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Modified Quick Convergent Inflow Algorithm for Solving Linear Programming Problems

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Abstract

A Modified Quick Convergent Inflow Algorithm (MQCIA) for Solving Linear Programming problems, based on variance of predicted response, is presented. The method adds a point of maximum variance to an initial design thus leading to a maximizer of the response function in a maximization problem. Similarly, a point of minimum variance is added to an initial design thus leading to a minimizer of the response function in a minimization problem. Effectiveness of the method has been demonstrated and the results show that by improving an existing experimental design, the optimizer of the response function is approached. Analytical justification for the MQCIA has also been established.

Keywords: Quick Convergent Inflow Algorithm, linear programming problems, variance of predicted response

1. Introduction

Line search algorithms have been effectively used in solving Linear Programming (LP) problems. One of such algorithms is the Quick Convergent Inflow Algorithm (QCIA) of Odiakosa and Iwundu (2013). The algorithm relies on adding the point reached by the line equation, at each iteration, to an existing design. The addition of such point(s) guarantees convergence of the algorithm to the required optimizer of the response or objective function. The algorithm compares favourably well with other line search algorithms that utilize the principles of experimental design such as the Maximum Norm Exchange Algorithm of Umoren (1999), the Quadratic Exchange Algorithm of Umoren (2002) and the Modified Super Convergent Line Series Algorithm of Etukudo and Umoren (2008). Iwundu and Hezekiah (2014) applied the QCIA to solving constrained linear programming problems on segmented regions. However, problem arises when the point reached by the line equation does not satisfy the linear inequality constraints and hence cannot be used as an admissible point of the experimental design. We propose in this work that the addition of a point of maximum variance to an initial design will lead to the maximizer of the response function in a maximization problem. Similarly, the addition of a point of minimum variance to an initial design will lead to a minimizer of the response function in a minimization problem.

The motivation for the modified Quick Convergent Inflow algorithm stems from the fact that since Linear Programming (LP) problems can be solved sequentially using experimental design principles, then by improving an existing design the optimizer of the LP problem can be approached.

Many techniques exist for constructing a better design from an existing design. They include augmentation technique and variance exchange technique both of which could rely on variance of predicted response, Atkinson and Donev (1992), Fedorov (1971). It has been established in Atkinson and Donev (1992, p. 117) that relationship often exists between the experimental design and the variance of predicted response in optimal design construction. By sequentially adding a trial at the point where the variance of prediction is a maximum, to an existing design, leads to the construction of a near-optimum design. This was the idea behind the construction of D-optimum continuous design of Wynn (1970) on an irregular geometric area and is also a fundamental principle in the construction of D-optimal designs using variance exchange algorithms.

The fundamental idea used in this work is as in Odiakosa and Iwundu (2013) but the modification is that instead of adding the point reached by the line equation, at each iteration, to an existing design, we add to an existing design a point in the design region having a maximum (or minimum) variance of prediction in a maximization (or minimization) problem. At each iteration, the determinant value of information matrix of the associated design and the value of the objective function of the optimizer are observed. The sequence terminates when no further improvement is observed. Hence for given LP problem of the standard form, we seek to obtain an optimizer of the linear objective function using the modification provided in this work.

2. Method

The method used in this work is similar to that of Odiakosa and Iwundu (2013) but draws its strength from the fact that an initial or existing design can be improved by adding to it a point in the design region having maximum (or minimum) varince of prediction in a maximization (or minimization) problem.

2.1 Algorithm

For given Linear Programming problem of the form:

$$\left.\begin{aligned} &\text{minimize (maximize) } f(x_1,\cdots,x_n) = \underset{\sim}{c}'\underset{\sim}{x} = \textstyle\sum_{i=1}^n c_i x_i \\ &\text{subject to } A\underset{\sim}{x} \le \underset{\sim}{b};\ \underset{\sim}{x} \ge 0 \end{aligned}\right\} \tag{1}$$

where $\underset{\sim}{x}$ is the vector of variables sought for, A is a matrix of known coefficients, $\underset{\sim}{c}$ and $\underset{\sim}{b}$ are vectors of known coefficients, the sequential steps that make up the Modified Quick Convergent Inflow Algorithm are;

(i) Obtain $\tilde{N}$ grid of points $\underset{\sim}{x}^{(1)}$, $\underset{\sim}{x}^{(2)}$, $\cdots$ $\underset{\sim}{x}^{(\tilde{N})}$ from the feasible region to make up the candidate set S = $\{\underset{\sim}{x}^{(1)}, \underset{\sim}{x}^{(2)}, \cdots \underset{\sim}{x}^{(\tilde{N})}\}$ from which design points will be selected into the design measure. The feasible region comprising of a continuum of points is discretized into $\tilde{N}$ grid of points following the recommendation of Hebble and Mitchell (1972).

(ii) From the $\tilde{N}$ grid of points, select an N-point (N $\le \tilde{N}$), n-variate non-singular initial design, ξ_N.

Without loss of generality we write

$$\xi_N = \begin{pmatrix} \underset{\sim}{x}^{(1)} \\ \underset{\sim}{x}^{(2)} \\ \vdots \\ \underset{\sim}{x}^{(N)} \end{pmatrix}, \qquad X_N = \begin{pmatrix} x_{11} & \dots & x_{1n} \\ x_{12} & \dots & x_{2n} \\ & \vdots & \\ x_{N1} & \dots & x_{Nn} \end{pmatrix}$$

X_N is the design matrix associated with the N-point design and the corresponding information matrix is $M_N = X_N'X_N$.

(iii) Obtain the starting point of search as the average of the initial design points

$$\underset{\sim}{\overline{x}}_N = \left(\frac{\sum x_{i1}}{N},\ \frac{\sum x_{i2}}{N},\ \dots\ \frac{\sum x_{in}}{N}\right) \equiv (\overline{x}_{1N},\ \overline{x}_{2N},\ \dots \overline{x}_{nN})$$

Since the region of search is convex, the starting point of search is a feasible point of the problem and consequently satisfies the constraints in (1).

(iv) Obtain the direction vector, d. The direction of search is ∇f which is $\underset{\sim}{c} \equiv \underset{\sim}{g}$, the function being linear in the variables.

Here $\underset{\sim}{g} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$ is the vector of coefficients of the objective function.

The normalized direction vector, d_k^* , at the k^{th} iteration is such that $d_k^{*'} d_k^* = 1$. Here k = 0.

(v) Evaluate the step-length of search. The step-length is taken as $\rho^{(0)}$ where

$$\rho^{(0)} = \min\left\{\left|\frac{\sum_{j=1}^n a_{ij}\underset{\sim}{\overline{x}}_{jN} - b_i}{u_i}\right|;\quad |u_i| \ne 0\right\};\quad i = 1,2,\dots m;\quad u_i = \sum_{j=1}^n a_{ij}\, d_j;\quad d_j = \frac{c_i}{||c||}$$

(vi) Make a move to the next point of search

$\overline{x}^{(1)} = \underline{\overline{x}}_N - \rho^{(0)}c'$ for minimization problem

and

$\overline{x}^{(1)} = \underline{\overline{x}}_N + \rho^{(0)}c'$ for maximization problem.

The objective function has the value f ($\overline{x}^{(1)}$) at this point.

(vii) At each grid point in the candidate set S = $\{\underline{x}^{(1)}, \underline{x}^{(2)}, \cdots, \underline{x}^{(\tilde{N})}\}$ evaluate the variance of predicted response, namely, $v_j^{(N)} = \underline{x}^{(j)}\ M_N^{-1}\left(\underline{x}^{(j)}\right)^T$; j = 1, 2, ..., $\tilde{N}$.

At this point we add $\underline{x}^{(N+1)}$ additional design point to ξ_N and thus form the design measure

$$\xi_{N+1} = \begin{pmatrix} \underline{x}^{(1)} \\ \underline{x}^{(2)} \\ \vdots \\ \underline{x}^{(N)} \\ \underline{x}^{(N+1)} \end{pmatrix}$$

and compute $\underline{\overline{x}}_{N+1} = [\frac{\sum x_{i1}}{N+1}, \frac{\sum x_{i2}}{N+1}, \ldots \frac{\sum x_{in}}{N+1}] \equiv (\overline{x}_{1,N+1}, \overline{x}_{2,N+1}, \ldots \overline{x}_{n,N+1})$.
In a minimization problem $\underline{x}^{(N+1)}$ is such that

$$\underline{x}^{(N+1)} M_N^{-1}\left(\underline{x}^{(N+1)}\right)^T = \min\{\underline{x}^{(j)} M_N^{-1}\left(\underline{x}^{(j)}\right)^T\}; \quad j = 1, 2, \ldots \tilde{N}.$$

Similarly, in a maximization problem $\underline{x}^{(N+1)}$ is such that

$$\underline{x}^{(N+1)} M_N^{-1}\left(\underline{x}^{(N+1)}\right)^T = \max\{\underline{x}^{(j)} M_N^{-1}\left(\underline{x}^{(j)}\right)^T\}; \quad j = 1, 2, \ldots \tilde{N}.$$

(viii) At the (k+1)st iteration, make a move to the next point of search

$\overline{x}^{(k+1)} = \underline{\overline{x}}_{N+k} - \rho^{(k)}c'$ for minimization problem

and

$\overline{x}^{(k+1)} = \underline{\overline{x}}_{N+k} + \rho^{(k)}c'$ for maximization problem.

The objective function has the value f ($\overline{x}^{(k+1)}$) at this point.

(ix) Stop at (k+1)st iteration if

f($\overline{x}^{(k+1)}$) > f($\overline{x}^{(k)}$) in minimization problem

or if

f($\overline{x}^{(k+1)}$) < f($\overline{x}^{(k)}$) in maximization problem.

(x) The required optimizer is $\overline{x}^{(k)} = \underline{x}_g^*$.

2.2 Stopping Rule

The algorithm stops if the sequence converges as proposed by Odiakosa and Iwundu (2013) or when an addition of a point to an existing design does not improve the design as measured by the value of the objective function of the optimizer at the current iteration. However, if there is no justification to terminate the search at the current iteration, the process continues.

2.3 Notation

For the purpose of comparing numerical illustrations of the MQCIA with the QCIA of Odiakosa and Iwundu (2012), the notations used in the QCIA shall be employed in section 3 as;

ξ_N^K = N-point design at the k^{th} iteration;

M$_k$ = M(ξ_N^K) = Information matrix associated with ξ_N^K;

$\underline{\overline{x}}_k^*$ = Starting point of search at the kth iteration;

d_0 = Direction vector;

d_0^* = Normalized direction vector;

ρ_{kj} = Step-length evaluated at the kth iteration using the jth constraint;

ρ_0^* = Optimal step-length;

$\underline{x}_k^*$ = The point located at the kth iteration;

v_i = The ith predictive variance.

3. Results

We consider the problem of maximizing Z = 5x$_1$ + 4x$_2$ subject to

$$6x_1 + 4x_2 = 24$$

$$x_1 + 2x_2 = 6$$

$$-x_1 + x_2 = 1$$

$$x_2 = 2$$

$$x_1, x_2 \geq 0.$$

Following the sequential steps of the algorithm we obtain the candidate set as

S = {(0,0), (1,1), (1,2), (0,1), (2,2), (4,0), (3.2,1), (3,1.5), (1,0), (2,0), (2,1), (3,0), (3,1)}

With a 2-point initial design $\xi_2^0 = \begin{pmatrix} (1, & 2) \\ (3, & 0) \end{pmatrix}$; det $M(\xi_2^0) = 9.0000$, the starting point is $\underline{\overline{x}}_0^* = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Using the model and the initial design, the design matrix is $X_0 = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$ and the information matrix is $M_0 = \begin{pmatrix} 10 & 2 \\ 2 & 4 \end{pmatrix}$.

The vector of coefficient of the objective function is $\underline{g} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$.

The direction vector is $d_0 = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$.

The normalized direction vector is $d_0^* = \begin{pmatrix} 0.7808688094 \\ 0.6246950476 \end{pmatrix}$.

The computation of the step-length is as follows;

Using the first constraint,

$$\rho_{01} = \frac{\begin{pmatrix} 6 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} - 24}{\begin{pmatrix} 6 & 4 \end{pmatrix}\begin{pmatrix} 0.7808688094 \\ 0.6246950476 \end{pmatrix}} = -1.113586824$$

Using the second constraint,

$$\rho_{02} = \frac{\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} - 6}{\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 0.7808688094 \\ 0.6246950476 \end{pmatrix}} = -0.9850960363$$

Using the third constraint,

$$\rho_{03} = \frac{\begin{pmatrix} -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} - 1}{\begin{pmatrix} -1 & 1 \end{pmatrix}\begin{pmatrix} 0.7808688094 \\ 0.6246950476 \end{pmatrix}} = 12.80624848$$

Using the fourth constraint,

$$\rho_{04} = \frac{\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} - 2}{\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}0.7808688094\\0.6246950476\end{pmatrix}} = -1.600781059$$

The optimal step-length is $\rho_0^* = 0.9850960363$

With $\underline{\overline{x}}_0^*$, d_0^* and ρ_0^* a move is made to $\underline{x}_1^* = \underline{\overline{x}}_0^* + \rho_0^* d_0^* = \begin{pmatrix}2.769230769\\1.615384615\end{pmatrix} \cong \begin{pmatrix}2.7692\\1.6154\end{pmatrix}$.

The variance of predicted response at the point $\underline{x}_1^*$ is 2.1598 and the value of the objective function at $\underline{x}_1^*$ is 20.3076.

We observe that $\underline{x}_1^*$ satisfies the constraints.

The variances of predicted response at the $\tilde{N}$ grid points

S = {(0,0), (1,1), (1,2), (0,1), (2,2), (4,0), (3.2,1), (3,1.5), (1,0), (2,0), (2,1), (3,0), (3,1)}

are respectively;

$$v_1 = 0.0000$$
$$v_2 = 0.5555$$
$$v_3 = 2.0000$$
$$v_4 = 0.5555$$
$$v_5 = 2.2222$$
$$v_6 = 3.5555$$
$$v_7 = 2.1200$$
$$v_8 = 2.2500$$
$$v_9 = 0.2222$$
$$v_{10} = 0.8888$$
$$v_{11} = 1.0000$$
$$v_{12} = 2.0000$$
$$v_{13} = 1.8888.$$

We add to the initial design, the point in S having maximun variance of predicted response and hence form a new design

$$\xi_3^1 = \begin{pmatrix}1 & 2\\3 & 0\\4 & 0\end{pmatrix}$$

$\det M(\xi_3^1) = 11.1111$.

The starting point of search at this iteration is

$$\underline{\overline{x}}_1^* = \begin{pmatrix}2.6667\\0.6667\end{pmatrix}.$$

Using the model and the new design, the design matrix is $X_1 = \begin{pmatrix}1 & 2\\3 & 0\\4 & 0\end{pmatrix}$ and the information matrix is $M_1 = \begin{pmatrix}26 & 2\\2 & 4\end{pmatrix}$.

The vector of coefficients of the objective function is $\underline{g} = \begin{pmatrix}5\\4\end{pmatrix}$.

The direction vector is $d_1 = \begin{pmatrix}5\\4\end{pmatrix}$.

The normalized direction vector is $d_1^* = \begin{pmatrix} 0.7809 \\ 0.6247 \end{pmatrix}$.

The computation of the step-length is as follows;

Using the first constraint,

$$\rho_{11} = \frac{\begin{pmatrix} 6 & 4 \end{pmatrix}\begin{pmatrix} 2.6667 \\ 0.6667 \end{pmatrix} - 24}{\begin{pmatrix} 6 & 4 \end{pmatrix}\begin{pmatrix} 0.7808688094 \\ 0.6246950476 \end{pmatrix}} = -0.7423448165$$

Using the second constraint,

$$\rho_{12} = \frac{\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 2.6667 \\ 0.6667 \end{pmatrix} - 6}{\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 0.7808688094 \\ 0.6246950476 \end{pmatrix}} = -0.9850467815$$

Using the third constraint,

$$\rho_{13} = \frac{\begin{pmatrix} -1 & 1 \end{pmatrix}\begin{pmatrix} 2.6667 \\ 0.6667 \end{pmatrix} - 1}{\begin{pmatrix} -1 & 1 \end{pmatrix}\begin{pmatrix} 0.7808688094 \\ 0.6246950476 \end{pmatrix}} = 19.20937272$$

Using the fourth constraint,

$$\rho_{14} = \frac{\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 2.6667 \\ 0.6667 \end{pmatrix} - 2}{\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 0.7808688094 \\ 0.6246950476 \end{pmatrix}} = -2.134321386$$

The optimal step-length is $\rho_1^* = 0.7423448165$.

With $\underline{\bar{x}}_1^*$, d_1^* and ρ_1^* a move is made to $\underline{x}_2^* = \underline{\bar{x}}_1^* + \rho_1^* \, d_1^* \cong \begin{pmatrix} 3.2464 \\ 1.1304 \end{pmatrix}$.

The variance of predicted response at the point $\underline{x}_2^*$ is 1.8210 and the value of the objective function at $\underline{x}_2^*$ is 20.7536.

We again observe that $\underline{x}_2^*$ satisfies the constraints.

The variances of predicted response at the $\tilde{N}$ grid points

S = {(0,0), (1,1), (1,2), (0,1), (2,2), (4,0), (3.2,1), (3,1.5), (1,0), (2,0), (2,1), (3,0), (3,1)}

are respectively;

$$v_1 = 0.0000$$
$$v_2 = 0.7800$$
$$v_3 = 3.0000$$
$$v_4 = 0.7800$$
$$v_5 = 3.1200$$
$$v_6 = 1.9200$$
$$v_7 = 1.6248$$
$$v_8 = 2.2950$$
$$v_9 = 0.1200$$
$$v_{10} = 0.4800$$
$$v_{11} = 1.0200$$
$$v_{12} = 1.0800$$

$$v_{13} = 1.5000.$$

In a similar way, we form a new design

$$\xi_4^1 = \begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 4 & 0 \\ 2 & 2 \end{pmatrix}$$

$\det M(\xi_3^1) = 12.75$.

The starting point of search at this iteration is $\underline{\overline{x}}_2^* = \begin{pmatrix} 2.5000 \\ 1.0000 \end{pmatrix}$.

Using the model and the new design, the design matrix is $X_2 = \begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 4 & 0 \\ 2 & 2 \end{pmatrix}$ and the information matrix is $M_2 = \begin{pmatrix} 30 & 6 \\ 6 & 8 \end{pmatrix}$.

The vector of coefficient of the objective function is $\underline{g} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$.

The direction vector is $d_2 = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$.

The normalized direction vector is $d_2^* = \begin{pmatrix} 0.7808688094 \\ 0.6246950476 \end{pmatrix}$.

The computation of the step-length is as follows;

Using the first constraint,

$$\rho_{21} = \frac{\begin{pmatrix} 6 & 4 \end{pmatrix}\begin{pmatrix} 2.5000 \\ 1.0000 \end{pmatrix} - 24}{\begin{pmatrix} 6 & 4 \end{pmatrix}\begin{pmatrix} 0.7808688094 \\ 0.6246950476 \end{pmatrix}} = -0.6959917649$$

Using the second constraint,

$$\rho_{22} = \frac{\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 2.5000 \\ 1.0000 \end{pmatrix} - 6}{\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 0.7808688094 \\ 0.6246950476 \end{pmatrix}} = -0.7388220272$$

Using the third constraint,

$$\rho_{23} = \frac{\begin{pmatrix} -1 & 1 \end{pmatrix}\begin{pmatrix} 2.5000 \\ 1.0000 \end{pmatrix} - 1}{\begin{pmatrix} -1 & 1 \end{pmatrix}\begin{pmatrix} 0.7808688094 \\ 0.6246950476 \end{pmatrix}} = 16.0078106$$

Using the fourth constraint,

$$\rho_{24} = \frac{\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 2.5000 \\ 1.0000 \end{pmatrix} - 2}{\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 0.7808688094 \\ 0.6246950476 \end{pmatrix}} = -1.600781059$$

The optimal step-length is $\rho_2^* = 0.6959917649$.

With $\underline{\overline{x}}_2^*$, d_2^* and ρ_2^* a move is made to $\underline{x}_3^* = \underline{\overline{x}}_2^* + \rho_2^* \, d_2^* = \begin{pmatrix} 3.043478261 \\ 1.434782609 \end{pmatrix} \cong \begin{pmatrix} 3.0435 \\ 1.4348 \end{pmatrix}$. The variance of predicted response at the point $\underline{x}_3^*$ is 1.6365 and the value of the objective function at $\underline{x}_3^*$ is 20.9567.

We again observe that $\underline{x}_3^*$ satisfies the constraints.

The variances of predicted response at the $\tilde{N}$ grid points

S = {(0,0), (1,1), (1,2), (0,1), (2,2), (4,0), (3.2,1), (3,1.5), (1,0), (2,0), (2,1), (3,0), (3,1)}
are respectively;

$$v_1 = 0.0000$$
$$v_2 = 0.5098$$
$$v_3 = 2.0392$$
$$v_4 = 0.5882$$
$$v_5 = 2.0392$$
$$v_6 = 2.5098$$
$$v_7 = 1.4415$$
$$v_8 = 1.6764$$
$$v_9 = 0.1568$$
$$v_{10} = 0.6274$$
$$v_{11} = 0.7450$$
$$v_{12} = 1.4117$$
$$v_{13} = 1.2941.$$

Again, we form a new design

$$\xi_5^1 = \begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 4 & 0 \\ 2 & 2 \\ 4 & 0 \end{pmatrix}$$

$\det M(\xi_5^1) = 13.28$.

The starting point of search at this iteration is $\underline{\overline{x}}_3^* = \begin{pmatrix} 2.8000 \\ 0.8000 \end{pmatrix}$.

Using the model and the new design, the design matrix is $X_3 = \begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 4 & 0 \\ 2 & 2 \\ 4 & 0 \end{pmatrix}$ and the information matrix is $M_3 = \begin{pmatrix} 46 & 6 \\ 6 & 8 \end{pmatrix}$.

The vector of coefficient of the objective function is $\underline{g} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$.

The direction vector is $d_3 = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$.

The normalized direction vector is $d_3^* = \begin{pmatrix} 0.7808688094 \\ 0.6246950476 \end{pmatrix}$.

The computation of the step-length is as follows;

Using the first constraint,

$$\rho_{31} = \frac{\begin{pmatrix} 6 & 4 \end{pmatrix}\begin{pmatrix} 2.8000 \\ 0.8000 \end{pmatrix} - 24}{\begin{pmatrix} 6 & 4 \end{pmatrix}\begin{pmatrix} 0.7808688094 \\ 0.6246950476 \end{pmatrix}} = -0.5567934119$$

Using the second constraint,

$$\rho_{32} = \frac{\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}2.8000\\0.8000\end{pmatrix} - 6}{\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}0.7808688094\\0.6246950476\end{pmatrix}} = -0.788076829$$

Using the third constraint,

$$\rho_{33} = \frac{\begin{pmatrix}-1 & 1\end{pmatrix}\begin{pmatrix}2.8000\\0.8000\end{pmatrix} - 1}{\begin{pmatrix}-1 & 1\end{pmatrix}\begin{pmatrix}0.7808688094\\0.6246950476\end{pmatrix}} = 19.20937272$$

Using the fourth constraint,

$$\rho_{34} = \frac{\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}2.8000\\0.8000\end{pmatrix} - 2}{\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}0.7808688094\\0.6246950476\end{pmatrix}} = -1.920937271$$

The optimal step-length is $\rho_3^* = 0.5567934119$

With $\underline{\bar{x}}_3^*$, d_3^* and ρ_3^* a move is made to $\underline{x}_4^* = \underline{\bar{x}}_3^* + \rho_3^*\, d_3^* = \begin{pmatrix}3.2348\\1.1478\end{pmatrix}$.

The variance of predicted response at the point $\underline{x}_4^*$ is 1.5024 and the value of the objective function at $\underline{x}_4^*$ is 20.7652. This value is less then the 20.9567, the value of the objective function obtained at the previous iteration, thus indicating that the search has moved away from the region of optimality. Hence, the optimizer is $\underline{x}_g^* = \begin{pmatrix}3.0435\\1.4348\end{pmatrix}$ and the optimal value of objective function is 20.9567.

Table 1 gives the summary statistics of the maximization problem using the Modified Quick Convergent Inflow Algorithm.

Table 1. Summary statistics of the iterative steps

Iterative step k	Design size N	Determinant of information matrix Det$\{M(\xi_N^k)\}$	Optimizer $\underline{x}_k^*$	Prediction variance of the optimizer d$(\underline{x}_k^*;\xi_N^k)$	Value of objective function of the optimizer f$(\underline{x}_k^*)$
0	2	9.0000	$\begin{pmatrix}2.7692\\1.6154\end{pmatrix}$	2.1598	20.3076
1	3	11.1111	$\begin{pmatrix}3.2464\\1.1304\end{pmatrix}$	1.8210	20.7536
2	4	12.7500	$\begin{pmatrix}3.0435\\1.4348\end{pmatrix}$	1.6365	20.9567
3	5	13.2800	$\begin{pmatrix}3.2348\\1.1478\end{pmatrix}$	1.5024	20.7652

4. Discussion

The maximizer of the objective function for the maximization problem considered in section 3 has been reported in Taha (2011) as $x_1 = 3.0000$ and $x_2 = 1.5000$. The corresponding value of objective function is 21.0000. The Modified Quick Convergent Inflow Algorithm locates $x_1 = 3.0435$ and $x_2 = 1.4348$ as the maximizer of the objective function with 20.9565 as the value of objective function. The norm of the difference between the exact vector of optimizer $\begin{pmatrix} 3.0000 \\ 1.5000 \end{pmatrix}$ and the approximate vector of optimizer $\begin{pmatrix} 3.0435 \\ 1.4348 \end{pmatrix}$ is 0.07837914263. The relative error of the difference is 0.02336814546 which is approximately 2.34%. These results indicate that the approximate solution obtained using the Modified Quick Convergent Inflow Algorithm is close to the exact value. The same algorithm can be used for minimization problems. For example, the algorithm when applied to the minimization of the objective function $f(x_1, x_2) = -x_1 + x_2$ subject to $\{-x_1 + 3x_2 \leq 10;\ x_1 + x_2 \leq 6;\ x_1 - x_2 \leq 2;\ x_1, x_2 \geq 0\}$ located $\underline{x}_g^* = \begin{pmatrix} 1.9999634 \\ -0.0000266 \end{pmatrix}$ as the minimizer of the objective function, when the search commenced with an initial design $\xi_2 = \begin{pmatrix} 2 & 4 \\ 2 & 0 \end{pmatrix}$. Obviously, the value -0.0000266 is due to round up error and can be taken as zero. Hence the optimizer is approximately $\underline{x}_g^* = \begin{pmatrix} 1.9999634 \\ 0.0000000 \end{pmatrix}$ and the associted value of the objective function is f($\underline{x}_g^*$) = -2.0000166. This solution is very close to the exact solution namely, $x_1 = 2.0$ and $x_2 = 0$ with the value of objective function as -2.0. The norm of the difference between the exact vector of optimizer $\begin{pmatrix} 2.0 \\ 0 \end{pmatrix}$ and the approximate vector of optimizer $\begin{pmatrix} 1.9999634 \\ 0.0000000 \end{pmatrix}$ is 3.66 $\times 10^{-5}$ and the relative error is 1.83 $\times 10^{-5}$ which is approximately 0.00183%.

Without loss of generality, the Modified Quick Convergent Inflow Algorithm offers approximate solutions to maximization as well as minimization problems in Linear Programming. In a maximization problem the function increases monotonically until it reaches the optimum value after which it changes direction. Similarly, in a minimization problem the function decreases monotonically until it reaches the optimum value after which it changes direction.

4.1 Analytical Justification for Modified Quick Convergent Inflow Algorithm (MQCIA)

In this section we present the problem and design considered, the reason for choosing added design points and the convergence of the Modified Quick Convergent Inflow Algorithm.

4.1.1 Problem and Design

Consider the problem

Optimize f($x_1, \cdots x_n$) $= \underline{c}'\underline{x} = \sum_{i=1}^{n} c_i x_i$

subject to A$\underline{x} \leq \underline{b}$; $\underline{x} \geq 0$.

Let S = {$\underline{x}^{(1)}$, $\underline{x}^{(2)}$, $\cdots$ $\underline{x}^{(\bar{N})}$} be grid points in the feasible region of the problem and let

$$\xi_N = \begin{pmatrix} \underline{x}^{(1)} \\ \vdots \\ \underline{x}^{(N)} \end{pmatrix}$$

be an N-point design measure. Then

$$X_N = \begin{pmatrix} \underline{x}^{(1)} \\ \vdots \\ \underline{x}^{(N)} \end{pmatrix}$$

is an (Nxn) design matrix obtained using ξ_N and the objective function and whose row, $\underline{x}^{(i)}$; i = 1, 2, . . . N is a (1xn) vector of support points spanned by the model parameters. The matrix $M_n = X_N' X_N$ is called the information matrix.

At the initial iteration, the predictive variance is defined by

$$v_j^{(N)} = \underline{x}^{(j)} M_N^{-1} \left(\underline{x}^{(j)}\right)^T; \quad \text{j = 1, 2, \dots N}$$

The notation $(.)^T$ represents transpose.

4.1.2 Reason for Choosing Added Design Points

Now $\underline{\overline{x}}_N = \left(\frac{\sum x_{i1}}{N}, \frac{\sum x_{i2}}{N}, \ldots \frac{\sum x_{in}}{N}\right); i = 1, 2, \ldots N$, the average of the design points, is also a feasible point of the problem since the feasible set is convex. By increasing the design points at the next iteration by one, adding say $\underline{x}^{(N+1)}$, we have the design measure

$$\xi_{N+1} = \begin{pmatrix} \underline{x}^{(1)} \\ \vdots \\ \underline{x}^{(N+1)} \end{pmatrix}.$$

The predictive variance of the jth design point in the expanded design measure is defined by

$$v_j^{(N+1)} = \underline{x}^{(j)} M_{N+1}^{-1} \left(\underline{x}^{(j)}\right)^T ; \quad \text{j} = 1, 2, \ldots, \text{N} + 1$$

where the corresponding information matrix M_{N+1} is given by

$$M_{N+1} = \begin{pmatrix} X_N \\ \underline{x}^{(N+1)} \end{pmatrix}^T \begin{pmatrix} X_N \\ \underline{x}^{(N+1)} \end{pmatrix} = X_N^T X_N + \left(\underline{x}^{(N+1)}\right)^T \left(\underline{x}^{(N+1)}\right) = M_n + \left(\underline{x}^{(N+1)}\right)^T \left(\underline{x}^{(N+1)}\right)$$

$$\begin{aligned} M_{N+1}^{-1} &= M_N^{-1} - M_N^{-1} \left(\underline{x}^{(N+1)}\right)^T \left(I + \underline{x}^{(N+1)} M_N^{-1} \left(\underline{x}^{(N+1)}\right)^T\right)^{-1} \underline{x}^{(N+1)} M_N^{-1} \\ &= M_N^{-1} - \frac{M_N^{-1} \left(\underline{x}^{(N+1)}\right)^T \underline{x}^{(N+1)} M_N^{-1}}{I + \underline{x}^{(N+1)} M_N^{-1} \left(\underline{x}^{(N+1)}\right)^T} \\ \therefore\ v_j^{(N+1)} &= \underline{x}^{(j)} M_{N+1}^{-1} \left(\underline{x}^{(j)}\right)^T = \underline{x}^{(j)} M_N^{-1} \left(\underline{x}^{(j)}\right)^T - \frac{\underline{x}^{(j)} M_N^{-1} \left(\underline{x}^{(N+1)}\right)^T \underline{x}^{(N+1)} M_N^{-1} \left(\underline{x}^{(j)}\right)^T}{I + \underline{x}^{(N+1)} M_N^{-1} \left(\underline{x}^{(N+1)}\right)^T} \\ &= \underline{x}^{(j)} M_N^{-1} \left(\underline{x}^{(j)}\right)^T - \frac{\left\{\underline{x}^{(j)} M_N^{-1} \left(\underline{x}^{(N+1)}\right)^T\right\}^2}{I + \underline{x}^{(N+1)} M_N^{-1} \left(\underline{x}^{(N+1)}\right)^T}; j = 1, 2, \cdots N + 1 \end{aligned}$$

If j = N + 1, then

$$v_{N+1}^{(N+1)} = \frac{\underline{x}^{(N+1)} M_N^{-1} \left(\underline{x}^{(N+1)}\right)^T}{1 + \underline{x}^{(N+1)} M_N^{-1} \left(\underline{x}^{(N+1)}\right)^T}$$

In general $v_j^{(N+1)} \leq \underline{x}^{(j)} M_N^{-1} \left(\underline{x}^{(j)}\right)^T$ for all j = 1, 2, $\cdots$, N+1.

Using Cauchy Schwarz inequality we see that

$$\left\{\underline{x}^{(j)} M_N^{-1} \left(\underline{x}^{(N+1)}\right)^T\right\}^2 \leq \left\{\underline{x}^{(j)} M_N^{-1} \left(\underline{x}^{(j)}\right)^T\right\} \left\{\underline{x}^{(N+1)} M_N^{-1} \left(\underline{x}^{(N+1)}\right)^T\right\}.$$

Consequently,

$$v_j^{(N+1)} \geq \underline{x}^{(j)} M_N^{-1} \left(\underline{x}^{(j)}\right)^T - \frac{\left\{\underline{x}^{(j)} M_N^{-1} \left(\underline{x}^{(j)}\right)^T\right\} \left\{\underline{x}^{(N+1)} M_N^{-1} \left(\underline{x}^{(N+1)}\right)^T\right\}}{1 + \underline{x}^{(N+1)} M_N^{-1} \left(\underline{x}^{(N+1)}\right)^T}$$

i.e $v_j^{(N+1)} \geq \frac{\underline{x}^{(j)} M_N^{-1} \left(\underline{x}^{(j)}\right)^T}{1 + \underline{x}^{(N+1)} M_N^{-1} \left(\underline{x}^{(N+1)}\right)^T}$ for all j.

This minimum value is attained by adding $\underline{x}^{(N+1)}$.

Remark

(1) We have seen that

$$\frac{\underline{x}^{(j)} M_N^{-1} \left(\underline{x}^{(j)}\right)^T}{1 + \underline{x}^{(N+1)} M_N^{-1} \left(\underline{x}^{(N+1)}\right)^T} \leq v_j^{(N+1)} \leq \underline{x}^{(j)} M_N^{-1} \left(\underline{x}^{(j)}\right)^T \text{ for all j} = 1, 2, \cdots \text{N} + 1.$$

The predictive variance $v_{N+1}^{(N+1)}$ has this minimun value. That is, additional design point will have the smaller predictive variance from the set of design points considered.

(2) $v_j^{(N+1)}$ can be made as large (small) as possible by making $\underline{x}^{(N+1)} M_N^{-1} \left(\underline{x}^{(N+1)}\right)^T$ as large (small) as possible. Consequently for a maximization (minimization) problem we will choose the additional design point $\underline{x}^{(N+1)}$ from the set of $\tilde{N}$ design points that has maximum (minimum) predictive variance.

4.1.3 Convergence

We conclude this section by showing that the algorithm converges in the feasible region. To show that the algorithm converges we observe that from

$$\overline{x}^{(k+1)} = \underline{\overline{x}}_{N+k} + d\,\rho^{(k)}, \quad f\left(\overline{x}^{(k+1)}\right) > f\left(\overline{x}^{(k)}\right)$$

implies

$$f\left(\underline{\overline{x}}_{N+k}\right) + \rho^{(k)} d > f\left(\underline{\overline{x}}_{N+k-1}\right) + \rho^{(k-1)} d$$

i.e.

$$f\left(\underline{\overline{x}}_{N+k}\right) + \rho^{(k)} f(d) > f\left(\underline{\overline{x}}_{N+k-1}\right) + \rho^{(k-1)} f(d)$$

which implies

$$f\left(\underline{\overline{x}}_{N+k}\right) - f\left(\underline{\overline{x}}_{N+k-1}\right) > f(d)\left(\rho^{(k-1)} - \rho^{(k)}\right) > 0$$

if $\rho^{(k)}$ is decreasing.

Similarly for minimization problem

$$f\left(\overline{x}^{(k+1)}\right) < f\left(\overline{x}^{(k)}\right)$$

implies

$$f\left(\underline{\overline{x}}_{N+k}\right) - \rho^{(k)} d < f\left(\underline{\overline{x}}_{N+k-1}\right) - \rho^{(k-1)} d$$

i.e.

$$f\left(\underline{\overline{x}}_{N+k}\right) - \rho^{(k)} f(d) < f\left(\underline{\overline{x}}_{N+k-1}\right) - \rho^{(k-1)} f(d)$$

which implies

$$f\left(\underline{\overline{x}}_{N+k}\right) - f\left(\underline{\overline{x}}_{N+k-1}\right) < -f(d)\left(\rho^{(k-1)} - \rho^{(k)}\right) < 0$$

if $\rho^{(k)}$ is decreasing.

So the problem of convergence of the algorithm reduces to the proof that $\rho^{(k)}$ is a decreasing function in the feasible region of the problem.

Theorem *Let* $e_i^{(N)} = \sum_{j=1}^{n} a_{ij}\,\overline{x}_{j,N} - b_i$, *1* $\le i \le m$. *Then* $\left|e_i^{(N+1)}\right| \le \left|e_i^{(N)}\right|$ *for all i = 1, 2,* $\cdots$ *m in the feasible region.*

Proof. Let $\underline{a}^{(i)}$ be the i^{th} row of A. Since $\underline{\overline{x}}_N = \frac{1}{N}\sum_{j=1}^{N} \underline{x}^{(j)} = \frac{1}{N}\,\underline{1}'_N X_N$.

We see that

$$e_i^{(N)} = \sum_{j=1}^{n} a_{ij}\overline{x}_{j,N} - b_i = \frac{1}{N} a^{(i)} X'_N \underline{1}_N.$$

Here $\underline{1}_N$ is a column vector of all ones.

$$\begin{aligned}
e_i^{(N+1)} &= \tfrac{1}{N+1} a^{(i)} X'_{N+1}\,\underline{1}_{N+1} - b_i \\
&= \tfrac{1}{N+1} a^{(i)} X'_N\,\underline{1}_N + \tfrac{1}{N+1} a^{(i)} \underline{x}^{(N+1)'} - b_i \\
&= \tfrac{N}{N+1}\left\{\tfrac{a^{(i)}}{N} X'_N\,\underline{1}_N - b_i\right\} + \tfrac{1}{N+1} a^{(i)} \underline{x}^{(N+1)'} - \left(1 - \tfrac{N}{N+1}\right) b_i \\
&= \tfrac{N}{N+1}\left\{\tfrac{a^{(i)}}{N} X'_N\,\underline{1}_N - b_i\right\} + \tfrac{1}{N+1}\left(a^{(i)} \underline{x}^{(N+1)'} - b_i\right) \\
&= \tfrac{N}{N+1} e_i^{(N)} + \tfrac{1}{N+1}\left(a^{(i)} \underline{x}^{(N+1)'} - b_i\right)
\end{aligned}$$

where

$$X'_{N+1} = \begin{pmatrix} X'_N & \underline{x}^{(N+1)'} \end{pmatrix}$$

Since $\underline{x}^{N+1}$ is in the feasible region, it satisfies $a^{(i)} \underline{x}^{(N+1)'} \le b_i$, the constraints of the problem.

$$\therefore \left|e_i^{(N+1)}\right| \le \frac{N}{N+1}\left|e_i^{(N)}\right| \le \left|e_i^{(N)}\right|$$

The following corollary is immediate. □

Corollary

$$\rho^{(k)} = \min_{1 \le i \le m} \frac{\left| e_i^{(N+1)} \right|}{\left| a^{(i)} d \right|} \; ; \quad \left| a^{(i)} d \right| \neq 0$$

is a non-increasing function of k in the feasible region.

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