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Chi-Square Mixture of Transformed/Inverse Transformed Gamma Family

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Abstract

In this study, we define the chi-square mixture of transformed gamma distribution which contained some special submodels namely, the chi-square mixture of gamma, Weibull, and exponential mixture distributions. Also, the chi-square mixture of inverse transformed gamma distribution is defined and a class of submodels are deduced, that is, the chi-square mixture of inverse gamma, inverse Weibull, and inverse exponential mixture distributions. For both classes, statistical properties are investigated, that is, mean, variance, skewness and kurtosis using rth raw moment. The limiting behavior and special cases are also given to established relationships.

Keywords: mixture distribution, inverse/transformed gamma distribution, rth raw moments, skewness, kurtosis

1. Introduction

Mixtures models have continued to receive increasing attention over the years from both practical and theoretical point of view. Fields in which mixture models have been successfully applied include, but not limited to, fisheries (Fleischman & Burwen, 2003), economics (Alexander, 2004), medicine (Schlattmann, 2009), genetics (Schork et al., 1996), psychology (Ram & Grimm, 2009), palaeontology (Hunt & Chapman, 2001), archaeology (Dong, 1997), electrophoresis (Melnykov et al., 2011), sedimentology (Sylvester, 2007), geology (Coli et al., 2012), botany (Gutierrez et al., 1995), agriculture (Xu et al., 2010), zoology (Baral et al., 2013), communication theory (Yang & Zwolinski, 2011) and engineering (Liu et al., 2008).

One of the oldest known applications of mixture model was that of Karl Pearson. Pearson (1894) successfully fitted a mixture of two univariate normal densities to the crab forehead breadth data provided by Weldon who speculated the presence of two new crab subspecies in the sample. Unfortunately, Pearson's method on moments suffered some computational difficulties where it involved solving of a ninth degree polynomial equation–an undaunted task at that time.

Mixture distribution occurs naturally when the population consists of several homogeneous subpopulations. In a finite mixture distribution, the density function is a convex combination of probability density functions (pdf) of the form $p_i(x|\theta)$ which can be represented as follows:

$$m(x|\Theta) = \sum_{i=1}^{k} \pi_i p_i(x|\theta), \tag{1}$$

where $0 \leq \pi_i \leq 1$ for $i = 1, 2, \ldots, k$ and $\sum_{i=1}^{k} \pi_i = 1$. The pdf $p_i(x|\theta)$, called a *component density*, is the pdf of the ith component for $i = 1, 2, \ldots, n$ associated with π_i called a *mixing weight*. Moreover, Θ defines the set of parameter given as $\Theta = \{\pi_1, \pi_2, \ldots \pi_n, \theta_1, \theta_2, \ldots, \theta_k\}$.

Finite mixture distribution has been extensively studied (Böhning, 2007). An infinite analogue can also be formulated. If the parameter space Θ is absolutely continuous random variable having pdf $g(\theta)$, then we have a continuous mixture of densities $f(x|\theta)$ with weight function $g(\theta)$ of the following form:

$$m(x|\Theta) = \int_{\Theta} f(x|\theta) g(\theta) d(\theta). \tag{2}$$

Recently, studies on infinite mixture distributions have surfaced (Adnan & Kiser, 2012; Rezaul et al., 2011). In this article, we define the chi-square mixture distributions of transformed gamma and inverse transformed gamma distributions. This leads to a new family of chi-square mixture distribution that generalize the classical transformed/inverse transformed gamma family (Panjer, 2006). Furthermore, this study provides an analogue to similar studies done by Roy et al. (2005, 2006a, 2006b).

The rest of this article is organized as follows. Section 2 of the paper presents the mixtures of transformed Gamma family, mean, variance, skewness and kurtosis are obtained. Section 3 gives the mixtures of inverse transformed gamma family. Similary, mean, variance, skewness and kurtosis are obtained. The results and conclusions are given in Section 4.

2. Mixture of Transformed Gamma Family

This section presents the mixing of chi-square distribution with the transformed gamma family. Here, transformed gamma, Weibul, gamma and exponential distributions were considered as weight function. Relationships of the resulting mixture distributions are also shown. We have the following results.

Theorem 2.1 *The chi-square mixture of exponential function given by*

$$f(x;v,\theta)=\int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})}\frac{1}{\theta+\chi^2}e^{-\frac{x}{\theta+\chi^2}}d\chi^2, \qquad 0<x<\infty \tag{3}$$

where θ and v are positive constants is a probability density function.

Proof. Function (3) is nonnegative, since the integrand is a product of two nonnegative functions, namely the chi-square and exponential distribution, and so integrating over $[0,\infty]$ becomes nonnegative.

It is sufficient to show that $\int_{-\infty}^{\infty} f(x;v,\theta)dx = 1$. Hence,

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x;v,\theta)dx &= \int_0^\infty\int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})}\frac{1}{\theta+\chi^2}e^{-\frac{x}{\theta+\chi^2}}d\chi^2dx\\
&= \lim_{a\to\infty}\int_0^a\left[\lim_{b\to\infty}\int_0^b \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})}\frac{1}{\theta+\chi^2}e^{-\frac{x}{\theta+\chi^2}}dx\right]d\chi^2\\
&= \lim_{a\to\infty}\int_0^a \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})}\left[\lim_{b\to\infty}\int_0^b\frac{1}{\theta+\chi^2}e^{-\frac{x}{\theta+\chi^2}}dx\right]d\chi^2\\
&= 1,
\end{aligned}$$

where $\Gamma(\cdot)$ is the usual gamma function. The proof of Theorem 1 is now complete. □

Definition 2.1 A random variable X is said to have a chi-square mixture of exponential distribution with v degrees of freedom and parameter θ if its density is given by function (3) above.

We now have the following theorem.

Theorem 2.2 *Let X be a random variable which follows a chi-square mixture of exponential distribution with v degrees of freedom and parameter θ. Then the rth raw moment about the origin is given by*

$$\mu_r' = \Gamma(r+1)\int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})}\cdot(\theta+\chi^2)^r d\chi^2,$$

for $r>-1$. Furthermore, the mean is

$$\theta+v,$$

and the variance is

$$(\theta+v)^2+4v,$$

The skewness, β_1, is

$$\frac{[2\theta^3+2v^3+6\theta v^2+24v^2+6\theta^2v+24\theta v+48v]^2}{[\theta^2+2\theta v+v^2+4v]^3}.$$

and the kurtosis, β_2, is

$$\frac{9\theta^4 + 9v^4 + 36\theta v^3 + 168v^3 + 54\theta^2 v^2 + 336\theta v^2 + 864v^2 + 36\theta^3 v + 168\theta^2 v + 576\theta v + 1152v}{[\theta^2 + 2\theta v + v^2 + 4v]^2}.$$

Proof. Let X be a random variable with density of chi-square mixture of exponential distribution. The *rth* raw moment, μ_r', is given by

$$\begin{aligned}\mu_r' &= \int_0^\infty \int_0^\infty \frac{e^{-\frac{\chi^2}{2}} \chi^{2\,\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})} \cdot (\theta + \chi^2)^{-1} e^{-\frac{x}{(\theta+\chi^2)}} x^r dx d\chi^2 \\ &= \Gamma(r+1) \int_0^\infty \frac{e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})} \cdot (\theta + \chi^2)^r d\chi^2.\end{aligned} \tag{4}$$

So, if $r = 1$, the mean, μ_1', is equal to

$$\theta + v. \tag{5}$$

To solve the variance, let $r = 2$ so that from Equation (4) we have

$$\mu_2' = 2\,(\theta + v)^2 + 4v. \tag{6}$$

Therefore the variance, μ_2, is

$$(\theta + v)^2 + 4v. \tag{7}$$

In computing for skewness, β_1, given as:

$$\frac{\mu_3^2}{\mu_2^3} \tag{8}$$

we need to obtain the third central moment, μ_3, as given by the formula:

$$\mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3. \tag{9}$$

So for $r = 3$, Equation (4) yields

$$\begin{aligned}\mu_3' &= \Gamma(4) \int_0^\infty \frac{e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})} \cdot (\theta + \chi^2)^3 d\chi^2 \\ &= 3\Big(\theta^3 + 6\theta^2\left(\frac{v}{2}\right) + 12\theta\left(\frac{v}{2}\right)\left(\frac{v}{2}+1\right) + 8\left(\frac{v}{2}+2\right)\left(\frac{v}{2}+1\right)\left(\frac{v}{2}\right)\Big) \\ &= 3\theta^3 + 9\theta^2 v + 9\theta v^2 + 18\theta v + 3v^3 + 18v^2 + 24v.\end{aligned}$$

Hence,

$$\mu_3' = 3\theta^3 + 9\theta^2 v + 9\theta v^2 + 18\theta v + 3v^3 + 18v^2 + 24v. \tag{10}$$

By substituting Equation (5), (6) and (10) to Equation (9) we have

$$\mu_3 = 6v^2 + 24v + 6\theta v - \left(\theta^3 + 3\theta^2 v + 9\theta v + 7v^3\right). \tag{11}$$

Therefore, by substituting Equations (7) and (11) to Equation (8) yields the following expression for skewness:

$$\frac{\left[6v^2 + 24v + 6\theta v - \left(\theta^3 + 3\theta^2 v + 9\theta v + 7v^3\right)\right]^2}{\left(\theta^2 + 2\theta v + 3v^2 + 4v\right)^3}. \tag{12}$$

Finally, we compute for kurtosis, β_2, as given by the formula:

$$\frac{\mu_4}{\mu_2^2}. \tag{13}$$

First, we need the fourth central moment, μ_4, given by

$$\mu_4' - 4\mu_1'\mu_3' + 6\mu_2'{\mu_1'}^2 - 3{\mu_1'}^4. \tag{14}$$

We proceed by letting $r = 4$, then Equation (4) becomes

$$\mu'_4 = 4\theta^4 + 16\theta^3 v + 24\theta^2 v^2 + 48\theta^2 v + 16\theta v^3 + 96\theta v^2 + 128\theta v + 4v^4 + 48v^3 + 176v^2 + 192v. \tag{15}$$

Consequently,

$$\mu_4 = \theta^4 + 4\theta^3 v + 18\theta^2 v^2 + 28\theta v^3 - 48\theta v^2 + 32\theta v + 13v^4 + 12v^2 + 192v. \tag{16}$$

By substituting Equations (16) and (7) to Equation (13), we have the following expression for coefficient of kurtosis:

$$\frac{\theta^4 + 4\theta^3 v + 18\theta^2 v^2 + 28\theta v^3 - 48\theta v^2 + 32\theta v + 13v^4 + 12v^2 + 192v}{(\theta^2 + 2\theta v + 3v^2 + 4v)^2}.$$

The proof is now complete. □

As special case of this theorem: When $v \to 0$, the limiting distribution of chi-square mixture of exponential is the exponential distribution with parameter θ.

In the succeding results, proofs are omitted as it follow similar idea in the preceding theorem.

Theorem 2.3 *The chi-square mixture of Weibull function defined by*

$$f(x; v, \theta, \tau) = \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})} \frac{\tau(\frac{x}{\theta+\chi^2})^\tau e^{-(\frac{x}{\theta+\chi^2})^\tau}}{x} d\chi^2, \qquad 0 < x < \infty \tag{17}$$

where parameters θ, τ and v are positive real numbers is a probability density function.

Definition 2.2 A random variable X is said to have a chi-square mixture of Weibull distribution with parameters θ, τ and v if it has a density function given by (17) above.

We now have the following result.

Theorem 2.4 *Let X be a random variable which follows a chi-square mixture of Weibull distribution with parameters v, θ and τ. Then the rth raw moment about the origin is given by,*

$$\mu'_r = \mathbb{E}[X^r] = \Gamma(1 + \frac{r}{\tau}) \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})} (\theta + \chi^2)^r d\chi^2,$$

for $\tau > -r$. Furthermore the mean, μ'_1, is

$$\Gamma(1 + \frac{1}{\tau})(\theta + v)$$

variance, σ^2, is

$$\left[(\theta + v)^2 + 2v\right]\Gamma(1 + \frac{2}{\tau}) - \mu'_1$$

while the skewness is

$$\beta_1 = \frac{\left(\Gamma(1 + \frac{3}{\tau})((\theta + v)^3 + 2v(3\theta + 3v + 4)) - 3\mu'_1\sigma^2 + 2{\mu'_1}^3\right)^2}{\sigma^6}$$

and the kurtosis

$$\beta_2 = \frac{\Gamma(1 + \frac{4}{\tau})\Big((\theta + v)^4 + 12v(\theta + v)^2 + 4v(8\theta + 11v + 12)\Big) - 4{\beta_1}^{1/2}\mu'_1\sigma^3 - 6{\mu'_1}^2\sigma^2 + 6{\mu'_1}^3 + 5{\mu'_1}^4}{\sigma^4}.$$

Some special cases of chi-square mixture of Weibull distribution:

(1) When $\tau = 1$, the chi-square mixture of Weibull distribution reduces to chi-square mixture of exponential distribution with parameters v and θ.

(2) When $\tau = 1$, in addition as $v \to 0$, the limiting distribution of chi-square mixture of Weibull distribution is the exponential distribution with parameter θ.

(3) The limit of a Chi-square mixture of Weibull distribution as $v \to 0$ is the Weibull distribution with parameters τ and θ.

Theorem 2.5 *The chi-square mixture of gamma function given by*

$$f(x;v,\theta,\alpha) = \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})} \frac{(\frac{x}{\theta+\chi^2})^\alpha e^{-(\frac{x}{\theta+\chi^2})}}{x\Gamma(\alpha)} d\chi^2, \qquad 0 < x < \infty \tag{18}$$

where v, θ and α are positive real numbers is a probability density function.

Definition 2.3 A random variable X is said to have a chi-square mixture of gamma distribution with v degrees of freedom and parameters θ, α if it has a probability density function given by (18) above.

Theorem 2.6 *Let X be a random variable that follows a Chi-square mixture of gamma distribution with parameters θ, τ and $v > 0$. Then the rth raw moment about the origin is given by*

$$\mu'_r = \mathbb{E}[X^r] = \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})}(\theta+\chi^2)^r d\chi^2,$$

for $r > -\alpha$. Then the mean, μ'_1, is

$$\alpha(\theta+v),$$

variance, σ^2, is

$$\mu'_1(\theta+v) + 2v(\alpha+1)\alpha$$

and the skewness, β_1, is

$$\frac{\left((\alpha^3+3\alpha^2+2\alpha)\left[(\theta+v)^3+2v(3\theta+3v+4)\right] - 3(\alpha+1)[{\mu'_1}^2(\theta+v)+2\mu'_1\alpha v] + 2{\mu'_1}^3\right)^2}{\sigma^6}.$$

Finally the kurtosis, β_2, is equal to

$$\begin{aligned}&\left(\frac{\Gamma(\alpha+4)}{\Gamma(\alpha)}\big((\theta+v)^4+12v(\theta+v)^2+4v(8\theta+11v+12)\big)\right.\\&-4{\mu'}_1\Big(\frac{\Gamma(\alpha+3)}{\Gamma(\alpha)}((\theta+v)^3+2v(3\theta+3v+4))\Big)\\&\left.+6{\mu'_1}^2\Big(\frac{\Gamma(\alpha+2)}{\Gamma(\alpha)}\left[(\theta+v)^2+2v\right]\Big)-3{\mu'_1}^4\right)\Big/\sigma^4.\end{aligned}$$

Some special cases of chi-square mixture of gamma distribution:

(1) When $\tau = 1$, the chi-square mixture of gamma distribution reduces to chi-square mixture of exponential distribution with parameters v and θ.

(2) When $\tau = 1$, in addition as $v \to 0$, the limiting distribution of a Chi-square mixture of gamma distribution is the exponential distribution with parameter θ.

(3) The limit of Chi-square mixture of gamma distribution as $v \to 0$ is the gamma distribution with parameters α and θ.

Theorem 2.7 *The chi-square mixture of transformed gamma function given by*

$$f(x;v,\theta,\alpha,\tau) = \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})} \frac{\tau(\frac{x}{\theta+\chi^2})^{\tau\alpha} e^{-(\frac{x}{\theta+\chi^2})^\tau}}{x\Gamma(\alpha)} d\chi^2, \qquad 0 < x < \infty \tag{19}$$

where θ, τ, α, and v all are positive real numbers is a probability density function.

Definition 2.4 A random variable X is said to have a chi-square mixture of transformed gamma distribution with v degrees of freedom and parameters θ, τ, and α if its probability density function is given by function (19) above.

The following result is a generalization of the previous results.

Theorem 2.8 *Let X be a random variable that follows a chi-square mixture of transformed gamma distribution with v degrees of freedom and positive parameters α, τ and θ. Then the rth raw moment about the origin is given by*

$$\mu'_r = \mathbb{E}[X^r] = \frac{\Gamma(\alpha+\frac{r}{\tau})}{\Gamma(\alpha)} \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})}(\theta+\chi^2)^r d\chi^2$$

for $\alpha > -\frac{r}{\tau}$. Thus the mean, μ'_1, is

$$\frac{\Gamma\left(\alpha+\frac{1}{\tau}\right)}{\Gamma(\alpha)}(\theta+v)$$

while the variance, σ^2,

$$\frac{(\theta+v)^2}{\Gamma(\alpha)}\left[\Gamma(\alpha+\frac{2}{\tau})-\frac{\left[\Gamma(\alpha+\frac{1}{\tau})\right]^2}{\Gamma(\alpha)}\right]+\frac{2v\Gamma(\alpha+\frac{2}{\tau})}{\Gamma(\alpha)},$$

the skewness, β_1, is

$$\Gamma(\alpha)\Bigg(\Gamma(\alpha+\frac{3}{\tau})\left((\theta+v)^3+6v(\theta+v)+8v\right)-3\mu_1\Gamma(\alpha+\frac{2}{\tau})\left[(\theta+v)^2+2v\right]+2\mu_1^2$$
$$\Gamma(\alpha+\frac{1}{\tau})(\theta+v)\Bigg)^2\Bigg/\left((\theta+v)^2\left[\Gamma(\alpha+\frac{2}{\tau})-\frac{\left[\Gamma(\alpha+\frac{1}{\tau})\right]^2}{\Gamma(\alpha)}\right]+2v\Gamma(\alpha+\frac{2}{\tau})\right)^3$$

while the kurtosis, β_2, is

$$\Gamma(\alpha)\Bigg[\Gamma(\alpha+\frac{4}{\tau})\left((\theta+v)^4+12v(\theta+v)^2+4v(8\theta+11v+12)\right)-4\mu_1$$
$$\left(\Gamma(\alpha+\frac{3}{\tau})\left((\theta+v)^3+2v(3\theta+3v+4)\right)\right)+6\mu_1^2\left(\Gamma(\alpha+\frac{2}{\tau})\left[(\theta+v)^2+2v\right]\right)$$
$$-3\mu_1^3\Gamma(\alpha+\frac{1}{\tau})(\theta+v)\Bigg]\Bigg/\left((\theta+v)^2\left[\Gamma(\alpha+\frac{2}{\tau})-\frac{\left[\Gamma(\alpha+\frac{1}{\tau})\right]^2}{\Gamma(\alpha)}\right]+2v\Gamma(\alpha+\frac{2}{\tau})\right)^2.$$

Some special cases of chi-square mixture of transformed gamma distribution:

(1) When $\alpha = 1$, the chi-square mixture of transformed gamma distribution reduces to chi-square mixture of Weibull distribution with parameters v, τ and θ. The limit of chi-square mixture of transformed gamma distribution as $v \to 0$ is the transformed gamma distribution.

(2) When $\tau = 1$, the chi-square mixture of transformed gamma distribution reduces to Chi-square mixture of gamma distribution with parameters v, α, and θ.

(3) When $\alpha = \tau = 1$, the chi-square mixture of transformed gamma distribution reduces to chi-square mixture of exponential distribution with parameters v and θ.

3. Mixtures of Inverse Transformed Gamma Family

Proofs are omitted whenever they are similar to proofs given in last section.

Theorem 3.1 *The chi-square mixture of inverse exponential distribution function given by*

$$f(x;v,\theta)=\int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})}(\theta+\chi^2)\frac{e^{-\frac{(\theta+\chi^2)}{x}}}{x^2}d\chi^2 \qquad 0<x<\infty, \tag{20}$$

where v and θ are positive real numbers is a probability density function.

Definition 3.1 A random variable X is defined to have a chi-square mixture of inverse exponential distribution with v degrees of freedom and parameter θ if its probability density function is given by function (20) above.

Theorem 3.2 *Let X be a random variable which follows a chi-square mixture of inverse exponential distribution with v degrees of freedom and parameter θ. Then the rth raw moment about the origin is given by,*

$$\mu'_r=\Gamma(1-r)\int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})}\cdot(\theta+\chi^2)^r d\chi^2,$$

which exist for $r > 1$.

Hence, a chi-square mixture of inverse exponential has infinite mean (and higher moments), indicating a heavy tail, which is also true for its unmixed distribution, the inverse exponential distribution.

Note that the the inverse exponential distribution with parameter θ is the limiting distribution of chi-square mixture of inverse exponential distribution as $v \to 0$.

Theorem 3.3 *The chi-square mixture of inverse Weibull distribution function given by*

$$f(x; v, \theta, \tau) = \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})} \frac{\tau(\frac{\theta+\chi^2}{x})^\tau e^{-(\frac{\theta+\chi^2}{x})^\tau}}{x} d\chi^2, \qquad 0 < x < \infty \tag{21}$$

where v, θ, and τ are positive real numbers is a probability density function.

Definition 3.2 A random variable X is said to have a chi-square mixture of inverse Weibull distribution with v degrees of freedom and parameters θ and τ if its probability density function is given by function (21) above.

Theorem 3.4 *Let X be a random variable which follows a chi-square mixture of inverse Weibull distribution with v degrees of freedom and parameters τ, θ. Then the rth raw moment about the origin is given by*

$$\mu'_r = \mathbb{E}[X^r] = \Gamma(1 - \frac{r}{\tau}) \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})}(\theta + \chi^2)^r d\chi^2, \tag{22}$$

for $\tau > r$. Hence the mean is given by

$$\mu'_1 = \Gamma(1 - \frac{1}{\tau})(\theta + v) \qquad \textit{for } \tau > 1,$$

and the variance

$$\sigma^2 = (\theta + v)^2 \left[\Gamma(1 - \frac{2}{\tau}) - \left(\Gamma(1 - \frac{1}{\tau})\right)^2\right] + 2v\Gamma(1 - \frac{2}{\tau}) \qquad \textit{for } \tau > 2.$$

Furthermore the skewness, β_1, is

$$\frac{\left(\Gamma(1 - \frac{3}{\tau})\left((\theta + v)^3 + 2v(3\theta + 3v + 4)\right) - 3\mu_1\Gamma(1 - \frac{2}{\tau})\left[(\theta + v)^2 + 2v\right] + 2\mu_1^3\right)^2}{\sigma^6} \qquad \textit{for } \tau > 3$$

while the kurtosis, β_2, is

$$\begin{aligned}
&\Gamma(1 - \frac{4}{\tau})\left((\theta + v)^4 + 12v(\theta + v)^2 + 4v(8\theta + 11v + 12)\right) \\
&- 4\mu'_1\left(\Gamma(1 - \frac{3}{\tau})\left((\theta + v)^3 + 2v(3\theta + 3v + 4)\right)\right) \\
&+ 6{\mu'_1}^2\left(\Gamma(1 - \frac{2}{\tau})\left[(\theta + v)^2 + 2v\right]\right) - 3{\mu'_1}^4 \Big/ \sigma^4 \qquad \text{for } \tau > 4.
\end{aligned}$$

Some special cases of chi-square mixture of inverse Weibull distribution:

(1) When $\tau = 1$, the chi-square mixture of inverse Weibull distribution reduces to chi-square mixture of inverse exponential distribution with parameters v and θ.

(2) The limit of chi-square mixture of inverse Weibull distribution as $v \to 0$ is the inverse Wiebull distribution with parameters τ and θ.

Theorem 3.5 *The chi-square mixture of inverse gamma distribution function given by*

$$f(x; v, \alpha, \theta) = \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})} \frac{(\frac{\theta+\chi^2}{x})^\alpha e^{-(\frac{\theta+\chi^2}{x})}}{x\Gamma(\alpha)} d\chi^2, \qquad 0 < \chi^2 < \infty \tag{23}$$

where α, θ, v are positive real numbers is a probability density function.

Definition 3.3 A random variable X is said to have a chi-square mixture of inverse gamma distribution with v degrees of freedom and parameters α, θ if its probability density function is given by function (23) above.

Theorem 3.6 *Let X follows a chi-square mixture of inverse gamma distribution with parameters α, θ and v degrees of freedom. Then the rth raw moment about the origin is given by*

$$\mu'_r = \frac{\Gamma(\alpha - r)}{\Gamma(\alpha)} \int_0^\infty \frac{e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}} \Gamma(\frac{v}{2})} (\theta + \chi^2)^r d\chi^2, \qquad for\ \alpha > r.$$

Then the mean and variance are respectively given by

$$\mu'_1 = \frac{(\theta + v)}{\alpha - 1},$$

and

$$\sigma^2 = (\theta + v)^2 \Big(\frac{1 + 2v(\alpha - 1)}{(\alpha - 1)^2 (\alpha - 2)} \Big).$$

Also, the skewness and kurtosis are respectively given by

$$\beta_1 = \frac{16 \left((\theta + v)^2 + 3v(\alpha - 1) \right)^2 (\alpha - 2)}{(\alpha - 3)(\theta + v)^4 (1 + 2v(\alpha - 1))^3}$$

and

$$\beta_2 = \Bigg((\theta + v)^4 + 12v(\theta + v)^2 + 4v(8\theta + 11v + 12) - 4\mu'_1(\alpha - 4)((\theta + v)^3 + 2v(3\theta + 3v + 4))$$
$$+ 6{\mu'_1}^2 (\alpha - 3)(\alpha - 4)((\theta + v)^2 + 2v) - 3{\mu'_1}^4 (\alpha - 1)(\alpha - 2)(\alpha - 3)(\alpha - 4) \Bigg) \Bigg/ \Big((\alpha - 1)(\alpha - 2)(\alpha - 3)(\alpha - 4)\sigma^4 \Big).$$

Some special cases of chi-square mixture of inverse gamma distribution:

(1) When $\alpha = 1$, the chi-square mixture of inverse gamma distribution reduces to chi-square mixture of inverse exponential distribution with parameters v and θ.

(2) The limit of chi-square mixture of inverse gamma distribution, as $v \to 0$, is the inverse gamma distribution with parameters α and θ.

Theorem 3.7 *The chi-square mixture of inversed transformed gamma distribution function given by*

$$f(x; v, \theta, \alpha, \tau) = \int_0^\infty \frac{e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}} \Gamma(\frac{v}{2})} \frac{\tau(\frac{\theta+\chi^2}{x})^{\tau\alpha} e^{-(\frac{\theta+\chi^2}{x})^\tau}}{x\Gamma(\alpha)} d\chi^2, \qquad 0 < x < \infty \tag{24}$$

where θ, τ, α, and v all are positive real numbers is a probability density function.

Definition 3.4 A random variable X is said to have a chi-square mixture of inversed transformed gamma distribution with v degrees of freedom and parameters θ, τ, and α if its probability density function is given by function (24) above.

We have the following result.

Theorem 3.8 *Let X follows a chi-square mixture of inverse transformed gamma distribution with v degrees of freedom and parameters τ, θ and α. Then the rth raw moment about the origin is given by*

$$\mu'_r = \mathbb{E}[X^r] = \frac{\Gamma\Big(\alpha - \frac{r}{\tau}\Big)}{\Gamma(\alpha)} \int_0^\infty \frac{e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}} \Gamma(\frac{v}{2})} (\theta + \chi^2)^r d\chi^2$$

for $\alpha\tau > r$. Therefore, the mean and the variance are respectively given by

$$\mu'_1 = \frac{\Gamma\Big(\alpha - \frac{1}{\tau}\Big)}{\Gamma(\alpha)} (\theta + v) \qquad for\ \alpha > \frac{1}{\tau},$$

and

$$\sigma^2 = \frac{(\theta+v)^2}{\Gamma(\alpha)}\left[\Gamma\left(\alpha-\frac{2}{\tau}\right)-\frac{\left[\Gamma\left(\alpha-\frac{1}{\tau}\right)\right]^2}{\Gamma(\alpha)}\right]+\frac{2v\Gamma\left(\alpha-\frac{2}{\tau}\right)}{\Gamma(\alpha)} \qquad for\ \alpha > \frac{2}{\tau},$$

Similary, the skewness and the kurtosis is also respectively given by

$$\beta_1 = \frac{\left(\frac{\Gamma\left(\alpha-\frac{3}{\tau}\right)}{\Gamma(\alpha)}((\theta+v)^3+2v(3\theta+3v+4))-3\mu_1'\frac{\Gamma\left(\alpha-\frac{2}{\tau}\right)}{\Gamma(\alpha)}[(\theta+v)^2+2v]+2{\mu_1'}^3\right)^2}{\sigma^6} \qquad for\ \alpha > \frac{3}{\tau},$$

and

$$\begin{aligned}\beta_2 = & \frac{\Gamma\left(\alpha-\frac{4}{\tau}\right)}{\Gamma(\alpha)}\Big((\theta+v)^4+12v(\theta+v)^2+4v\,(8\theta+11v+12)\Big)-4\mu_1'\left(\frac{\Gamma\left(\alpha-\frac{3}{\tau}\right)}{\Gamma(\alpha)}((\theta+v)^3+2v(3\theta+3v+4))\right)\\ & +6{\mu_1'}^2\left(\frac{\Gamma\left(\alpha-\frac{2}{\tau}\right)}{\Gamma(\alpha)}\left[(\theta+v)^2+2v\right]\right)-3{\mu_1'}^4\Big/\sigma^4 \qquad \text{for } \alpha > \frac{4}{\tau}.\end{aligned} \tag{25}$$

Some special cases of chi-square mixture of inverse transformed gamma distribution:

(1) When $\alpha = 1$, the chi-square mixture of inverse transformed gamma distribution reduces to chi-square mixture of inverse Weibull distribution with parameters v, τ and θ.

(2) When $\tau = 1$, chi-square mixture of inverse transformed gamma distribution reduces to chi-square mixture of inverse gamma distribution with parameters v, α and θ.

(3) When $\alpha = \tau = 1$, chi-square mixture of inverse transformed gamma distribution reduces to chi-square mixture of inverse exponential distribution with parameters v and θ.

4. Results and Conclusions

In this article, we introduce two models of the chi-square mixture distribution, namely, the chi-square mixture of transform gamma distribution and the chi-square mixture of inverse transform gamma distribution. These models being studied are generalization of the classical transform gamma distribution and inverse transform gamma distribution.

Special submodels of the chi-square mixture of transform gamma distribution, which are deduced and defined, include the chi-square mixture of gamma, Weibull, and exponential mixture distributions. Various properties including mean, variance, skewness, kurtosis are derived. Similarly, out of chi-square mixture of inverse transform gamma distribution, special submodels are also deduced and defined. These are chi-square mixture of inverse gamma, inverse Weibull, and inverse exponential mixture distributions. Also, same statistical properties of each distributions are obtained. Finally, relationship of the two families of chi-square distribution are established via limiting behavior and some special cases.

To best summarized the results, the following figures are helpful for readers understanding. The figure below is for the class of chi-square mixture of transformed gamma distribution.

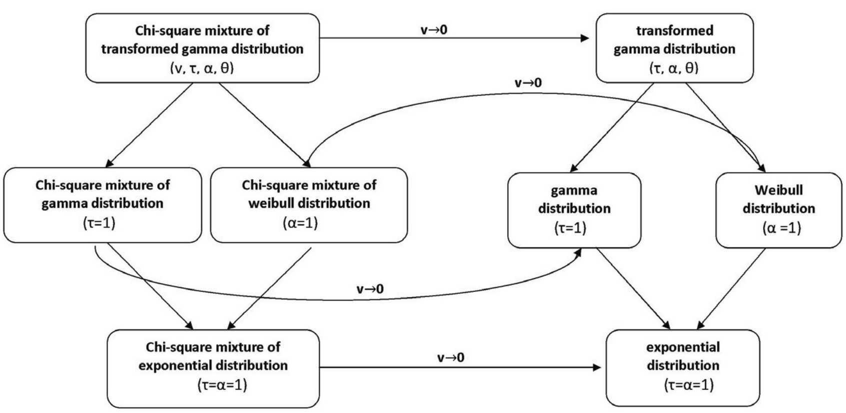

Figure 1. Chi-square mixture of transformed gamma distribution

Similarly, for the class of chi-square mixture of inverse transformed gamma distribution, we have the figure below.

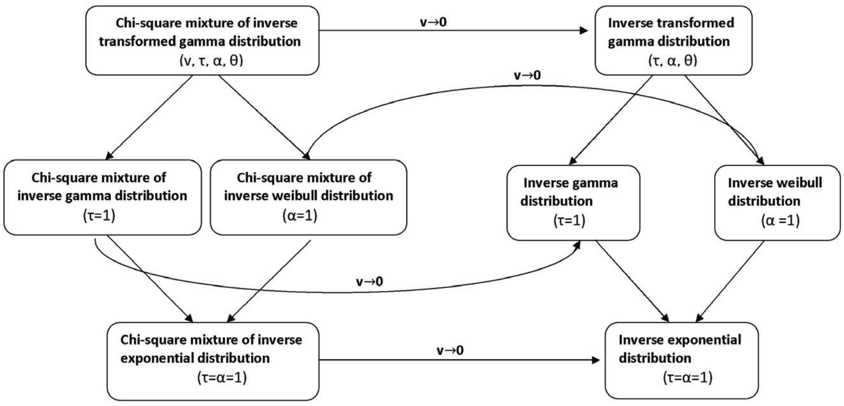

Figure 2. Chi-square mixture of inverse transformed gamma distribution

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