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Some New Characterizations of Markov-Bernoulli Geometric Distribution Related to Random Sums

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Abstract

The Markov-Bernoulli geometric distribution is obtained when a generalization, as a Markov process, of the independent Bernoulli sequence of random variables, is introduced. In this paper, new characterizations of the Markov-Bernoulli geometric distribution, as the distribution of the summation index of randomly truncated non-negative integer valued random variables, are given in terms of moment relations of the sum and summands. The achieved results generalize the corresponding characterizations concerning the usual geometric distribution.

Keywords: Markov-Bernoulli geometric distribution, random sum, characterization, random truncation, moments, Euler differential equation

1. Introduction

Many basic counting distributions are defined on a se-quence of independent identically distributed (iid) Bernoulli random variables (rv's). Many other distributions are defined by compounding and mixing. Another way of obtaining new discrete distributions is to define the counting distributions related to some Markov chain. Assuming some dependency in the sequence of Bernoulli rv's gives an additional parameter by which the Bernoulli model could be a more realistic model in practice. Edwards (1960) proposed a generalization of the sequence of independent Bernoulli trials by considering the success probability evolves over time according to a Markov chain. In other words, let $X_1, X_2, \ldots$ be a sequence of Bernoulli rv's with the following one step transition probabilities matrix

$$\begin{array}{cc} & \begin{array}{cc} & X_{i+1} \\ 0 & 1 \end{array} \\ X_i \begin{array}{c} 0 \\ 1 \end{array} & \left[\begin{array}{cc} 1-(1-\rho)\,p & (1-\rho)\,p \\ (1-\rho)(1-p) & \rho+(1-\rho)\,p \end{array} \right] \end{array} \tag{1.1}$$

and initial distribution:

$$P(X_1 = 1) = p = 1 - P(X_1 = 0),$$

where $p \in [0,\ 1]$ and $\rho \in [0,\ 1]$. The additional parameter ρ is the correlation coefficient between X_i and X_{i+1}, and is, also, called the persistent indicator of the initial state of the system (1.1) (Wang, 1981).

The sequence $X_1, X_2, \ldots$ with the transition matrix (1.1) and the given initial distribution is called the Markov-Bernoulli model(MBM) or the Markov modulated Bernoulli process (Özekici, 1997). Numerous researchers have studied the MBM from the various aspects of probability, statistics and their applications, in particular the classical problems related to the usual Bernoulli model (Anis & Gharib, 1982; Arvidsson & Francke, 2007; Čekanavičius & Vellaisamy, 2010; Gharib & Yehia, 1987; Inal, 1987; Maillart et al., 2008; Minkova & Omey, 2011; Omey et al., 2008; Özekici, 1997; Özekici & Soyer, 2003; Pacheco et al., 2009; Pedler, 1980; Pires & Diniz, 2012; Satheesh et al., 2002; Xekalaki & Panaretos, 2004 and others). Further, due to the fact that the MBM operates in a random environment depicted by a Markov chain so that the probability of success at each trial depends on the state of the environment, this model has a wide variety of applications include, but not limited, reliability modeling (where system and components function in a randomly changing environment), non-life insurance, matching DNA-sequences, disease clustering, traffic modeling, the occupation and waiting times problems in two state Markov chains, reconstructing patterns from sample data and statistical ecology (Arvidsson & Francke, 2007; Chang & Zeiterman, 2002; John, 1971; Özekici, 1997; Özekici & Soyer, 2003; Pacheco et al., 2009; Pedler, 1980; Pires & Diniz, 2012; Switzer, 1967, 1971; Wang, 1981; Xekalaki & Panaretos, 2004).

The Markov-Bernoulli geometric (MBG) distribution has been obtained by Anis and Gharib (1982) in an earlier detailed study of MBM. If E_i; $i = 0; 1$ are the states of the system (1.1) and if N is a rv representing the number of trials necessary for the system to be in state E_1 for the first time, then the probability mass function (pmf) of the rv N is given by:

$$P(N = n) = \begin{cases} p, & n = 1 \\ (1-p)a^{-1}(1-a^{-1})^{n-2}, & n \geq 2, \end{cases} \tag{1.2}$$

where, $a = 1/[(1-\rho)p] = E(N) + \rho/(1-\rho)$.

The distribution (1.2) is the MBG distribution. It represents a generalization of the usual geometric distribution ($\rho = 0$). In the past works on the MBM some characterizations for the MBG distribution (1.2) are achieved (Minkova & Omey, 2011; Yehia & Gharib, 1993). The random sum of independent identically distributed (iid) nonnegative rv's; where the summation index is a geometric rv; is called a geometric compounding of the underlying rv's. Such a compounding mechanism is closely related to the rarefaction (or the thinning version) of a renewal process and has practical applications to the traffic theory, reliability and ecology problems involving rare events (Gnedenko & Korolev, 1996). These applications motivated many researchers to characterize such random sums, see e.g. Gharib et al. (2003), Khalil et al. (1991), Milne and Yeo (1989) and the references therein.

Let $\{X_n; n \geq 1\}$ be a sequence of iid integer-valued rv's with pmf: $p_k = P(X_1 = k) < 1$, $k = 0, 1, 2, ...$; and let $\{Y_n; n \geq 1\}$ be another sequence of iid integer-valued rv's with pmf: $q_k = P(Y_1 = k) < 1$, $k = 0, 1, 2, ...$, $\sum_{k=0}^{\infty} q_k = 1$. The two sequences $\{X_n\}$ and $\{Y_n\}$ are assumed independent. Define the integer-valued rv $N = \inf\{n \geq 1 : X_n < Y_n\}$. Clearly $N-1$ has a geometric distribution with pmf: $P(N-1 = k) = p^k(1-p)$, $k = 0, 1, 2, ...$, where $p = P(X_1 \geq Y_1)$ and $p \in (0, 1)$.

In this paper, we are interested in characterizing the random sum:

$$Z = \sum_{k=0}^{N-1} Y_k + X_N, \quad Y_0 = 0, \tag{1.3}$$

introduced by Khalil et al. (1991).

The rv Z represents the truncated sum until the moment where for the first time the process $\{Y_n; n \geq 1\}$ has greater jump than the corresponding jump of the process $\{X_n; n \geq 1\}$. The sequence $\{Y_n\}$ is called the truncating process (Gharib et al., 2014). In queuing systems with unreliable server, Z can be interpreted as the total time duration of the unreliable server until the successful finish of the service if the corresponding duration without breakdowns is previously known (Dimitrov et al., 1991). The random sum (1.3) also arises in some models such as unreliable computing systems, multiclients computer service, data transmissions, constructing data and so on, where in a random environment some undesirable events (breakdowns, interruptions, etc.) arise and lead to process interruption or to incorrect final results (Duda, 1983; Koren et al., 1986).

Throughout the paper, we write $P_U(s) = E(s^U)$; $|s| \leq 1$, to denote the probability generating function (pgf) of an integer-valued rv U, $W \sim MBG(\alpha, c)$ to denote a MBG distributed rv W with pmf:

$$P(W = k) = \begin{cases} 1-\alpha, & k = 0 \\ \alpha c^{-1}(1-c^{-1})^{k-1}, & k \geq 1, \end{cases}$$

where, $c = 1/[(1-\rho)(1-\alpha)]$, $\alpha \in (0, 1)$, $\rho \in [0, 1]$, and $U \stackrel{d}{=} V$ to denote a rv U having the same distribution as a rv V. Let $I(A)$ be the indicator of a set A and put

$$P_1(s) = E[s^{X_1} I(X_1 < Y_1)] = \sum_{k=0}^{\infty} p_k s^k \sum_{\gamma=k+1}^{\infty} q_\gamma, \tag{1.4}$$

$$Q_1(s) = E[s^{Y_1} I(X_1 \geq Y_1)] = \sum_{k=0}^{\infty} q_k s^k \sum_{\gamma=k}^{\infty} p_\gamma. \tag{1.5}$$

The pgf of Z in (1.3) is given by:

$$P_Z(s) = P_1(s)/[1 - Q_1(s)] \tag{1.6}$$

(cf. Khalil et al., 1991).

In the following, we list some results obtained by Gharib et al. (2014) which are of direct relevance to the development of the results of the present paper. Consider the random sum Z defined by (1.3).

Lemma 1.1 *Assume that* $Y_1 \sim MBG(\alpha, \rho)$*, for* $\alpha \in (0, 1)$ *and* $\rho \in [0, 1]$*, Then the pgf of* Z *is given by*

$$P_Z(s) = \frac{[(1-ts)P_{X_1}(ts)]}{[1-\{1-(1-t)P_{X_1}(ts)\}s]}, \quad |s| \le 1 \tag{1.7}$$

and

$$E(Z) = P_Z^{'}(1) = \frac{[1-P_{X_1}(t)]}{[(1-t)P_{X_1}(t)]}, \tag{1.8}$$

where $t = \rho + (1-\rho)\alpha$.

Lemma 1.2 *Assume* $X_1 \sim MBG(\beta, \rho_1)$*; for some parameters* $\beta \in (0; 1)$ *and* $\rho_1 \in [0, 1]$*, and* Y_1 *is arbitrarily non-negative integer-valued rv with* $q_0 = P(Y_1 = 0) < 1$*, then* $Z \stackrel{d}{=} X_1$.

Using the results of Lemma 1.1 and Lemma 1.2, Gharib et al. (2014) gave the following two characterizations of the MBG distribution.

Theorem 1.1 *Assume* $Y_1 \sim MBG(\alpha, \rho)$*, for some parameters* $\alpha \in (0; 1)$ *and* $\rho \in [0, 1]$*, and* X_1 *is an arbitrarily non-negative integer- valued rv with* $p_0 = P(X_1 = 0) < 1$*, Then* $Z \stackrel{d}{=} X_1$ *if and only if* $X_1 \sim MBG(\beta, \rho_1)$*, for some* $\beta \in (0; 1)$ *and* $\rho_1 \in [0, 1]$.

Theorem 1.2 *Assume* $Y_1 \sim MBG(\alpha, \rho)$*; for some* $\alpha \in (0; 1)$ *and* $\rho \in [0, 1]$ *and* X_1 *is an arbitrarily non-negative integer-valued rv with* $p_0 = P(X_1 = 0) < 1$*: Then* $X_1 \sim MBG(\beta, \rho_1)$*, for some* $\beta \in (0;1)$ *and* $\rho_1 \in [0,1]$*, if and only if* $E(Z) = \beta/(1-D)$ *where* $D = \rho_2 + (1-\rho_2)\beta$*,* $\rho_2 = \rho_1 t$ *and* $t = \rho + (1-\rho)\alpha$.

The organization of this paper is as follows. In section 2, we derive certain differential equations associated with the random sum (1.3). Also, the unique solutions of these differential equations are given under certain initial conditions. In section 3, we provide three new characterizations of the MBG distribution related to the random sum (1.3) based on certain relations between the moments of Z and/or X_1. Finally some concluding remarks are given.

2. Preliminaries

The following lemma derives the main differential equations that will be used in the proofs of the new characterizations of the MBG distribution.

Lemma 2.1 *Let* $\phi(t) = -1/P_{X_1}(t)$*,* $t = \rho + (1-\rho)\alpha$*,* $\rho \in [0, 1]$*,* $\alpha \in (0, 1)$*, and* $\mu_r = E(Z^r)$*,* $r = 1, 2, 3, 4$*. Then*

(i) $t\phi'(t) + \mu_1\phi(t) = -\frac{1}{2}(\mu_2 + \mu_1)(1-t)$.

(ii) $t^2\phi''(t) + 2\mu_1 t\phi'(t) + (\mu_2 - \mu_1)\phi(t) = -\frac{1}{3}(\mu_3 - \mu_1)(1-t)$.

(iii) $t^3\phi'''(t) + 3\mu_1 t^2\phi''(t) + 3(\mu_2 - \mu_1)t\phi'(t) + (\mu_3 - 3\mu_2 + 2\mu_1)\phi(t) = -\frac{1}{4}(\mu_4 - 2\mu_3 - \mu_2 + 2\mu_1)(1-t)$.

Proof. First, rewrite (1.7) as:

$$(1 - s[1-(1-t)P_{X_1}(ts)])P_Z(s) - (1-ts)P_{X_1}(ts) = 0, \tag{2.1}$$

where $t = \rho + (1-\rho)\alpha$, $\alpha \in (0, 1)$, $\rho \in [0, 1]$ and $|s| \le 1$.

(i) Differentiating (2.1) twice w.r.t. s, then setting $s = 1$, and using the relations $P_Z(1) = 1$, $P'_Z(1) = \mu_1$, $P''_Z(1) = \mu_2 - \mu_1$, we obtain:

$$2t[1 + (1-t)\mu_1]P_{X_1}^{'}(t) + (1-t)(\mu_1 + \mu_2)P_{X_1}(t) - 2\mu_1 = 0. \tag{2.2}$$

Now, using (1.8), we have:

$$\phi'(t) = \frac{P_{X_1}^{'}(t)}{P_{X_1}^2(t)} = [1 + (1-t)\mu_1]\frac{P_{X_1}^{'}(t)}{P_{X_1}(t)}.$$

Hence, using the last relation, Equation (2.2) reduces to

$$t\phi'(t) + \mu_1\phi(t) = -\frac{1}{2}(\mu_2 + \mu_1)(1 - t).$$

Proving (i).

(ii) Differentiating (2.1) three times w.r.t. s, then setting $s = 1$, and using the relations $P_Z(1) = 1, P'_Z(1) = \mu_1 P''_Z(1) = \mu_2 - \mu_1$, $P'''_Z(1) = \mu_3 - 3\mu_2 + 2\mu_1$, we obtain:

$$t^2[1 + (1 - t)\mu_1]P''_{X_1}(t) + t(1 - t)(\mu_2 + \mu_1)P'_{X_1}(t) - \frac{1}{3}(\mu_1 - \mu_3)(1 - t)P_{X_1}(t) - (\mu_2 - \mu_1) = 0. \tag{2.3}$$

Now, using (1.8), we have:

$$\phi''(t) = \frac{P''_{X_1}(t)}{P^2_{X_1}(t)} - 2\frac{[P'_{X_1}(t)]^2}{P^3_{X_1}(t)} = \frac{[1 + (1 - t)\mu_1]P''_{X_1}(t) - 2\phi'(t)P'_{X_1}(t)}{P_{X_1}(t)}.$$

From which we get:

$$[1 + (1 - t)\mu_1]P_{X_1}''(t) = \phi''(t)P_{X_1}(t) + 2\phi'(t)P'_{x_1}(t). \tag{2.4}$$

Also, using (i), we have:

$$(\mu_2 + \mu_1)(1 - t) = -2t\phi'(t) - 2\mu_1\phi(t). \tag{2.5}$$

Now, using (2.4) and (2.5), Equation (2.3) reduces to

$$t^2\phi''(t)P_{X_1}(t) - 2\mu_1 t\phi(t)P'_{X_1}(t) + \frac{1}{3}(\mu_3 - \mu_1)(1 - t)P_{X_1}(t) - (\mu_2 - \mu_1) = 0. \tag{2.6}$$

Finally, since $\phi(t)P'_{X_1}(t) = -\phi'(t)P_{X_1}(t)$ and $t = \rho + (1 - \rho)\alpha$, then Equation (2.6) can be written as

$$t^2\phi''(t) + 2\mu_1 t\phi'(t) + (\mu_2 - \mu_1)\phi(t) = -\frac{1}{3}(\mu_3 - \mu_1)(1 - t).$$

Proving (ii).

(iii) Proof of this part is similar to that of (i) and (ii). □

Remark 2.1 Lemma 2.1 reduces, for $\rho = 0$, to Lemma 3 of Ghitany and Gharib (2005).

Remark 2.2 The differential equations derived in Lemma 2.1 has the following general form:

$$a_k t^k \phi^{(k)}(t) + a_{k-1}t^{k-1}\phi^{(k-1)}(t) + ... + a_1 t\phi^{(1)}(t) + a_0\phi(t) = g(t),$$

where $a_k, a_{k-1}, ..., a_0$ are constants, $\phi^{(k)}(t)$ denotes the k^{th} derivative of $\phi(t)$, and $g(t)$ is a non-zero function of t. This differential equation is known as the k^{th} order nonhomogeneous Euler differential equation (Zwillinger, 1992, p. 235).

3. Characterization Results

The first characterization of the MBG distribution related to the random sum (1.3) is expressed in terms of the first two moments of Z.

Theorem 3.1 *Assume $Y_1 \sim MBG(\alpha,\rho)$, for some $\alpha \in (0; 1)$ and $\rho \in [0,1]$, and X_1 is arbitrary non-negative integer-valued rv with $p_0 = P(X_1 = 0) < 1$. Then $X_1 \sim MBG(\beta,\rho_1)$, for some $\beta \in (0; 1)$ and $\rho_1 \in [0, 1]$, if and only if*

$$\mu_2 - \mu_1(2\mu_1 + 1) = \frac{2\beta\rho_1 t}{(1 - \beta)(1 - \rho_1 t)^2} \tag{3.1}$$

where, $t = \rho + (1 - \rho)\alpha$, and $E(Z^i) = \mu_i$, $i = 1, 2$ with $\mu_2 < \infty$.

Proof. Necessity: Suppose $X_1 \sim MBG(\beta,\rho_1)$, for some $\beta \in (0; 1)$ and $\rho_1 \in [0, 1]$. Then, by using Lemma 1.2, $Z \sim MBG(\beta,\rho_2)$, for some $\beta \in (0; 1)$ and $\rho_2 = \rho_1 t$; $t = \rho + (1 - \rho)\alpha$. Now, it is straight forward to see that condition (3.1) is satisfied upon substituting

$$\mu_1 = \frac{\beta}{1 - D}, \quad \mu_2 = \frac{\mu_1(1 + D)}{1 - D},$$

where, $D = \rho_2 + (1-\rho_2)\beta$.

Sufficiency: Suppose condition (3.1) is satisfied. This and (i) of Lemma 2.1 imply that

$$t\phi^{'}(t) + \frac{\beta}{(1-\beta)(1-\rho_1 t)}\phi(t) = -\frac{\beta}{(1-\beta)^2(1-\rho_1 t)^2}(1-t), \quad t \in (0;1), \text{ and } \rho_1 \in [0,1), \tag{3.2}$$

subject to the initial condition $\phi(1) = -1$.

Now, the general solution of (3.2) is:

$$\phi(t) = C_1 t^{-\frac{\beta}{(1-\beta)}}(-1+\rho_1 t)^{\frac{\beta}{(1-\beta)}} - \frac{(1-\ell t)}{(1-\beta)(1-\rho_1 t)},$$

where C_1 is an arbitrary constant and $\ell = \rho_1 + (1-\rho_1)\beta$.

Now, using the initial condition $\phi(1) = -1$, we obtain $C_1 = 0$. Hence, the solution of (3.2) is given by:

$$\phi(t) = -\frac{(1-\ell t)}{(1-\beta)(1-\rho_1 t)}, \quad t \in (0;1), \beta \in (0;1), \text{ and } \rho_1 \in [0,1).$$

(For the uniqueness of this solution, see Zwillinger, 1992, p. 51).

Consequently, we have uniquely ($\phi(s) = -1/P_{X_1}(s)$).

$$P_{X_1}(s) = \frac{(1-\beta)(1-\rho_1 s)}{(1-\ell s)}, |s| \le 1, \beta \in (0;1), \text{ and } \rho_1 \in [0,1).$$

Therefore, $X_1 \sim MBG(\beta, \rho_1)$. This completes the proof of Theorem 3.1. □

Remark 3.1 If $\rho_1 = 0$, Theorem 3.1 reduces to Theorem 3 of Ghitany and Gharib (2005) concerning the case of geometric distribution.

The second characterization of the MBG distribution related to the random sum (1.3) is given in terms of the first three moments of Z and the first moment of X_1.

Theorem 3.2 *Assume $Y_1 \sim MBG(\alpha, \rho)$, for some $\alpha \in (0;1)$ and $\rho \in [0,1]$, and X_1 is arbitrary non-negative integer-valued rv with $p_0 = P(X_1 = 0) < 1$: Then $X_1 \sim MBG(\beta, \rho_1)$, for some $\beta \in (0;1)$, and $\rho_1 \in [0,1]$, if and only if*

$$(2\mu_3 + \mu_1)\mu_1 - 3\mu_2^2 = 0, \tag{3.3}$$

and

$$\mu_2 - \mu_1 - 2\mu_1\nu_1 = \frac{2\beta\rho_1(t-D)}{(1-\rho_1)(1-D)^2}, \tag{3.4}$$

where $D = \rho_1 t + (1-\rho_1 t)\beta$, $t = \rho + (1-\rho)\alpha$, $\alpha \in (0;1)$, $\rho \in [0,1]$, $E(X_1) = \nu_1$, and $E(Z^i) = \mu_i$, $i = 1,2,3$ with $\mu_3 < \infty$.

Proof. Necessity: Suppose $X_1 \sim MBG(\beta, \rho_1)$ for some $\beta \in (0;1)$, and $\rho_1 \in [0,1]$. Then according to Lemma 1.2, $Z \sim MBG(\beta, \rho_2)$, for some $\beta \in (0;1)$ and $\rho_2 = \rho_1 t$, $t = \rho + (1-\rho)\alpha$. Now, it is straight forward to see that each of conditions (3.3) and (3.4) is satisfied upon substituting

$$\mu_1 = \frac{\beta}{1-D}, \quad \mu_2 = \frac{\mu_1(1+D)}{1-D}, \quad \mu_3 = \frac{\mu_1(D^2+4D+1)}{(1-D)^2} \quad \nu_1 = \frac{\beta}{1-\ell},$$

where $D = \rho_1 t + (1-\rho_1 t)\beta$, $t = \rho + (1-\rho)\alpha$ and

$$\ell = \rho_1 + (1-\rho_1)\beta.$$

Sufficiency: Suppose conditions (3.3) and (3.4) are satisfied. These and (ii) of Lemma 2.1, imply that

$$t^2\phi^{''}(t) + \frac{2\beta t}{(1-D)}\phi^{'}(t) + \frac{2\beta D}{(1-D)^2}\phi(t) = -\frac{2\beta D}{(1-D)^2}(1-t), \quad t \in (0;1), \beta \in (0;1), \tag{3.5}$$

subject to the initial conditions $\phi(1) = -1$, $\phi^{'}(1) = \nu_1 = \frac{\beta}{1-\ell}$, where $\ell = \rho_1 + (1-\rho_1)\beta$, and $\rho_1 \in [0,1)$.

Now, the general solution of Equation (3.5) is given by

$$\phi(t) = t^{r_1}(-1+\rho_1 t)^{r_4} C_1 + \frac{(1-\beta)(-1+\rho_1 t)^{r_3} t^{r_2}}{\sqrt{\beta^2-6\beta+1}} C_2 - \frac{(1-\ell t)}{(1-\beta)(1-\rho_1 t)}$$

where, C_1 and C_2 are arbitrary constants, $t = \rho + (1-\rho)\alpha$, $\ell = \rho_1 + (1-\rho_1)\beta$, $\rho_1 \in [0, 1)$, $r_{1,2} = \frac{1-3\beta\pm\sqrt{\beta^2-6\beta+1}}{2(1-\beta)}$ and $r_{3,4} = \frac{1+\beta\pm\sqrt{\beta^2-6\beta+1}}{2(1-\beta)}$.

Now, using the initial conditions of Equation (3.5), we get the following two algebraic equations in C_1 and C_2

$$(-1+\rho_1)^{r_4} C_1 + \frac{(1-\beta)(-1+\rho_1)^{r_3}}{\sqrt{\beta^2-6\beta+1}} C_2 = 0,$$

and

$$(-1+\rho_1)^{r_4-1}(\rho_1 - r_1) C_1 + \frac{(1-\beta)(-1+\rho_1)^{r_3-1}(\rho_1 - r_2)}{\sqrt{\beta^2-6\beta+1}} C_2 = 0$$

Solving these two algebraic equations by the elimination method, for C_1 and C_2 , we get that $C_1 = C_2 = 0$.

Therefore, Equation (3.5) has the unique solution (for the uniqueness of the solution, see Zwillinger, 1992, p. 51):

$$\phi(t) = -\frac{(1-\ell t)}{(1-\beta)(1-\rho_1 t)}, \quad t \in (0;1), \beta \in (0;1), \text{ and } \rho_1 \in [0,1).$$

Consequently, we have uniquely ($\phi(s) = -1/P_{X_1}(s)$)

$$P_{X_1}(s) = \frac{(1-\beta)(1-\rho_1 s)}{(1-\ell s)}, \quad |s| \leq 1, \beta \in (0;1), \text{ and } \rho_1 \in [0,1).$$

Therefore, $X_1 \sim MBG(\beta, \rho_1)$. This completes the proof of Theorem 3.2. □

Remark 3.2 If $\rho_1 = 0$; Theorem 3.2 reduces to Theorem 4 of Ghitany and Gharib (2005) concerning the case of geometric distribution.

The third characterization of the Markov-Bernoulli geometric distribution related to the random sum (1.3) is expressed in terms of the first four moments of Z and the first two moments of X_1.

Theorem 3.3 *Let $Y_1 \sim MBG(\alpha, \rho)$; for some $\alpha \in (0; 1)$ and $\rho \in [0,1]$, and let X_1 be an arbitrary non-negative integer-valued rv with $p_0 = P(X_1 = 0) < 1$. Then $X_1 \sim MBG(\beta, \rho_1)$, for some $\beta \in (0; 1)$, and $\rho_1 \in [0, 1]$, if and only if*

$$3(\mu_4 - 2\mu_3 - \mu_2 + 2\mu_1)(\mu_2 - \mu_1) - 4(\mu_3 - 3\mu_2 + 2\mu_1)(\mu_3 - \mu_1) = 0, \tag{3.6}$$

$$\mu_3 - 3\mu_2 + 2\mu_1 - 3(\mu_2 - \mu_1)\nu_1 = \frac{6\beta D\rho_1(t-D)}{(1-\ell)(1-\rho_1 t)(1-D)^2}, \tag{3.7}$$

and

$$\nu_2 - \nu_1(2\nu_1 + 1) = \frac{2\beta\rho_1}{(1-\ell)(1-\rho_1 t)}, \tag{3.8}$$

where $\mu_i = E(Z^i)$, $i = 1, 2, 3, 4$, with $\mu_4 < \infty$, $\nu_j = E(X_1^j)$, $j = 1; 2$, $D = \rho_1 t + (1-\rho_1 t)\beta$, $t = \rho + (1-\rho)\alpha$ and $\ell = \rho_1 + (1-\rho_1)\beta$.

Proof. Necessity: Suppose X_1˜MBG (β, ρ_1) for some $\beta \in (0; 1)$, and $\rho_1 \in [0, 1]$. Then according to Lemma 1.2, Z ~MBG (β, ρ_2), for some $\beta \in (0; 1)$ and $\rho_2 = \rho_1\, t$, $t = \rho + (1-\rho)\alpha$. Now, it is straight forward to see that each of conditions (3.6)-(3.8) is satisfied upon substituting

$$\mu_1 = \frac{\beta}{1-D}, \quad \mu_2 = \frac{\mu_1(1+D)}{1-D}, \quad \mu_3 = \frac{\mu_1(D^2+4D+1)}{(1-D)^2}, \quad \mu_4 = \frac{\mu_1(D^3+11D^2+11D+1)}{(1-D)^3},$$

$$\nu_1 = \frac{\beta}{1-\ell}, \quad \nu_2 = \frac{\nu_1(1+\ell)}{1-\ell},$$

where, $D = \rho_1 t + (1-\rho_1 t)\beta$, $t = \rho + (1-\rho)\alpha$ and $\ell = \rho_1 + (1-\rho_1)\beta$.

Sufficiency: Suppose conditions (3.6)-(3.8) are satisfied. These and (iii) of Lemma 2.1, imply that

$$t^3\phi'''(t) + 3\mu_1 t^2\phi''(t) + 3(\mu_2 - \mu_1)t\phi'(t) + (\mu_3 - 3\mu_2 + 2\mu_1)\phi(t) = -\frac{1}{4}(\mu_4 - 2\mu_3 - \mu_2 + 2\mu_1)(1-t), \quad t \in (0;1) \quad (3.9)$$

subject to the initial conditions $\phi(1) = -1$, $\phi'(1) = \nu_1 = \frac{\beta}{1-\ell}$,

$$\phi''(1) = \nu_2 - \nu_1(2\nu_1 + 1) = \frac{2\beta\rho_1}{(1-\ell)(1-\rho_1)},$$

where $\ell = \rho_1 + (1-\rho_1)\beta$, $\rho_1 \in [0,1)$, $\mu_3 - 3\mu_2 + 2\mu_1 = E[Z(Z-1)(Z-2)] > 0$, $\mu_2 - \mu_1 > 0$ and $\mu_1 > 0$.

From conditions (3.3), (3.6) and (3.7), we obtain:

$$\mu_3 - \mu_1 = \frac{3(\mu_2-\mu_1)(\mu_2+\mu_1)}{2\mu_1}, \quad \mu_3 - 3\mu_2 + 2\mu_1 = \frac{6\beta D^2}{(1-D)^3}, \quad \mu_4 - 2\mu_3 - \mu_2 + 2\mu_1 = \frac{24\beta D^2}{(1-D)^4}.$$

Therefore, (3.9) can be rewritten as:

$$t^3\phi'''(t) + \frac{3\beta t^2}{(1-D)}\phi''(t) + \frac{6\beta D t}{(1-D)^2}\phi'(t) + \frac{6\beta D^2}{(1-D)^3}\phi(t) = -\frac{6\beta D^2}{(1-D)^4}(1-t), \quad t \in (0;1) \quad (3.10)$$

with initial conditions $\phi(1) = -1$, $\phi'(1) = \nu_1 = \frac{\beta}{1-\ell}$,

$$\phi''(1) = \nu_2 - \nu_1(2\nu_1 + 1) = \frac{2\beta\rho_1}{(1-\ell)(1-\rho_1)}.$$

Finally, Equation (3.10) with the above initial conditions has the unique solution (for the uniqueness of the solution, see Zwillinger, 1992, p. 51):

$$\phi(t) = -\frac{(1-\ell t)}{(1-\beta)(1-\rho_1 t)}, \quad t \in (0;1), \beta \in (0;1), \text{ and } \rho_1 \in [0,1).$$

Consequently, we have uniquely ($\phi(s) = -1/P_{X_1}(s)$)

$$P_{X_1}(s) = \frac{(1-\beta)(1-\rho_1 s)}{(1-\ell s)}, \quad |s| \le 1, \beta \in (0;1), \text{ and} \rho_1 \in [0,1).$$

Therefore, $X_1 \sim MBG(\beta, \rho_1)$. This completes the proof of Theorem 3.3.

Remark 3.3 If $\rho_1 = 0$; Theorem 3.3 reduces to Theorem 5 of Ghitany and Gharib (2005) concerning the case of geometric distribution.

4. Concluding Remarks

1) From the "if" part of Theorems 3.1-3.3 and according to Theorem 1.1, It follows that when X_1 has the MBG distribution then the random sum Z and the summands have distributions of the same type and in this case the summands are called N-sum stable (Satheesh et al., 2002). This result is valid, also, as a consequence of the fact that geometric random sums are stable in the same sense.

2) Each of the three nonhomogeneous differential equations used in the characterizations of the Markov-Bernoulli geometric distribution given in this paper has a linear nonhomogeneous term. The solution associated with each of these differential equations under certain initial condition(s) is essentially its particular solution.

3) We may note that the condition of Theorem 3.1 involves only the moments of Z while the conditions of Theorem 3.2 and Theorem 3.3 involve both the moments of Z and X_1: Indeed, this is due to the nature of the initial conditions used in the sufficiency part of the proofs of these theorems.

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