International Journal of Statistics and Probability; Vol. 2, No. 3; 2013
ISSN 1927-7032 E-ISSN 1927-7040
Published by Canadian Center of Science and Education

A Central Limit Theorem for a Nonparametric Maximum Conditional Hazard Rate Estimator in Presence of Right Censoring

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Received: May 5, 2013 Accepted: June 3, 2013 Online Published: July 15, 2013

doi:10.5539/ijsp.v2n3p110 URL: http://dx.doi.org/10.5539/ijsp.v2n3p110

The research is financed by the government of Togo

Abstract

In this paper, we consider a non-parametric kernel type estimator of the time where a hazard rate function is maximum in the presence of covariate and right censoring. Via a strong representation of the estimator, we establish weak convergence and asymptotic normality results.

Keywords: conditional hazard rate, maximum Conditional hazard, non-parametric estimation, strong representation, right censoring

1. Introduction

The estimation of the hazard rate and the related topics are important subjects in statistics because of the variety of their applications.Those subjects may be considered in several manners according to the data and there are many techniques used in the literature to estimate the hazard functions. In this paper we focus on the investigation of the maximum hazard rate with covariate. More precisely let T be a life time, Z a covariate and C a right censoring variable independent of T conditionally on Z. Assume that T, Z, and C are continuous and denote by $F(t|z)$ (resp. $G(t|z)$) the conditional distribution function of T (resp. C) given $Z = z$, $f(t|z)$ the conditional probability density of T and $f(z)$ the marginal density function of Z. Define $X = \min(T, C)$ and $\delta = I(T \leq C)$ where $I(A)$ is the indicator function of a Borel set A.

The conditional hazard rate function $\lambda(t|z)$ of T given $Z = z$ is defined by

$$\lambda(t|z) = \lim_{\Delta t \to 0} \frac{\mathbb{P}[T \leq t + \Delta t | T > t, Z = z]}{\Delta t} = \frac{f(t|z)}{1 - F(t|z)}, \quad F(t|z) \neq 1.$$

This function is very useful in statistical applications such as in survival analysis, medical follow up, industrial reliability studies or in earthquake studies. In this setting, knowing how to estimate the maximum of the instantaneous risk allows to predict the maximum risk when a new seismic series occurs and the knowledge of the maximum may arise when exploring relationship between responses and potential covariates. Denote by θ the time in an interval $[a_z, b_z]$ of $\mathbb{R}^+$ corresponding to the maximum of the conditional hazard rate function, that is,

$$\theta(z) = \mathrm{Argmax}_{a_z \leq t \leq b_z} \lambda(t|z). \tag{1}$$

Non-parametric estimation of the hazard rate function was first introduced in the statistical literature by Watson and Leadbetter (1964a, 1964b). The topic was developed by other authors among which Singpurwalla and Wong (1983), Tanner and Wong (1983) and Gneyou (1991). The conditional case was considered later by Van Keilegom and Veraverbeke (1997) and (2001).

The problem of estimating the maximum of a conditional hazard rate function is somewhat similar to the problem of estimating the conditional mode of a random variable. The methods employed here are inspired by the methods used to the treatment of the latter problem which has received much attention during the last twenty five years. For the references, see e.g. Collomb et al. (1987), Samanta and Thavaneswaran (1990), Ould-Saïd (1993), Quintela-del-Río and Vieu (1997), Louani and Ould-Saïd (1999), Ferraty et al. (2005), Dabo-Niang and Laksaci (2007) and Ezzahrioui and Ould Saïd (2008) for uncensored models. In censored data case, see e.g. Ould Saïd and Cai (2005) and Khardani et al. (2010, 2011).

Concerning the maximum hazard rate estimation, Quintela-del-Río (2006) considered a non-parametric estimator under dependence conditions in uncensored case. Gneyou (2012) considered a kernel-type estimator in the model of right censored data with covariate and establish strong uniform consistency results. The aim of this paper is to extend these results to the weak convergence. The paper is organized as follow. In Section 2 we recall the definitions of the non-parametric estimator of the conditional hazard rate function $\lambda(t|z)$ and the corresponding estimator $\theta_n(z)$ of its maximum value $\theta(z)$ as given in Gneyou (2012) and state the assumptions under which the results will be obtained. In Section 3 we establish a basic almost sure asymptotic representation for the estimator $\theta_n(z)$ which leads to some main results such as weak convergence and asymptotic normality. Detailed proofs are given in the appendix.

2. Definitions and Assumptions

Let $(T_i, Z_i, C_i)_{i=1}^n$ be a sample of size n of the random variables (T, Z, C). As it is often the case in clinical trials or industrial life tests, the lifetimes $T_1, T_2, \cdots$ are not completely observable due to the presence of right-censored variables. In presence of covariates Z_i and right-censoring C_i, the observable data consist of n observations $(X_i, \delta_i, Z_i)_{i=1}^n$ with $\delta_i = I(T_i \leq C_i)$, $i = 1, \cdots, n$.

2.1 Definitions

Denote by

$$\Lambda(t|z) = \int_0^t \lambda(s|z)ds \tag{2}$$

the conditional cumulative hazard function of T given $Z = z$ and define $H(t|z) = \mathbb{P}[X \leq t|Z = z]$ the conditional distribution function of X given $Z = z$ and $H_1(t|z) = \mathbb{P}[X \leq t, \delta = 1|Z = z] = \int_0^t (1 - G(s|z))dF(s|z)$ the conditional sub-distribution function of the uncensored observation $(X, \delta = 1)$ given $Z = z$. Since it is assumed that T and C are independent conditionally on Z, $\Lambda(t|z)$ can be written in the form

$$\Lambda(t|z) = \int_0^t \frac{dH_1(s|z)}{1 - H(s^-|z)}. \tag{3}$$

Hence, non-parametric kernel-type estimators of the functions $\Lambda(t|z)$, $\lambda(t|z)$ and $\theta(z)$ are respectively given by

$$\Lambda_n(t|z) = \int_0^t \frac{dH_{1n}(s|z)}{1 - H_n(s^-|z)}, \tag{4}$$

$$\lambda_n(t|z) = \sum_{i=1}^n \frac{W_i(h_n, z)\delta_i N_{a_n}(t - X_i)}{\sum_{j=1}^n W_j(h_n, z)I(X_j > X_i)} \tag{5}$$

and

$$\theta_n(z) = \mathrm{Argmax}_{a_z \leq t \leq b_z} \lambda_n(t|z) \tag{6}$$

where

$$H_n(t|z) = \sum_{i=1}^n \hat{W}_i(h_n, z)I(X_i \leq t) \quad \text{and} \quad H_{1n}(t|z) = \sum_{i=1}^n W_i(h_n, z)I(X_i \leq t, \delta_i = 1),$$

$W_i(h_n, z) = \dfrac{K_{h_n}(z - Z_i)}{\sum_{i=1}^n K_{h_n}(z - Z_i)}$, $i = 1, \cdots, n$ are Nadaraya-Watson weights associated to a kernel K on $\mathbb{R}^d$, N is a kernel on $\mathbb{R}$, (h_n) and (a_n), $(n \in \mathbb{N})$ are two sequences of positive non increasing real numbers and where K_h and N_a are defined by $K_h(x) = \frac{1}{h^d}K(\frac{x}{h})$, $N_a(s) = \frac{1}{a}N(\frac{s}{a})$, for all $x \in \mathbb{R}^d$, $h > 0$, $s \in \mathbb{R}$ and $a > 0$.

Note that $H_n(t|z)$ and $H_{1n}(t|z)$ are kernel estimators of $H(t|z)$ and $H_1(t|z)$ respectively obtained by regression.

Let $\tau_z = \sup\{t \in \mathbb{R}^+/F(t|z) < 1\}$. In applications, τ_z is typically not know in advance, but may be chosen such that the size of the observed risk set is sufficiently large.

For later use, introduce the Fourier transforms

$$\begin{aligned}
k(u) &= \int_{\mathbb{R}} e^{iux} N(x)dx \\
\varphi(u|z) &= \int_{\mathbb{R}} e^{iux} d\Lambda(x|z) = \int_{\mathbb{R}} e^{iux} \lambda(x|z)dx \\
\varphi_n(u|z) &= \int_{\mathbb{R}} e^{iux} d\Lambda_n(x|z) = \int_{\mathbb{R}} e^{iux} \frac{dH_{1n}(x|z)}{1 - H_n(x|z)}, \quad u \in \mathbb{R}
\end{aligned}$$

and for a general conditional (sub-distribution) function $L(t|z)$, $t \in \mathbb{R}^+$, $z \in \mathbb{R}^d$, denote by $L_z(t)$ the function $t \mapsto L(t|z)$; $L_z^{'}(t), L_z^{''}(t)$, its first and second derivatives with respect (w.r.) to t and $L^{(i,j)}(t|z) = \dfrac{\partial^{i+j} L(t|z)}{\partial t^i \partial z^j}$ its derivative of order $i + j$ w.r. to t and z, for all $(i, j) \in \mathbb{N}^2$, whenever all those derivatives exist. For a sequence of conditionals $L_n(t|z)$ denote by $L_n^z(t)$, $L_n'^z(t)$, $L_n''^z(t)$ for $L_n(t|z)$, $\frac{\partial}{\partial t} L_n(t|z)$, $\frac{\partial^2}{\partial t^2} L_n(t|z)$ respectively.

2.2 Assumptions

The following assumptions are needed throughout the proofs of the main results:

(F1.) (i) the r.v. Z takes values in a compact subset Δ of $\mathbb{R}^d$ and the variables T and C are independent conditionally on Z;

(ii) the marginal density function f of Z is a continuous function with bounded derivative at each $z \in \Delta$.

(F2.) There exists an interval $[a_z, b_z] \subset [0, \tau_z]$ containing a unique $\theta = \theta(z)$ satisfying $\lambda_z(\theta) = \max_{a_z \le t \le b_z} \lambda_z(t)$.

(F3.) The function $t \mapsto \lambda_z(t)$ is of class C^2 with respect to t such that

(i) $\lambda_z'(\theta) = 0$;

(ii) $d_z = \inf_{a_z \le t \le b_z} | \lambda_z''(t) | > 0$.

(F4.) There exists a positive constant η such that $\inf_{z \in \Delta}(1 - H(\tau_z|z)) \ge \eta$.

(F5.) The conditional sub-distribution functions $(t, z) \mapsto H(t|z)$ and $(t, z) \mapsto H_1(t|z)$ are of class C^2 and their first and second partial derivatives are continuous in $(t, s) \in [a_z, b_z] \times \Delta$ and are uniformly bounded.

(K1.) K is a symmetric Kernel of bounded variation on $\mathbb{R}^d$ with compact support satisfying

(i) $\displaystyle\int_{\mathbb{R}^d} K(z)dz = 1$,

(ii) $\displaystyle\int_{\mathbb{R}^d} z_j K(z)dz = 0, \forall j = 1, \cdots d$,

(iii) $\displaystyle\int_{\mathbb{R}^d} ||z||^2 K(z)dz = \alpha(K) > 0$ where $||z||$ is any norm on $\mathbb{R}^d$.

(K2.) N is a symmetric Kernel of bounded variation on $\mathbb{R}$ vanishing outside the interval $[-M, +M]$ for some $M > 0$ and satisfying

(i) $\displaystyle\int_{\mathbb{R}} N(u)du = 1$,

(ii) $\displaystyle\int_{\mathbb{R}} uN(u)du = 0$,

(iii) $\displaystyle\int_{\mathbb{R}} u^2 N(u)du = \alpha(N) > 0$,

(iv) N is two times derivable, the derivative N' is of bounded variation and satisfies $\int_{\mathbb{R}} N'^2(u)du < +\infty$.

(H.) $(h_n)_{n\in\mathbb{N}}$ and $(a_n)_{n\in\mathbb{N}}$ are two non increasing sequences of positive real numbers such that as $n \to +\infty$,

(H1.) (i) $h_n \to 0$, (ii) $nh_n^d \to +\infty$, (iii) $\dfrac{nh_n^d}{\left|\log h_n\right|} \to +\infty$.

(H2.) (i) $a_n \to 0$, (ii) $a_n^{-1/2} h_n^d \to 0$, (iii) $\dfrac{n a_n^2 h_n^d}{|\log h_n|} \to +\infty$, (iv) $\dfrac{n a_n^{4/3} h_n^d}{\log n} \to +\infty$.

(H3.) (i) $n a_n h_n^d \to +\infty$, (ii) $n a_n^6 h_n^d \to +\infty$.

(KH) $k(u)$ is absolutely integrable and $\limsup_{n\to+\infty} \int_{\mathbb{R}} u^2 \left|k(a_n u)\varphi_n^z(u)\right| du < +\infty$.

The assumptions $(F1)-(F5)$, $(K1)-(K2)$ and $(H1)$ are quite standard. $(F1)-(F5)$, $(K1)$ and $(H1)$ ensure the strong uniform convergence of the estimators $H_n(t|z)$ and $H_{1n}(t|z)$ to $H(t|z)$ and $H_1(t|z)$ respectively as in Bordes and Gneyou (2011b) while the assumptions $(K2)$ and $(H2)$ ensure the strong uniform convergence of $\lambda_n(t|z)$ and $\theta_n(z)$ to $\lambda(t|z)$ and $\theta(z)$ respectively. Assumptions $(H3)$ and (KH) ensure the asymptotic normality of the estimator $\theta_n(z)$ to $\theta(z)$.

3. Main Results

Gneyou (2012) proved the uniform convergence of the estimators $\lambda_n(t|z)$ and $\theta_n(z)$. In what follows we establish weak convergence and asymptotic normality of the estimator $\theta_n(z)$. For that, we need to consider the process

$$\begin{aligned} l^z(t, X, \delta) &= \frac{I(X \le t, \delta = 1) - H_1(t|z)}{1 - H(t^-|z)} - \int_0^t \frac{I(X \le s, \delta = 1) - H_1(s|z)}{(1 - H(s^-|z))^2} dH(s|z) \\ &+ \int_0^t \frac{I(X \le s) - H(s|z)}{(1 - H(s^-|z))^2} dH_1(s|z) \quad 0 \le t \le \tau_z. \end{aligned} \tag{7}$$

$l^z(t, X, \delta)$ is a centred random process which plays a major role in our investigations. The following theorem establishes a strong representation of the maximum hazard rate estimator $\theta_n(z)$. We apply it to derive a weak convergence leading to the asymptotic normality of the estimator θ_n.

Theorem 1 *Under the assumptions* $(F1)$-$(F5)$, $(K1)$-$(K2)$ *and* $(H1)$-$(H2)$, *we have, for all* $z \in \Delta$

$$\theta_n(z) - \theta(z) = \frac{1}{\lambda_n''^z(\theta_n^*)} \sum_{i=1}^n W_i(h_n, z)\zeta^z(\theta, X_i, \delta_i) + \tilde{r}_n^z(\theta) + O(a_n^2), \quad a.s. \tag{8}$$

where θ_n^* *is between* θ *and* θ_n,

$$\zeta^z(t, X_i, \delta_i) = \frac{1}{a_n^2} \int_{\mathbb{R}} l_i^z(t - a_n u) N''(u) du, \tag{9}$$

with $l_i^z(t)$ *as in (7) and*

$$\sup_{t \in [a_z, b_z]} \left|\tilde{r}_n^z(t)\right| \to 0 \quad \text{a.s.} \tag{10}$$

The proof of Theorem 1 is given in the next section. Let $D[0, \tau_z]$ be the standard Skorohod space on $[0, \tau_z]$. We have

Theorem 2 *If the assumptions of Theorem 1 hold then as* $n \to +\infty$, *for all* $z \in \Delta$, *the process*

$$\sqrt{n a_n^3 h_n^d} \sum_{i=1}^n W_i(h_n, z)\zeta^z(t, X_i, \delta_i), \quad 0 \le t \le \tau_z, \tag{11}$$

converges weakly in $D[0, \tau_z]$ *to a Gaussian process with covariance function given by*

$$\Gamma_z(t, s) = \| K \|_2^2 \, \lambda_z^*(s) \int_{\mathbb{R}} N'(v) N'(v + t - s) dv, \tag{12}$$

where $\lambda_z^*(t) = \lambda^*(t|z) = \dfrac{\lambda(t|z)}{1 - H(t|z)}$.

As a consequence of this theorem in conjunction with the following proposition, we obtain the asymptotic normality of the maximum conditional hazard rate estimator θ_n.

Proposition 1 *Assume that assumptions* $(H3)(iii)$ *and* (HK) *hold. Then for all* $z \in \Delta$ $\lambda_n''^z(t)$ *converges in probability to* $\lambda_z''(t)$ *uniformly in* t.

The proof of this proposition is postponed to the next section.

Theorem 3 *Assume that the assumptions* $(F1)$-$(F5)$*,* $(K1)$-$(K2)$*,* $(H1)$-$(H3)$ *and* (HK) *hold. Then*

1) *If in addition* $na_n^5h_n^{5d} \to 0$*, then*

$$\sqrt{na_n^3h_n^d}(\theta_n(z) - \theta(z)) \overset{\mathcal{L}}{\rightsquigarrow} \mathcal{N}(0, \frac{\sigma_z^2(\theta)}{(\lambda_z''(\theta))^2}) \tag{13}$$

2) *If* $na_n^5h_n^{5d} \to \gamma > 0$*, then,*

$$\sqrt{na_n^3h_n^d}(\theta_n(z) - \theta(z)) \overset{\mathcal{L}}{\rightsquigarrow} \mathcal{N}(\frac{m_z(\theta)}{\lambda_z''(\theta)}, \frac{\sigma_z^2(\theta)}{(\lambda_z''(\theta))^2}) \tag{14}$$

where

$$\begin{aligned}
\sigma_z^2(t) &= \frac{\lambda_z(t)}{1 - H_z(t)} \|K\|_2^2 \left\|N'\right\|_2^2, \\
m_z(t) &= \sqrt{\gamma}[f'(z)L^{(1,1)}(t|z) + \frac{1}{2}f(z)L^{(1,2)}(t|z)] \int_{\mathbb{R}^d} \|x\|^2 K(x)dx, \\
L(t|s) &= \frac{h_1(t|s) - h_1(t|z)}{1 - H(t^-|z)} + \frac{(H(t|s) - H(t|z))h_1(t|z)}{(1 - H(t^-|z))^2},
\end{aligned}$$

with $h_1(t|s) = H_1^{(1,0)}(t|s)$ *and* $h(t|s) = H^{(1,0)}(t|s)$*.*

4. Appendix: Proofs of the Results

The following lemmas are needed to prove the main results.

Lemma 1 *Let* $\ell : \mathbb{R}^d \to \mathbb{R}$ *be a function continuous at z. Then*

A. *Under assumptions* $(F1)(ii)$*,* $(K1)(i)$ *and* $(H1)(i)$*,*

$$\int_{\mathbb{R}^d} K_{h_n}(z - s)\ell(s)ds \to \ell(z), \quad \text{as } n \to +\infty.$$

B. *If* ℓ *is a function twice continuously differentiable at z then, under assumptions* $(K1)$ *and* $(H1)(i)$*, we have*

$$\left|\int_{\mathbb{R}^d} K_{h_n}(z - s)\ell(s)ds - \ell(z)\right| = O(h_n^{2d}), \quad \text{as } n \to +\infty.$$

The proof of this lemma is analogue to the proofs of Lemmas 4 and 5 in Bordes and Gneyou (2011a), hence we omit it.

Lemma 2 *Define* $E_i^z(t) = \int_{\mathbb{R}} l_i^z(t - a_nu)dN'(u), 0 \le t \le \tau_z$*. Then, for all* $0 \le t \le \tau_z$*,*

$$\mathbb{E}(E_i^z(t)|Z = s) = \int_{\mathbb{R}} Q^z(t - a_nu, s)dN'(u)$$

and if the assumptions $(K2)$ *and* $(H2) - (ii)$ *are satisfied then,*

$$\operatorname{cov}(E_i^z(t), E_i^z(t')|Z = s) = a_n\left[\lambda^*(t|s)\int_{\mathbb{R}} N'(v)N'(v + \frac{t - t'}{a_n})dv\right] + o(1)$$

where

$$Q^z(t, s) = \int_0^t \frac{d(H_1(v|s) - H_1(v|z)}{1 - H(v^-|z)} + \int_0^t \frac{H(v|s) - H(v|z)}{(1 - H(v^-|z))^2}dH(v|z). \tag{15}$$

Proof. By Fubini's Theorem, it is easily seen that

$$\mathbb{E}[E_i^z|Z = s] = \int_{\mathbb{R}} \mathbb{E}l^z(t - a_nu, X_i, \delta_i, s)dN'(u)$$

where

$$\begin{aligned} l^z(t, X_i, \delta_i, s) &= \frac{I(X_i \le t, \delta_i = 1|Z_i = s) - H_1(t|z)}{1 - H(t^-|z)} \\ &- \int_0^t \frac{I(X_i \le v, \delta_i = 1|Z_i = s) - H_1(v|z)}{(1 - H(v^-|z))^2} dH(v|z) \\ &+ \int_0^t \frac{I(X_i \le v|Z_i = s) - H(v|z)}{(1 - H(v^-|z))^2} dH_1(v|z) \\ &= l^{*z}(t, X_i, \delta_i, s) + Q^z(t, s), \end{aligned} \tag{16}$$

with

$$\begin{aligned} l^{*z}(t, X_i, \delta_i, s) &= \frac{I(X_i \le t, \delta_i = 1|Z_i = s) - H_1(t|s)}{1 - H(t^-|z)} \\ &- \int_0^t \frac{I(X_i \le v, \delta_i = 1|Z_i = s) - H_1(v|s)}{(1 - H(v^-|z))^2} dH(v|z) \\ &+ \int_0^t \frac{I(X_i \le v|Z_i = s) - H(v|s)}{(1 - H(v^-|z))^2} dH_1(v|z). \end{aligned} \tag{17}$$

It is easy to check that $\mathbb{E}l^{*z}(t, X_i, \delta_i, s) = 0$ and hence $\mathbb{E}l^z(t, X_i, \delta_i, s) = Q^z(t, s)$. Thus the first part of the lemma is proved. For the second part, we have by (16),

$$\begin{aligned} G(t, t', s) &= \operatorname{cov}(E_i^z(t), E_i^z(t')|Z = s) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E}[l^{*z}(t - a_n u, X_i, \delta_i, s) l^{*z}(t' - a_n v, X_i, \delta_i, s)] dN'(u) dN'(v) \end{aligned}$$

for all $t, t' \in [0, \tau_z]$. By (17), set

$$l^{*z}(t, X_i, \delta_i, s) = A - B + C \quad \text{and} \quad l^{*z}(t', X_i, \delta_i, s) = A' - B' + C'. \tag{18}$$

Then the expectation under the last integral equals

$$\mathbb{E}AA' - \mathbb{E}AB' + \mathbb{E}AC' - \mathbb{E}BA' + \mathbb{E}BB' - \mathbb{E}BC' + \mathbb{E}CA' - \mathbb{E}CB' + \mathbb{E}CC'.$$

By Fubini's theorem, we check that the eight last expectations equal respectively to zero while the first one $\mathbb{E}AA'$ equals to $d^*(t \wedge t'|z)$ where

$$d^*(t|z) = \int_0^t \frac{dH_1(v|z)}{(1 - H(v^-|z))^2} = \int_0^t \frac{\lambda(v|z)}{1 - H(v^-|z)} dv \tag{19}$$

Thus integrating by parts under the assumption $(K2)$, we check that

$$\begin{aligned} G_z(t, t', s) &= \int_{\mathbb{R}} \int_{u > v + \frac{t-t'}{a_n}} d^*(t - a_n u|z) dN'(u)] dN'(v) + \int_{\mathbb{R}} \int_{u < v + \frac{t-t'}{a_n}} d^*(t' - a_n v|z) dN'(u)] dN'(v) \\ &= a_n \int_{\mathbb{R}} \int_{v + \frac{t-t'}{a_n}}^{M} \lambda^*(t - a_n u|z) N'(u) du dN'(v) \\ &= a_n \int_{\mathbb{R}} N'(u) N'(v + \frac{t - t'}{a_n}) \lambda^*(t' - a_n v|z) dv \end{aligned} \tag{20}$$

where M is the upper boundary of the support of the kernel N. Developing the function $\lambda^*(t' - a_n v|z)$ by Taylor's theorem in order one at a neighbourhood of t' yields

$$G_z(t, t', s) = a_n \lambda^*(t|z) \int_{\mathbb{R}} N'(u) N'(v + \frac{t - t'}{a_n}) dv + o(a_n). \tag{21}$$

which ends the proof of the Lemma 2. □

Proof of Theorem 1. Notice that by definition of θ and θ_n, $\lambda_z'(\theta) = \lambda_n'^z(\theta_n) = 0$, $\lambda_z''(\theta) \le 0$ and $\lambda_n''^z(\theta_n) \le 0$. Hence by Taylor's expansion of order one in a neighbourhood of θ , we have

$$0 = \lambda_n'^z(\theta_n) = \lambda_n'^z(\theta) + (\theta_n - \theta) \lambda_n''^z(\theta_n^*) \tag{22}$$

where θ_n^* is between θ and θ_n. It follows from (22) that

$$\theta_n - \theta = -\frac{\lambda_n'^z(\theta)}{\lambda_n''^z(\theta_n^*)} = -\frac{1}{\lambda_n''^z(\theta_n^*)}(\lambda_n'^z(\theta) - \lambda_z'(\theta)). \tag{23}$$

By Assumption ($K2$) (i) it is easily seen that

$$\lambda_n(t|z) - \lambda(t|z) = \frac{1}{a_n}\int_{\mathbb{R}} N(\frac{t-s}{a_n})d(\Lambda_n(s|z) - \Lambda(s|z)) \ + O(a_n^2) \tag{24}$$

Hence by the derivation theorem and an integration by parts under the assumption ($K2$) we have

$$\begin{aligned}\lambda_n'(t|z) - \lambda'(t|z) &= \frac{1}{a_n^2}\int_{\mathbb{R}} N'(\frac{t-s}{a_n})d(\Lambda_n(s|z) - \Lambda(s|z)) \ + O(a_n^2)\\ &= -\frac{1}{a_n^2}\int_{\mathbb{R}}[\Lambda_n(t - a_n u|z) - \Lambda(t - a_n u|z)]dN'(u) \ + O(a_n^2).\end{aligned} \tag{25}$$

By Lemma 4.1 in Gneyou (2012)

$$\Lambda_n(t|z) - \Lambda(t|z) = A_n(t|z) + R_n(s|z) \tag{26}$$

where $A_n(t|z)$ is a process which can be written in the form

$$A_n(t|z) = \sum_{i=1}^n W_i(h_n, z) l_i^z(t) \tag{27}$$

and $R_n(t|z)$ is a remainder term which vanishes almost surely under assumptions ($H1$) and ($H2$) (see in Gneyou (2012)). It follows that

$$\lambda_n'(t|z) - \lambda'(t|z) = -\sum_{i=1}^n W_i(h_n, z)\eta(t, z, X_i, \delta_i) \ + \ \tilde{r}_n(t, z) \ + \ O(a_n^2) \tag{28}$$

where

$$\eta(t, z, X_i, \delta_i) = -\frac{1}{a_n^2}\int_{\mathbb{R}} l_i^z(t - a_n u)dN'(u), \tag{29}$$

$l_i^z(t)$ as in (7) and

$$\tilde{r}_n(t, z) = \frac{1}{a_n^2}\int_{\mathbb{R}} R_n(t - a_n u, z)dN'(u) = \tilde{U}_{1n}(t, z) + \tilde{U}_{2n}(t, z) \tag{30}$$

with

$$\begin{aligned}\tilde{U}_{n1}(t, z) &= \frac{1}{a_n^2}\int_{\mathbb{R}}\int_0^{t-a_n u}\frac{H_n(s|z) - H(s|z)}{(1 - H(s^-|z))^2}d(H_{1n}(s|z) - H_1(s|z))dN'(u)\\ \tilde{U}_{n2}(t, z) &= \frac{1}{a_n^2}\int_{\mathbb{R}}\int_0^{t-a_n u}\frac{(H_n(s|z) - H(s|z))^2}{(1 - H(s^-|z))^2(1 - H_n(s^-|z))}dH_{1n}(s|z)dN'(u)\end{aligned}$$

By Assumptions ($F4$), ($F5$), ($K1$) and ($H1$), we show that $\sup_{t\in[a_z,b_z]}|\tilde{r}_n(t, z)| \to 0$ almost surely by showing that $\sup_{t\in[a_z,b_z]}\left|\tilde{U}_{in}(t, z)\right| \to 0$ almost surely, for $i = 1, 2$. Since $\theta_n - \theta = -\frac{1}{\lambda_n''^z(\theta_n^*)}(\lambda_n'^z(\theta) - \lambda_z'(\theta))$, Theorem 1 follows from the representation (28) and (30). □

Proof of Theorem 2. Recall the notations $\zeta^z(t, X_i, \delta_i) = \frac{1}{a_n^2}\int_{\mathbb{R}} l_i^z(t - a_n u)dN'(u)$ and

$$W_i(h_n, z) = \frac{K_{h_n}(z - Z_i)}{\sum_{i=1}^n K_{h_n}(z - Z_i)} = \frac{K_{h_n}(z - Z_i)}{nf_n(z)}$$

where f_n is a consistent kernel estimator of the probability density f of the r.v. Z_i.

For all n large enough, we have

$$W_i(h_n, z)\zeta^z(t, X_i, \delta_i) = \frac{K_{h_n}(z - Z_i)}{na_n^2 f(z)}\int_{\mathbb{R}} l_i^z(t - a_n u)dN'(u).$$

Thus for all $0 \leq t \leq \tau_z$ we can write

$$\sqrt{na_n^3 h_n^d} f(z) \sum_{i=1}^{n} W_i(h_n, z)\zeta^z(t, X_i, \delta_i) \simeq \frac{1}{\sqrt{n}} \sum_{i=1}^{n} f_{n,t}(z, X_i, \delta_i, Z_i) \tag{31}$$

where

$$f_{n,t}(z, X_i, \delta_i, Z_i) = K(\frac{z - Z_i}{h_n}) \frac{1}{\sqrt{a_n h_n^d}} \int_{\mathbb{R}} l_i^z(t - a_n u) dN'(u). \tag{32}$$

We show the theorem by applying Theorem 2.11.23 of van der Vaart and Wellner (1996) to the class of function

$$\mathcal{F}_n = \{f_{n,t} : t \in T_z\} \quad \text{with} \quad T_z = [0, \tau_z]. \tag{33}$$

Let us calculate first the mean and the covariance functions of the process $\{f_{n,t} : t \in T_z\}$. It is straightforward that

$$\mathbb{E} f_{n,t}(z, X_i, \delta_i, Z_i) = \frac{1}{\sqrt{a_n h_n^d}} \int_{\mathbb{R}^d} K(\frac{z - s}{h_n}) \Phi^z(s) f(s) ds \tag{34}$$

where by Lemma 2, $\Phi^z(s) = \mathbb{E}[E_i^z(t)|Z_i = s] = \int_{\mathbb{R}} Q^z(t - a_n u, s) dN'(u)$ with $\Phi^z(z) = 0$. Set $\Psi(s) = \Phi^z(s) f(s)$. Develop $\Psi(s)$ by Taylor's theorem in the order two in a neighbourhood of z under the assumption $(F5)$. Since $\Psi(z) = 0$ we get under the assumption $(K1)$

$$\begin{aligned}
\int_{\mathbb{R}^d} K(\frac{z - s}{h_n}) \Psi(s) ds &= \Psi(z) \int_{\mathbb{R}^d} K(\frac{z - s}{h_n}) ds + \Psi'(z) \int_{\mathbb{R}^d} (s - z) K(\frac{z - s}{h_n}) ds \\
&+ \frac{1}{2} \Psi''(z) \int_{\mathbb{R}^d} (s - z)^2 K(\frac{z - s}{h_n}) ds + O((s - z)^3) \\
&= \frac{1}{2} \Psi''(z) h_n^{3d} \int_{\mathbb{R}^d} \|x\|^2 K(x) dx + O(h_n^{3d})
\end{aligned} \tag{35}$$

with $\Psi''(z) = 2f'(z)\Phi'(z) + f(z)\Phi''(z)$ and $\Phi(s) = \Phi^z(s)$. Hence

$$\mathbb{E} f_{n,t}(z, X_i, \delta_i, Z_i) = \frac{1}{\sqrt{a_n h_n^d}} h_n^{3d} [f'(z)\Phi'(z) + \frac{1}{2} f(z)\Phi''(z)] \int_{\mathbb{R}^d} \|x\|^2 K(x) dx + O(\frac{h_n^{2d}}{\sqrt{a_n h_n^d}}). \tag{36}$$

Recalling that

$$\begin{aligned}
\Phi(s) = \Phi^z(s) &= \int_{\mathbb{R}} \int_0^{t - a_n u} \frac{d(H_1(v|s) - H_1(v|z)}{1 - H(v^-|z)} dN'(u) + \int_{\mathbb{R}} \int_0^{t - a_n u} \frac{H(v|s) - H(v|z)}{(1 - H(v^-|z))^2} dH_1(v|z) dN'(u) \\
&= \int_{\mathbb{R}} \phi(u, s) N''(u) du
\end{aligned}$$

where

$$\phi(u, s) = \int_0^{t - a_n u} \left[\frac{h_1(v|s) - h_1(v|z)}{1 - H(v^-|z)} + \frac{(H(v|s) - H(v|z)) h_1(v|z)}{(1 - H(v^-|z))^2} \right] dv,$$

an integration by part yields

$$\Phi(s) = a_n^2 \int_{\mathbb{R}} N_{a_n}(t - x) L^{(1,0)}(x|s) dx$$

where

$$L(t|s) = \frac{h_1(t|s) - h_1(t|z)}{1 - H(t^-|z)} + \frac{(H(t|s) - H(t|z)) h_1(t|z)}{(1 - H(t^-|z))^2} \text{ and } L^{(i,j)}(t|z) = \frac{\partial^{i+j} L(t|z)}{\partial t^i \partial z^j}.$$

Since by the assumption $(F5)$ the function $t \mapsto L^{(1,0)}(t|s)$ is continuous at t, apply Lemma 3.1 of Bordes and Gneyou (2011 b) with the kernel N and have

$$\Phi(s) = a_n^2 (L^{(1,0)}(t|s) + O(1)).$$

It follows that

$$\mathbb{E} f_{n,t}(z, X_i, \delta_i, Z_i) = \sqrt{a_n^3 h_n^{5d}} [f'(z) L^{(1,1)}(t|z) + \frac{1}{2} f(z) L^{(1,2)}(t|z)] \int_{\mathbb{R}^d} \|x\|^2 K(x) dx + O(\sqrt{a_n^3 h_n^{5d}}). \tag{37}$$

Let us check now the covariance function of the process $\{f_{n,t} : t \in T_z\}$. Set

$$\Gamma_z(t,t') = \mathrm{cov}(f_{n,t}(z,X_i,\delta_i,Z_i), f_{n,t'}(z,X_i,\delta_i,Z_i))$$

By (32) and (16) we have for all $t,t' \in T_z$,

$$\Gamma_z(t,t') = \frac{1}{a_n^2}\int_{\mathbb{R}^d} K_{h_n}^2(z-s)G_z(t,t',s)f(s)ds \tag{38}$$

where

$$G_z(t,t',s) = \mathrm{cov}(E_i^z(t), E_i^z(t')|Z=s).$$

Using the kernel $\dfrac{K}{\int_{\mathbb{R}^d} K^2(x)dx}$, it follows from (38), (21) and Lemma 2 that

$$\begin{aligned}\Gamma(t,t'|z) &= \frac{1}{a_n h_n^d}\lambda^*(t'|z)\int_{\mathbb{R}^d}\frac{1}{h_n^d}K^2(\frac{z-s}{h_n})f(s)ds\int_{\mathbb{R}}N'(v)N'(v+\frac{t-t'}{a_n})dv + O(\frac{1}{a_n h_n^d})\\ &= \frac{1}{a_n h_n^d}\lambda^*(t'|z)f(z)\int_{\mathbb{R}^d}K^2(x)dx\int_{\mathbb{R}}N'(v)N'(v+\frac{t-t'}{a_n})dv + O(\frac{1}{na_n h_n^d})\end{aligned} \tag{39}$$

It remains to check the three conditions of Theorem 2.11.23 of van der Vaart and Wellner (1996). Set

$$\mathcal{F}_n = \{f_{n,t} : t \in T_z\} \tag{40}$$

where $f_{n,t}$ is the real function defined on $\mathbb{R}^d \times \mathbb{R}^+ \times \{0,1\} \times \mathbb{R}^d$ by

$$f_{n,t}(z,y,x,s) = \frac{1}{\sqrt{a_n h_n^d}}\int_{\mathbb{R}} l^z(t-a_n u,y,x)dN'(u)K(\frac{z-s}{h_n}) \tag{41}$$

with $l^z(t,y,x)$ as in (7). Let us check the Lindberg conditions (2.11.21) of Van der Vaart and Wellner (1996).
(i) By the assumption ($K2$) (i), an integration by parts yield

$$\begin{aligned}\int_{\mathbb{R}} l^z(t-a_n u,y,x)dN'(u) &= a_n\int_{\mathbb{R}}N'(u)\frac{\partial l^z}{\partial t}(t-a_n u,y,x)du\\ &\le a_n\sup_{t\in[a_z,b_z]}\left|\frac{\partial l^z}{\partial t}(t,y,x)\right|\left\|N'\right\|_{L^1(\mathbb{R})}.\end{aligned}$$

Since

$$\begin{aligned}\frac{\partial l^z}{\partial t}(t,y,x) &= \frac{(I(x=1)\delta_t - H_1^{(1,0)}(t|z))(1-H(t|z)) + H^{(1,0)}(t|z)}{(1-H(t|z))^2}\\ &- \frac{I(y\le t, x=1) - H_1(t|z)}{(1-H(t|z))^2}dH(t|z) + \frac{I(y\le t) - H(t|z)}{(1-H(t|z))^2}dH_1(t|z)\end{aligned}$$

and the derivatives dH and dH_1 are bounded under the assumption ($F5$), we have under the assumption ($F4$)

$$\left|\frac{\partial l^z}{\partial t}(t,y,x)\right| \le \eta^{-2}(5 + 2A_1 + 2A_2) = m$$

where A_1 and A_2 are absolute constants. Consequently we have for n large enough,

$$\sup_{t\in[a_z,b_z]}\left|f_{n,t}\right| \le \sqrt{a_n}m\left\|N'\right\|_{L(\mathbb{R})}K(\frac{z-s}{h_n}) \le F_n = m_0\frac{1}{h_n^{d/2}}K(\frac{z-s}{h_n})$$

where $m_0 = m\left\|N'\right\|_{L(\mathbb{R})}$. Since $h_n \searrow 0$ and f is continuous at z by assumption ($F1$)(ii), we apply Lemma 1 with the kernel $K^2/\left\|K\right\|^2_{L(\mathbb{R}^d)}$ and get

$$\mathbb{E}F_n^2 = m_0^2\int_{\mathbb{R}^d}\frac{1}{h_n^d}K^2(\frac{z-s}{h_n})f(s)ds = m_0^2\left\|K\right\|^2_{L(\mathbb{R}^d)}f(z) + o(1) = O(1).$$

(ii) Since K is symmetric, bounded and $nh_n^d \to +\infty$ as $n \to +\infty$ by assumption $(H1)(ii)$, we have for all $\varepsilon > 0$,

$$\begin{aligned}\mathbb{E}[F_n^2 I(F_n \geq \sqrt{n}\varepsilon)] &= m_0^2 \int_{\mathbb{R}^d} \frac{1}{h_n^d} K^2(\frac{z-s}{h_n}) I\{K^2(\frac{z-s}{h_n}) \geq \sqrt{nh_n^d}\varepsilon\} f(s)ds \\ &= m_0^2 \int_{\mathbb{R}^d} K^2(u) I\{K(u) \geq \sqrt{nh_n^d}\varepsilon\} f(z+h_n u)du \to 0 \; as \; n \to +\infty.\end{aligned}$$

(iii) Consider the pseudo-distance ρ defined by

$$\rho(t,t') = \max\{\left|H(t|z) - H(t'|z)\right|, \left|\lambda(t|z) - \lambda(t'|z)\right|\}.$$

For $a_z \leq t \leq t' \leq b_z$ and $\rho(t,t') \leq \rho_n$ with $\rho_n \to 0$, set

$$D(t,t') = \mathbb{E}[(f_{n,t}(z, X_i, \delta_i, Z_i) - f_{n,t'}(z, X_i, \delta_i, Z_i))^2].$$

Then we have

$$D(t,t') = \frac{1}{a_n} \int_{\mathbb{R}^d} \frac{1}{h_n^d} K^2(\frac{z-s}{h_n}) \int_{\mathbb{R}} \mathbb{E}[(l^z(t - a_n u, X_i, \delta_i, s) - l^z(t - a_n u, X_i, \delta_i, s))^2] f(s)ds.$$

Let

$$\begin{aligned}W(t,t') &= \mathbb{E}[(l^z(t - a_n u, X_i, \delta_i, s) - l^z(t - a_n u, X_i, \delta_i, s))^2] \\ l(v) &= l^z(v - a_n u, X_i, \delta_i, s) \\ l^*(v) &= l^{*z}(v - a_n u, X_i, \delta_i, s) \\ c(v) &= Q^z(v - a_n u, s)\end{aligned}$$

and recall that by (16) $l^z(v, X_i, \delta_i, s) = l^{*z}(v, X_i, \delta_i, s) + Q^z(v, s)$. Then

$$l(t) - l(t') = l^*(t) - l^*(t') + [c(t) - c(t')]$$

furthermore

$$(l(t) - l(t'))^2 = (l^*(t) - l^*(t'))^2 + (c(t) - c(t'))^2 - 2(c(t) - c(t'))(l^*(t) - l^*(t')).$$

Since $l^{*z}(t - a_n u, X_i, \delta_i, s)$ is a centered random variable, the last term of this last equality has mean zero. It follows that

$$W(t,t') = \mathbb{E}[(l^{*z}(t - a_n u, X_i, \delta_i, s) - l^{*z}(t - a_n u, X_i, \delta_i, s))^2] + [Q^z(t - a_n u, s) - Q^z(t' - a_n u, s)]^2.$$

After the development of square in the first brackets, we get by similar computations as in (18),

$$\begin{aligned}W(t,t') &= d_z^*(t - a_n u) + d_z^*(t' - a_n u) - 2d_z^*(t \wedge t' - a_n u) \\ &+ [Q^z(t - a_n u, s) - Q^z(t' - a_n u, s)]^2 \\ &= d_z^*(t - a_n u) - d_z^*(t' - a_n u) + [Q^z(t - a_n u, s) - Q^z(t' - a_n u, s)]^2.\end{aligned}$$

Hence we have

$$\begin{aligned}D(t,t') &= \frac{1}{a_n} \int_{\mathbb{R}^d} \frac{1}{h_n^d} K^2(\frac{z-s}{h_n}) \int_{\mathbb{R}} [d_z^*(t - a_n u) - d_z^*(t' - a_n u)] f(s)dsdN'(u) \\ &+ \frac{1}{a_n} \int_{\mathbb{R}^d} \frac{1}{h_n^d} K^2(\frac{z-s}{h_n}) \int_{\mathbb{R}} [Q^z(t - a_n u, s) - Q^z(t' - a_n u, s)]^2 f(s)dsdN'(u) \\ &= I + II\end{aligned}$$

with

$$I = \frac{1}{a_n} \int_{\mathbb{R}^d} \frac{1}{h_n^d} K^2(\frac{z-s}{h_n}) f(s)ds \int_{\mathbb{R}} [d_z^*(t - a_n u) - d_z^*(t' - a_n u)] dN'(u)$$

$$II = \frac{1}{a_n} \int_{\mathbb{R}^d} \frac{1}{h_n^d} K^2(\frac{z-s}{h_n}) \int_{\mathbb{R}} [Q^z(t - a_n u) - Q^z(t' - a_n u)]^2 f(s)dsdN'(u)$$

Since by the assumption $(F1)(ii)$ f is a continuous function with bounded derivative at z, we apply Fubini's theorem and Lemma 1 with the kernel $K^2/\|K\|^2_{L(\mathbb{R}^d)}$ to get

$$I = \frac{1}{a_n}\|K\|^2_{L(\mathbb{R}^d)} f(z)\int_{\mathbb{R}}[d_z^*(t-a_nu)-d_z^*(t'-a_nu)]dN'(u)+O(1)$$

and

$$II = \frac{1}{a_n}\|K\|^2_{L(\mathbb{R}^d)} f(z)\int_{\mathbb{R}}[Q^z(t-a_nu,z)-Q^z(t'-a_nu,z)]dN'(u)+O(1).$$

But for all v, $Q^z(v,z)=0$ (see in (15)), hence $II=0$. Recalling that

$$d_z^*(t)=\int_0^t\frac{dH_1(v|z)}{(1-H(v^-|z))^2}=\int_0^t\frac{\lambda(v|z)}{1-H(v^-|z)}dv,$$

we have after integrating by parts

$$I=\|K\|^2_{L(\mathbb{R}^d)} f(z)\int_{\mathbb{R}}N'(u)\left[\frac{\lambda(t'-a_nu|z)}{1-H(t'^- -a_nu|z)}-\frac{\lambda(t-a_nu|z)}{1-H(t^- -a_nu|z)}\right]du+O(1).$$

Using now the fact that

$$\frac{a}{b}-\frac{a'}{b'}=\frac{a}{bb'}(b-b')+\frac{1}{b'}(a-a'),\quad a,a',b,b'>0$$

and that f is bounded on Δ, we have by assumption $(F4)$

$$\begin{aligned}
\left|D(t,t')\right|=|I| &\le f(z)\|K\|^2_{L(\mathbb{R}^d)}\eta^{-1}\sup_{|u|\le 1}\sup_{|t-t'|\le\rho_n}\left|\lambda(t'-a_nu|z)-\lambda(t-a_nu|z)\right|\left\|N'\right\|_{L(\mathbb{R})}\\
&+ f(z)\|K\|^2_{L(\mathbb{R}^d)}\eta^{-2}\sup_{|u|\le 1}\sup_{|t-t'|\le\rho_n}\left|H(t-a_nu|z)-H(t'-a_nu|z)\right|\left\|N'\right\|_{L(\mathbb{R})}\\
&\le 2f(z)\beta\|K\|^2_{L(\mathbb{R}^d)}\sup_{|u|\le 1}\sup_{t\in[a_z,b_z]}|\lambda(t-a_nu|z)-\lambda(t|z)|\left\|N'\right\|_{L(\mathbb{R})}\\
&+ 2f(z)\beta\|K\|^2_{L(\mathbb{R}^d)}[\sup_{|u|\le 1}\sup_{t\in[a_z,b_z]}|H(t-a_nu|z)-H(t|z)|+\rho(t,t')]\left\|N'\right\|_{L(\mathbb{R})}\\
&\le (o(1)+\rho_n)O(1)\longrightarrow 0\quad\text{if}\quad \rho_n\to 0,
\end{aligned}$$

where $\beta=\max\{\eta^{-1},\eta^{-2}\}$.

It remains to check the entropy condition. Let us consider the following brackets $[f_{n,t_{i-1}}, f_{n,t_i}]$ with

$$f_{n,t^-}(z,y,x,s)=\frac{1}{\sqrt{a_nh_n^d}}\int_{\mathbb{R}}l^z(t^- -a_nu,y,x)dN'(u)K(\frac{z-s}{h_n}),$$

where $l^z(t,y,x)$ is defined as in (7). After some computations similar to above ones we obtain for n large enough,

$$\mathbb{E}[(f_{n,t_i^-}(X_i,\delta_i,Z_i)-f_{n,t_{i-1}^-}(X_i,\delta_i,Z_i))^2]\le 2\varepsilon\beta f(z)\|K\|^2_{L(\mathbb{R}^d)}\left\|N'\right\|_{L(\mathbb{R})}.$$

Therefore it is straightforward that for n large enough,

$$N_{[]}(2\varepsilon\beta f(z)\|K\|^2_{L(\mathbb{R}^d)}\left\|N'\right\|_{L(\mathbb{R})},\mathcal{F}_n,L^2(\mathbb{P}))\le\frac{2}{\varepsilon}$$

which implies

$$N_{[]}(\varepsilon,\mathcal{F}_n,L^2(\mathbb{P}))\le\frac{4\beta f(z)\|K\|^2_{L(\mathbb{R}^d)}\|N'\|_{L(\mathbb{R})}}{\varepsilon}. \tag{42}$$

Finally we have by (42),

$$J_{[]}(\delta_n,\mathcal{F}_n,L^2(\mathbb{P}))=\int_0^{\delta_n}\sqrt{\log N_{[]}(\varepsilon,\mathcal{F}_n,L^2(\mathbb{P}))}d\varepsilon\longrightarrow 0$$

for every $\delta_n\searrow 0$.

All assumptions of Theorem 2.11.23 of van der Vaart and Wellner (1996) being satisfied we conclude that the process

$$\mathbb{G}_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} f_{n,t}(X_i, \delta_i, Z_i) - \mathbb{E} f_{n,t}(X_i, \delta_i, Z_i) \tag{43}$$

converges weakly to $\mathbb{G}$ where $\mathbb{G}$ is a tight centered process on $T_z = [0, \tau_z]$ with covariance function defined for $t, t' \in T_z$ by

$$\mathbb{E}[\mathbb{G}(t)\mathbb{G}(t')] = \lim_{n\to+\infty} (\mathbb{E}[f_{n,t} f_{n,t'}] - \mathbb{E} f_{n,t} \mathbb{E} f_{n,t'}).$$

By (39) it is straightforward that

$$\mathbb{E}[\mathbb{G}(t)\mathbb{G}(t')] = \lambda^*(t'|z) f(z) \int_{\mathbb{R}^d} K^2(x)dx \int_{\mathbb{R}} N'(v)N'(v + t - t')dv \tag{44}$$

and the proof of the theorem is complete. □

Proof of Proposition 1. By Fourier's inversion theorem, it is easy to check that

$$\frac{1}{2\pi} \int_{\mathbb{R}} e^{-iut} k(a_n u) \varphi_n^z(u) du = \lambda_n^z(t)$$

and also using Lemma 1 we have

$$\frac{1}{2\pi} \int_{\mathbb{R}} e^{-iut} k(a_n u) \varphi^z(u) du = \lambda^z(t) + O(a_n^2).$$

Applying the derivative theorem under assumption (KH), we have

$$\begin{aligned} -\frac{1}{2\pi} \int_{\mathbb{R}} u^2 e^{-iut} k(a_n u) \varphi_n^z(u) du &= \lambda_n''^z(t) \\ -\frac{1}{2\pi} \int_{\mathbb{R}} u^2 e^{-iut} k(a_n u) \varphi^z(u) du &= \lambda''^z(t) + O(a_n^2). \end{aligned}$$

It follows that

$$\mathbb{E}^{1/2}[\sup_{t\in[a_z,b_z]} \left|\lambda_n''^z(t) - \mathbb{E}(\lambda_n''^z(t))\right|^2] \leq (2\pi)^{-1} \int_{\mathbb{R}} u^2 \left|k(a_n u)\right| \mathrm{Var}(\varphi_n^z(u)) du \tag{45}$$

and

$$\left|\mathbb{E}(\lambda_n''^z(t)) - \lambda''^z(t)\right| \leq (2\pi)^{-1} \int_{\mathbb{R}} u^2 \left|k(a_n u)\right| \left|\mathbb{E}(\varphi_n^z(u)) - \varphi^z(u)\right| du + O(a_n^2). \tag{46}$$

Moreover

$$\begin{aligned} \left|\mathbb{E}(\varphi_n^z(u)) - \varphi^z(u)\right| &= \left|\int_{\mathbb{R}} e^{iux} (\mathbb{E} d\Lambda_n^z(x) - d\Lambda^z(x))\right| = \left|\int_{\mathbb{R}} e^{iux} (\mathbb{E}\lambda_n^z(x) - \lambda^z(x)) dx\right| \\ &\leq \frac{C}{|u|} \sup_{t\in[0,\tau_z]} \left|\mathbb{E}\lambda_n^z(t) - \lambda^z(t)\right|. \end{aligned} \tag{47}$$

where C is a constant. Since $\lambda_n^z(t)$ is a consistent estimator of $\lambda^z(t)$, the last term of the right hand side of the above inequality tends to 0 as $n \to +\infty$ for all $u \neq 0$. Hence $\mathbb{E}(\varphi_n^z(u)) - \varphi^z(u) \longrightarrow 0$ as $n \to +\infty$ for all $u \neq 0$ and this is also true if $u = 0$.

Let us evaluate $\sigma^2(\varphi_n^z) = \mathrm{Var}(\varphi_n^z)$. We have

$$\begin{aligned} \varphi_n^z(u) - \mathbb{E}(\varphi_n^z(u)) &= \int_{\mathbb{R}} e^{ius} \left(\frac{dH_{1n}^z(s)}{1 - H_n^z(s)} - \mathbb{E}(\frac{dH_{1n}^z(s)}{1 - H_n^z(s)}) \right) \\ (\varphi_n^z(u) - \mathbb{E}(\varphi_n^z(u))^2 &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{iux} e^{iuy} G_n(x, y) dx dy \end{aligned}$$

where

$$G_n(x, y) = \left(\frac{dH_{1n}^z(x)}{1 - H_n^z(x)} - \mathbb{E}(\frac{dH_{1n}^z(x)}{1 - H_n^z(x)}) \right) \left(\frac{dH_{1n}^z(y)}{1 - H_n^z(y)} - \mathbb{E}(\frac{dH_{1n}^z(y)}{1 - H_n^z(y)}) \right).$$

Hence

$$\sigma^2(\varphi_n^z) = \mathbb{E}[\varphi_n^z(u)) - \mathbb{E}(\varphi_n^z(u)]^2 = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{iux}e^{iuy}\operatorname{cov}\left(\frac{dH_{1n}^z(x)}{1-H_n^z(x)}, \frac{dH_{1n}^z(y)}{1-H_n^z(y)}\right)dxdy.$$

But by the definitions of $H_{1n}^z(t)$ and $H_n^z(t)$ we have

$$\sigma^2(\varphi_n^z) = \sum_{k=1}^n\sum_{l=1}^n \operatorname{cov}\left(W_k(h_n,z)\frac{\delta_k e^{iuX_k}}{1-H_n^z(X_k)}, W_l(h_n,z)\frac{\delta_l e^{iuX_l}}{1-H_n^z(X_l)}\right).$$

By independence of (X_k,δ_k) and (X_l,δ_l) for all $k\neq l$ and the almost sure convergence of $H_n(.|z)$ to $H(.|z)$ (see Lemma 4.2 in Gneyou(2012)), the above expression reduces to

$$\begin{aligned}\sigma^2(\varphi_n^z) &= \sum_{k=1}^n \operatorname{Var}(W_k^2(h_n,z)\frac{\delta_k e^{iuX_k}}{1-H_n^z(X_k)})\\ &\leq \eta^2\sum_{k=1}^n \operatorname{Var}(W_k^2(h_n,z)\delta_k) \leq n\eta^2\mathbb{E}(W_1^2(h_n,z)\delta_1),\ \text{for all } n \text{ big enough.}\end{aligned}$$

Since for all $i = 1,\cdots,n$

$$W_i(h_n,z) = \frac{1}{nh_n^d f_n(z)}K(\frac{z-Z_i}{h_n}) = (1+O(1))\frac{1}{nh_n^d f(z)}K(\frac{z-Z_i}{h_n}),$$

it readily follows that

$$\mathbb{E}[W_i^2(h_n,z)\delta_i] \leq \frac{C}{n^2h_n^d f(z)}\int_{\mathbb{R}^d}\frac{1}{h_n^d}K^2(\frac{z-s}{h_n})\int_0^{+\infty}(1-G(x|s)dF(x|s)f(s)ds$$

where C is a positive constant. The middle integral equals to $\mathbb{P}[Z\leq+\infty,\delta=1]$. Hence it is less than one. As a consequence, we have by Fubini's Theorem

$$\mathbb{E}[W_i^2(h_n,z)\delta_i] \leq \frac{C}{nh_n^d f(z)}\int_{\mathbb{R}^d}\frac{1}{h_n^d}K^2(\frac{z-s}{h_n})f(s)ds.$$

Since the function f is continuous at z, we apply again Lemma 1 with the kernel $K^2/\int_{\mathbb{R}^d}K^2(x)dx$ to have

$$\mathbb{E}[W_i^2(h_n,z)\delta_i] \leq \frac{C\,\|K\|^2_{L(\mathbb{R}^d)}}{n^2h_n^d}.$$

It follows that

$$\sigma^2(\varphi_n^z) \leq 2\eta^2 \leq \frac{C'}{nh_n^d} \tag{48}$$

where C' is a novel positive constant. Combining now (45), (47) and (48) we prove that

$$\mathbb{E}^{1/2}[\sup_{t\in[a_z,b_z]}\left|\lambda_n''^z(t)-\lambda''^z(t))\right|^2] \leq \frac{A}{2\pi\sqrt{na_n^6h_n^d}}\int_{\mathbb{R}}t^2\,|k(t)|\,dt$$

and hence $\mathbb{E}^{1/2}[\sup_{t\in[a_z,b_z]}\left|\lambda_n''^z(t)-\lambda''^z(t))\right|^2] \longrightarrow 0$ as $n\to+\infty$ by assumption $(H3)(ii)$ where A is a constant. This imply the convergence in probability of $\lambda_n''^z(t)$ to $\lambda_z''(t)$ uniformly in t. □

Proof of Theorem 3. By Theorem 1 it is straightforward that

$$\sqrt{na_n^3h_n^d}f(z)(\lambda_n''^z(\theta_n^*))(\theta_n(z)-\theta(z) \simeq U_n(z) \tag{49}$$

and by the proof of Theorem 2, $U_n(z)$ is a linear functional of the empirical process $\mathbb{G}_n$ given in (43) and hence, is asymptotically Gaussian with asymptotic variance σ_z^2 and mean function equal to

$$\frac{a_n\sqrt{n}}{f(z)}\mathbb{E}(f_{n,t}) = \sqrt{na_n^5h_n^{5d}}g_n(t|z)$$

where

$$g_n(t|z) = [f'(z)L^{(1,1)}(t|z) + \frac{1}{2}f(z)L^{(1,2)}(t|z)] \int_{\mathbb{R}^d} \|x\|^2 K(x)dx + O(1).$$

This mean function converges to zero if $na_n^5 h_n^{5d} \to 0$ and it converges to

$$m^z(t) = \sqrt{C}[f'(z)L^{(1,1)}(t|z) + \frac{1}{2}f(z)L^{(1,2)}(t|z)] \int_{\mathbb{R}^d} \|x\|^2 K(x)dx$$

if $na_n^5 h_n^{5d} \to C$. This completes the proof of Theorem 3. □

Acknowledgements

The author thanks the referee for pointing out a number of oversights and misprints, as well as showing the way to an improvement of the paper.

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