $P_{\mathcal{H}_0}(|\widehat{\Phi}_n^{(m)} - \Phi^{(m)}| \geq t_\alpha) = \alpha$ for $\rho = 0.8$

n	α=0.10	α=0.05	α=0.01
20	2.947	3.059	3.929
30	2.054	3.293	3.200
50	2.984	2.068	2.928
100	2.001	3.427	3.379

Our Monte Carlo Power study for bivariate normality compared $\widehat{\Phi}_n^{(m)}$ with the three bivariate tests described above for $n = 25, 50, 100$, at significance level $\alpha = 0.05$. A thousand Monte Carlo samples were generated from each of various alternative bivariate distributions. The following notations are used. $\mathcal{N}(0, 1)$, $\mathcal{U}(0, 1)$, $exp(1)$ denote the standard normal, uniform and exponential distributions; χ_k^2 is the Chi-square distribution with k degrees of freedom; $\Gamma(a, b)$ is the Gamma distribution with density $b^{-a}\Gamma(a)^{-1}x^{a-1}exp(\frac{-x}{b})$, $x > 0$; $B(a, b)$ stands for the beta distribution with density $\frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1}$, $0 < x < 1$. The product of two independent copies of F_1 is denoted by F_1^2. $NMIX_2(a, \delta, \rho_1, \rho_2)$ is the bivariate normal mixture:

$$aBVN(0, 0, 1, 1, \rho_1) + (1 - a)BVN(\delta, \delta, 1, 1, \rho_2).$$

Table 3. Percentage of reject of bivariate normality hypothesis by the test based on $\widehat{\Phi}_n^{(m)}$, MA, $MARD$ and SR, with $\alpha = 0.05$, based on sample size n and 1000 replications

Alternative	n	$\widehat{\Phi}_n^{(m)}$	MA	MARD	SR
$\mathcal{N}(0,1)^2$	25	7	10	1	6
$\mathcal{U}(0,1)^2$	25	100	05	12	14
$\Gamma(5,1)^2$	25	91	44	14	23
$(\chi_5^2)^2$	25	77	57	39	61
$B(1,1)^2$	25	100	4	16	33
$B(1,2)^2$	25	100	22	07	45
$exp(1)^2$	25	79	66	90	98
$\mathcal{N}(0,1)^2$	50	5	9	8	6
$\mathcal{U}(0,1)^2$	50	100	11	57	81
$\Gamma(5,1)^2$	50	91	65	60	61
$(\chi_5^2)^2$	50	90	81	88	88
$B(1,1)^2$	50	100	8	83	89
$B(1,2)^2$	50	100	45	30	81
$exp(1)^2$	50	85	78	93	100
$\mathcal{N}(0,1)^2$	100	05	6	6	5
$\mathcal{U}(0,1)^2$	100	100	35	94	97
$\Gamma(5,1)^2$	100	95	90	83	91
$(\chi_5^2)^2$	100	100	100	97	100
$B(1,1)^2$	100	100	10	100	100
$B(1,2)^2$	100	100	54	92	98
$exp(1)^2$	100	98	94	100	100
$NMIX_2(0.5, 2, 0, 0)$	25	91	13	1	4
$NMIX_2(0.5, 4, 0, 0)$	25	100	18	4	5
$NMIX_2(0.79, 3, 0, 0)$	25	16	13	1	35
$NMIX_2(0.5, 3, 0, 0)$	25	100	10	2	34
$NMIX_2(0.5, 0.5, 0.9, -0.9)$	25	1	16	3	8
$NMIX_2(0.5, 2, 0, 0)$	50	79	15	5	3
$NMIX_2(0.5, 4, 0, 0)$	50	100	11	4	4
$NMIX_2(0.79, 3, 0, 0)$	50	42	11	4	50
$NMIX_2(0.5, 3, 0, 0)$	50	100	17	4	55
$NMIX_2(0.5, 0.5, 0.9, -0.9)$	50	1	11	4	3
$NMIX_2(0.5, 2, 0, 0)$	100	67	14	5	1
$NMIX_2(0.5, 4, 0, 0)$	100	100	4	4	1
$NMIX_2(0.79, 3, 0, 0)$	100	61	6	3	73
$NMIX_2(0.5, 3, 0, 0)$	100	100	27	11	44
$NMIX_2(0.5, 0.5, 0.9, -0.9)$	100	1	9	7	5

According to Table 3 we immediately note that MA, MARD and SR are generally inferior to the $\widehat{\Phi}_n^{(m)}$ statistic, specially against alternatives with shorter tailed marginal like $B(1,1)^2$, $B(1,2)^2$. However SR is superior to $\widehat{\Phi}_n^{(m)}$ against alternatives $exp(1)^2$. The conclusion that can be drawn from the power study in Table 3 is that the $\widehat{\Phi}_n^{(m)}$ test is more powerful than MARD's, MA's and SR's tests. The test statistic SR is very sensitive against exponential alternatives.

Acknowledgements

Mouhamed Amine Niang is grateful for financial support from the Agence Universitaire de la Francophonie (AUF).

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