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Estimation Based on Generalized Order Statistics from a Mixture of Two Rayleigh Distributions

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Abstract

This article is concerned with the problem of estimating the parameters, reliability and hazard rate functions of the mixture of two Rayleigh distributions ($MTRD$) based on generalized order statistics (GOS). The maximum likelihood and Bayes methods of estimation are used for this purpose. The Markov chain Monte Carlo ($MCMC$) method is used for obtaining the Bayes estimates under the squared error loss and $LINEX$ loss functions. Our results are specialized to progressive Type-II censored order statistics and upper record values. Comparisons are made between Bayesian and maximum likelihood estimators via a Monte Carlo simulation study.

Keywords: generalized order statistics, Bayes estimation, progressive censoring, Markov chain Monte Carlo, Monte Carlo simulation

1. Introduction

The Rayleigh distribution (RD) was first derived by Lord Rayleigh in connection with a study of acoustical problems. Since then many investigators have used the RD or some related forms of it in a variety of engineering, wave propagation, radiation and analysis of target data studies. The RD is also used to model wave heights in oceanography, and in communication theory to describe hourly median and instantaneous peak power of received radio signals. Several such situations have been discussed by Polovko (1968), Takeshi Yamane (1998), Zhi Ren et al. (2011), and many others. The RD is a special case of two parameter Weibull distribution. A random variable T is said to have a RD with parameter θ if its probability density function (PDF) is given by

$$f(t) = 2\theta\, t\, e^{-\theta t^2},\ t > 0,\ (\theta > 0). \tag{1}$$

The cumulative distribution function (CDF), reliability function (RF) and the hazard rate function (HRF) are given, respectively, by

$$F(t) = 1 - e^{-\theta t^2},\ t > 0,\ (\theta > 0), \tag{2}$$

$$R(t) = e^{-\theta t^2}, \tag{3}$$

$$H(t) = 2\theta\, t, \tag{4}$$

where $H(\cdot) = \frac{f(\cdot)}{R(\cdot)}$.

Mixtures of distributions arise frequently in life testing, reliability, biological and physical sciences. Some of the most important references that discussed different types of mixtures of distributions are a monograph by Everitt and Hand (1981), Titterington et al. (1985) and McLachlan and Basford (1988). Bayesian inferences based on finite mixture distribution have been discussed by several authors. Papadapoulos and Padgett (1986) considered Bayesian estimation of the mixing parameter, mean and reliability function of a mixture of two exponential lifetime distributions based on right censored samples. Attia (1993) considered the $MTRD$ and obtained estimates of model parameters using maximum likelihood (ML) and Bayesian approach with censored sampling. Ahmad et al. (1997) derive approximate Bayes estimation for mixture of two Weibull distributions under Type-II censoring. Jaheen (2005b) considered estimation for the mixed exponential distribution based on record statistics. Soliman (2006) obtained the estimates of the parameters and functions of these parameters of the $MTRD$ based on progressively

Type-II censored samples when the mixing proportion p is known. Saleem and Aslam (2008a & b) use ordinary type I right censored data for Bayesian analysis of Rayleigh mixture. Saleem and Irfan (2010) studied some properties of the Bayes estimates of the Rayleigh mixture parameters. The PDF, CDF, RF and HRF of the $MTRD$ are given, respectively, by

$$f(t) = p_1 f_1(t) + p_2 f_2(t), \tag{1}$$

$$F(t) = p_1 F_1(t) + p_2 F_2(t), \tag{2}$$

$$R(t) = p_1 R_1(t) + p_2 R_2(t), \tag{3}$$

$$H(t) = \frac{f(t)}{R(t)}, \tag{4}$$

where, for $j = 1, 2$, the mixing proportions p_j are such that $0 \le p_j \le 1$, $p_1 + p_2 = 1$ and $f_j(t)$, $F_j(t)$, $R_j(t)$ are given from (1), (2), (3) after using θ_j instead of θ.

The property of identifiability is an important consideration on estimating the parameters in a mixture of distributions. Also, testing hypothesis, classification of random variables, can be meaning fully discussed only if the class of all finite mixtures is identifiable. Identifiability of mixtures has been discussed by several authors, including Teicher (1963), AL-Hussaini and Ahmad (1981) and Ahmad (1988).

The concept of GOS was introduced by Kamps (1995) to unify several important ordering concepts that were separately treated in statistical literature, such as ordinary order statistics, ordinary record values, progressive Type-II censored order statistics and sequential order statistics, among others. For various distributional properties of GOS, see Kamps (1995). The GOS have been considered extensively by many authors, among others, they are Ahsanullah (1996; 2000), Kamps and Gather (1997), Cramer and Kamps (2000), Habibullah and Ahsanullah (2000), Jaheen (2002; 2005a), AL-Hussaini and Ahmad (2003), AL-Hussaini (2004), Ahmad (2007; 2008), Aboeleneen (2010), Jaheen and Al Harbi (2010), Abu El Fotouh (2011) and Ateya and Ahmad (2011).

The purpose of this paper is to estimate the parameters, RF and HRF of the $MTRD$ based on GOS using ML and Bayes methods. The non-informative prior and the conjugate prior are assumed to carry out the Bayesian analysis. The results are specialized to progressive Type-II censored order statistics and record values. This paper is organized as follows: In Section 2, the ML estimators of parameters, RF and HRF of the $MTRD$ are derived. In Section 3, the $MCMC$ method is used for obtaining the Bayes estimators of parameters, RF and HRF of the $MTRD$ under the squared error loss and $LINEX$ loss functions. Comparisons between Bayesian and ML estimators via Monte Carlo simulation study are made in Section 4. Finally, Concluding remarks about comparisons between the estimators are considered in Section 5.

2. Maximum Likelihood Estimation

Let $T_{1;n,\widetilde{m},k}, T_{2;n,\widetilde{m},k}, ..., T_{n;n,\widetilde{m},k}$, $k > 0$, $\widetilde{m} = (m_1, ..., m_{n-1}) \in \Re^{n-1}, m_1, ..., m_{n-1} \in \Re$, are n GOS drawn from the $MTRD$. The likelihood function (LF) is given in (Kamps, 1995), for $-\infty < t_1 < ... < t_n < \infty$, by

$$L(\theta|\mathbf{t}) = k \prod_{i=1}^{n-1} \gamma_i \{ \prod_{i=1}^{n-1} [R(t_i)]^{m_i} f(t_i) \} [R(t_n)]^{k-1} f(t_n), \tag{5}$$

where $\mathbf{t} = (t_1, ..., t_n)$, $\theta \in \Theta$, Θ is the parameter space, and

$$\gamma_i = k + n - i + M_i > 0, M_i = \sum_{\nu=i}^{n-1} m_\nu.$$

Substituting (1), (1) in (5), the LF takes the form

$$L(\theta|\mathbf{t}) \propto \prod_{i=1}^{n-1} [p_1 R_1(t_i) + p_2 R_2(t_i)]^{m_i} \prod_{i=1}^{n} [p_1 f_1(t_i) + p_2 f_2(t_i)] [p_1 R_1(t_n) + p_2 R_2(t_n)]^{k-1}. \tag{6}$$

Take the logarithm of (6), we have

$$\ell(\theta) \equiv \ln L(\theta|\mathbf{t}) \propto \sum_{i=1}^{n-1} m_i \ln[p_1 R_1(t_i) + p_2 R_2(t_i)] + \sum_{i=1}^{n} \ln[p_1 f_1(t_i) + p_2 f_2(t_i)]$$

$$+(k-1)\ln[p_1R_1(t_n)+p_2R_2(t_n)], \tag{7}$$

where $p_1 = p$, $p_2 = 1-p$.

Differentiating (7) with respect to the parameters p and θ_j and equating to zero gives the following likelihood equations

$$\left.\begin{aligned}\frac{\partial \ell}{\partial p} &= \sum_{i=1}^{n-1} m_i\vartheta^*(t_i)+\sum_{i=1}^{n}\vartheta(t_i)+(k-1)\vartheta^*(t_n)=0,\\ \frac{\partial \ell}{\partial \theta_j} &= p_j\{\sum_{i=1}^{n}\xi_j(t_i)\psi_j(t_i)-\sum_{i=1}^{n-1} m_i\psi_j^*(t_i)-(k-1)\psi_j^*(t_n)\}=0, \quad j=1,2\end{aligned}\right\}, \tag{8}$$

where, for $j = 1, 2$

$$\left.\begin{aligned}\vartheta(t_i) &= \frac{f_1(t_i)-f_2(t_i)}{f(t_i)}, \qquad & \vartheta^*(t_i) &= \frac{R_1(t_i)-R_2(t_i)}{R(t_i)},\\ \psi_j(t_i) &= \frac{f_j(t_i)}{f(t_i)}, & \psi_j^*(t_i) &= \frac{{t_i}^2R_j(t_i)}{R(t_i)},\\ \xi_j(t_i) &= [\frac{1}{\theta_j}-{t_i}^2]\end{aligned}\right\}. \tag{9}$$

Equations (8) do not yield explicit solutions for p and θ_j, $j = 1, 2$, and have to be solved numerically to obtain the ML estimates of the three parameters, Newton-Raphson iteration is employed to solve (8). The corresponding ML estimates of the reliability function $R(t)$ and the Hazard rate function $H(t)$ are given respectively by (3) and (4) after replacing p, θ_1 and θ_2 by their ML estimates $\hat{p}, \hat{\theta}_1$ and $\hat{\theta}_2$, (the solution of the above nonlinear equations).

3. Prior, Posterior and Bayes Estimators

Recently, there has been a considerable amount of interest in the Bayesian approach in estimation and reliability studies. It has received frequent attention for analyzing failure data and other time-to-event data, and has been often proposed as a valid alternative to traditional statistical perspectives. The Bayesian approach to estimation of the parameters and reliability analysis allows prior subjective knowledge on lifetime parameters and technical information on the failure mechanism, as well as experimental data, to be incorporated into the inferential procedure. Bayesian methods usually require less sample data to achieve the same quality of inferences than methods based on sampling theory. In this section, we present the Bayesian estimation for the parameters, RF and HRF for $MTRD$ based on GOS. In this section, Bayesian estimation for the parameters of the $MTRD$ is considered under squared error and $LINEX$ (Linear-Exponential) loss functions.

3.1 Bayes Estimation Using Conjugate Prior

Let p, θ_1 and θ_2 are independent random variables such that $p \sim Beta(b_1, b_2)$ and for $j = 1, 2$, θ_j to follow an inverted gamma prior distribution with PDF

$$\pi(\theta_j) = \frac{1}{\Gamma(\alpha_j)}(\frac{\alpha_j}{\beta_j})^{\alpha_j}\theta_j^{\alpha_j-1}\exp[-\frac{\alpha_j}{\beta_j}\theta_j], \quad (\theta_j, \alpha_j, \beta_j > 0). \tag{10}$$

A joint prior density function of $\theta = (p, \theta_1, \theta_2)$ is then given by

$$\pi(\theta) = \pi_1(p)\pi_2(\theta_1)\pi_3(\theta_2),$$

$$\pi(\theta) \propto p_1^{b_1-1}p_2^{b_2-1}\prod_{j=1}^{2}\theta_j^{\alpha_j-1}\exp[-\Sigma_{j=1}^{2}\frac{\alpha_j}{\beta_j}\theta_j], \tag{11}$$

where $0 < p_1 < 1$, $p_2 = 1-p_1$ and for $j = 1, 2$ $\theta_j > 0$, $(b_j, \alpha_j, \beta_j) > 0$.
It follows, from (6) and (11), that the joint posterior density function is given by

$$\pi^*(\theta|\mathbf{t}) = A_1p_1^{b_1-1}p_2^{b_2-1}\prod_{j=1}^{2}\theta_j^{\alpha_j-1}\exp[-\Sigma_{j=1}^{2}\frac{\alpha_j}{\beta_j}\theta_j]\prod_{i=1}^{n-1}[p_1R_1(t_i)+p_2R_2(t_i)]^{m_i}$$

$$\times\prod_{i=1}^{n}[p_1f_1(t_i)+p_2f_2(t_i)][p_1R_1(t_n)+p_2R_2(t_n)]^{k-1}, \tag{12}$$

where

$$A_1^{-1} = \int_\theta \pi(\theta) L(\theta|\mathbf{t})\, d\theta. \tag{13}$$

Under the squared error loss and *LINEX* loss functions, the Bayes estimator of a function, say $\phi \equiv \phi(p, \theta_1, \theta_2)$, are given, respectively, by

$$\hat{\phi}_{BS} = E(\phi|\mathbf{t}) = \int_\theta \phi \quad \pi^*(\theta|\mathbf{t})\, d\theta, \tag{14}$$

$$\hat{\phi}_{BL} = -\frac{1}{a} \ln[E(e^{-a\phi}|\mathbf{t})] = -\frac{1}{a} \ln[\int_\theta e^{-a\phi} \quad \pi^*(\theta|\mathbf{t})\, d\theta], \tag{15}$$

where the integral is taken over the three dimensional space and $a \neq 0$. To compute the integral we propose to consider *MCMC* methods.

The conditional posterior distribution of the parameters p, θ_1 and θ_2 using conjugate prior can be computed and written, respectively, by

$$\pi^*(p|\theta_1, \theta_2, \mathbf{t}) \propto p_1^{b_1-1} p_2^{b_2-1} \prod_{i=1}^{n-1} [p_1 R_1(t_i) + p_2 R_2(t_i)]^{m_i} \prod_{i=1}^{n} [p_1 f_1(t_i) + p_2 f_2(t_i)] \times [p_1 R_1(t_n) + p_2 R_2(t_n)]^{k-1}, \tag{16}$$

$$\pi^*(\theta_1|p, \theta_2, \mathbf{t}) \propto \theta_1^{\alpha_1-1} e^{-\alpha_1\theta_1/\beta_1} \prod_{i=1}^{n-1} [p_1 R_1(t_i) + p_2 R_2(t_i)]^{m_i} \prod_{i=1}^{n} [p_1 f_1(t_i) + p_2 f_2(t_i)] \times [p_1 R_1(t_n) + p_2 R_2(t_n)]^{k-1}, \tag{17}$$

$$\pi^*(\theta_2|p, \theta_1, \mathbf{t}) \propto \theta_2^{\alpha_2-1} e^{-\alpha_2\theta_2/\beta_2} \prod_{i=1}^{n-1} [p_1 R_1(t_i) + p_2 R_2(t_i)]^{m_i} \prod_{i=1}^{n} [p_1 f_1(t_i) + p_2 f_2(t_i)] \times [p_1 R_1(t_n) + p_2 R_2(t_n)]^{k-1}. \tag{18}$$

3.2 Bayes Estimation Using Non-informative Prior

Assuming that all of the parameters consisting θ are positive and independent, and that we are indifferent about the prior information about θ so that we set improper non-informative prior to θ_j, $j = 1, 2$, and p as follows

$$\pi_1(p) \propto \frac{1}{p}, \qquad \pi_2(\theta_1) \propto \frac{1}{\theta_1}, \qquad \pi_3(\theta_2) \propto \frac{1}{\theta_2}.$$

so that

$$\pi(\theta) \propto \pi_1(p)\pi_2(\theta_1)\pi_3(\theta_2),$$

$$\pi(\theta) \propto (p\theta_1\theta_2)^{-1}, \quad (0 < p < 1, \theta_1, \theta_2 > 0). \tag{19}$$

The posterior density function can be obtained from (6) and (19), as

$$\pi^*(\theta|\mathbf{t}) = A_1 (p\theta_1\theta_2)^{-1} \prod_{i=1}^{n-1} [p_1 R_1(t_i) + p_2 R_2(t_i)]^{m_i} \times \prod_{i=1}^{n} [p_1 f_1(t_i) + p_2 f_2(t_i)][p_1 R_1(t_n) + p_2 R_2(t_n)]^{k-1}. \tag{20}$$

Under the squared error loss and *LINEX* loss functions, the Bayes estimator of a function, say $\phi \equiv \phi(p, \theta_1, \theta_2)$, are given, respectively, by

$$\hat{\phi}_{BS} = E(\phi|\mathbf{t}) = \int_\theta \phi \quad \pi^*(\theta|\mathbf{t})\, d\theta, \tag{21}$$

$$\hat{\phi}_{BL} = -\frac{1}{a} \ln[E(e^{-a\phi}|\mathbf{t})] = -\frac{1}{a} \ln[\int_\theta e^{-a\phi} \quad \pi^*(\theta|\mathbf{t})\, d\theta], \tag{22}$$

where the integral is taken over the three dimensional space and $a \neq 0$. To compute the integral we propose to consider *MCMC* methods.

The conditional posterior distribution of the parameters p, θ_1 and θ_2 using non-informative prior can be computed and written, respectively, by

$$\pi^*(p|\theta_1, \theta_2, \mathbf{t}) \propto p^{-1} \prod_{i=1}^{n-1} [p_1 R_1(t_i) + p_2 R_2(t_i)]^{m_i} \prod_{i=1}^{n} [p_1 f_1(t_i) + p_2 f_2(t_i)] \times [p_1 R_1(t_n) + p_2 R_2(t_n)]^{k-1}, \tag{23}$$

$$\pi^*(\theta_1|p,\theta_2,\mathbf{t}) \propto \theta_1^{-1}\prod_{i=1}^{n-1}[p_1R_1(t_i)+p_2R_2(t_i)]^{m_i}\prod_{i=1}^{n}[p_1f_1(t_i)+p_2f_2(t_i)]\times[p_1R_1(t_n)+p_2R_2(t_n)]^{k-1}, \quad (24)$$

$$\pi^*(\theta_2|p,\theta_1,\mathbf{t}) \propto \theta_2^{-1}\prod_{i=1}^{n-1}[p_1R_1(t_i)+p_2R_2(t_i)]^{m_i}\prod_{i=1}^{n}[p_1f_1(t_i)+p_2f_2(t_i)]\times[p_1R_1(t_n)+p_2R_2(t_n)]^{k-1}. \quad (25)$$

3.3 MCMC Method

In this subsection, the *MCMC* method is considered to compute the Bayes estimators of the parameters p, θ_1 and θ_2. We consider the *MCMC* techniques, namely the Metropolis-Hasting algorithm, to generate samples from the conditional posterior distributions and then compute the Bayes estimates. For more details about the *MCMC* methods see, for example, Upadhyay et al. (2001), Press (2003) and Upadhyay and Gupta (2010). The Metropolis-Hastings algorithm generate samples from an arbitrary proposal distribution (i.e. a Markov transition kernel). The following *MCMC* procedure is proposed to compute Bayes estimators of $\phi \equiv \phi(p,\theta_1,\theta_2)$ based on squared error and *LINEX* loss functions.

1) Start with initial guess of p ,θ_1 and θ_2 say p^0, θ_1^0 and θ_2^0, respectively.
2) Set $i = 1$.
3) Generate p from $\pi^*(p|\theta_1,\theta_2,\mathbf{t})$, θ_1 from $\pi^*(\theta_1|p,\theta_2,\mathbf{t})$. and θ_2 from $\pi^*(\theta_2|p,\theta_1,\mathbf{t})$.
4) Repeat steps 2-3 N times. Now calculate Bayes estimator of ϕ under squared and *LINEX* loss functions, respectively, by

$$E(\phi|\mathbf{t}) = (1/(N-\nu))\Sigma_{i=\nu+1}^{N}\phi(p^i,\theta_1^i,\theta_2^i), \quad (26)$$

$$E(exp(-a\phi|\mathbf{t}) = (1/(N-\nu))\Sigma_{i=\nu+1}^{N}exp(-a\phi(p^i,\theta_1^i,\theta_2^i)), \quad (27)$$

where ν is the burn-in period.

4. Numerical Computations

A comparison between *ML* and Bayes estimators, under either a squared error or a *LINEX* loss functions, is made using a Monte Carlo simulation study in the two following cases:

4.1 Progressive Type-II Censored Order Statistics

The progressive Type-II censored order statistics can be obtained from the *GOS* as a special case by taking $m_i = r_i$ for $i = 1,2,...,m-1$ and $k = r_m + 1$. Therefore, the estimation results obtained in the above sections can be specialized to the progressive Type-II censored order statistics. Estimates of parameters, *RF* and *HRF* are computed and compared based on Monte Carlo simulation study according to the following steps:

1) For given values of p, θ_1 and θ_2, we generate progressively Type-II censored samples from the *MTRD* by using the algorithm described in Balakrishnan and Sandhu (1995), as follows:
   - using p, θ_1 and θ_2, with different chooses of n, m and k. We take in our consideration that the progressive censored order statistics $T^R_{1:m:n:k}, T^R_{2:m:n:k}, ..., T^R_{m:m:n:k}$ is a progressively Type-II censored sample from a population with *CDF* (6).
   - generate m independent Uniform (0, 1) observations $w_1, w_2, ..., w_m$.
   - determine the values of the censored scheme r_i, for $i = 1,2,...,m$.
   - set $E_i = 1/(i+\Sigma_{j=m-i+1}^{m} r_j)$ for $i = 1,2,...,m$.
   - set $V_i = w_i^{E_i}$ for $i = 1,2,...,m$.
   - set $U_{i,m,n} \equiv U_i = 1 - V_m.V_{m-1}...V_{m-i+1}$ for $i = 1,2,...,m$. Then $U_1, U_2, ..., U_m$ is the progressively Type-II right censored sample from the Uniform (0,1) distribution.
   - for given values of parameters θ_1, θ_2 and mixing proportion p, set:

$$U_i = p[1-e^{-\theta_1 t_i^2}] + (1-p)[1-e^{-\theta_2 t_i^2}]. \quad (28)$$

   For $i = 1,2,...,m$, the values of t_i for each U_i can be obtained numerically from (32). Then the resulting set $t_1, t_2, ..., t_m$ is the required progressively Type-II right censored sample from the *MTRD*.

Using the algorithm described above, a progressively Type-II censored samples of size m with different censored schemes are randomly generated from sample of size n simulated from the *MTRD*.

2) The ML estimates of the parameters p, θ_1 and θ_2 are obtained by solving the nonlinear equations (8), with $m_i = r_i, k = r_m + 1, i = 1, 2, ..., m - 1$, numerically.
3) Based on squared error and $LINEX$ loss functions the Bayes estimates of the parameters, reliability and Hazard rate functions are computed, from (26) and (27), according to the above $MCMC$ method.

The above steps are repeated 500 times. The estimated risks (ER) are computed by averaging the squared deviations over the 500 repetitions.

4.2 Upper Record Values

The upper record values can be obtained from the GOS by taking $m_i = -1$ for $i = 1, 2, ..., n - 1$ and $k = 1$. In this case, ML and Bayes estimates of parameters, RF and HRF are computed and compared based on Monte Carlo simulation study according to the following steps:

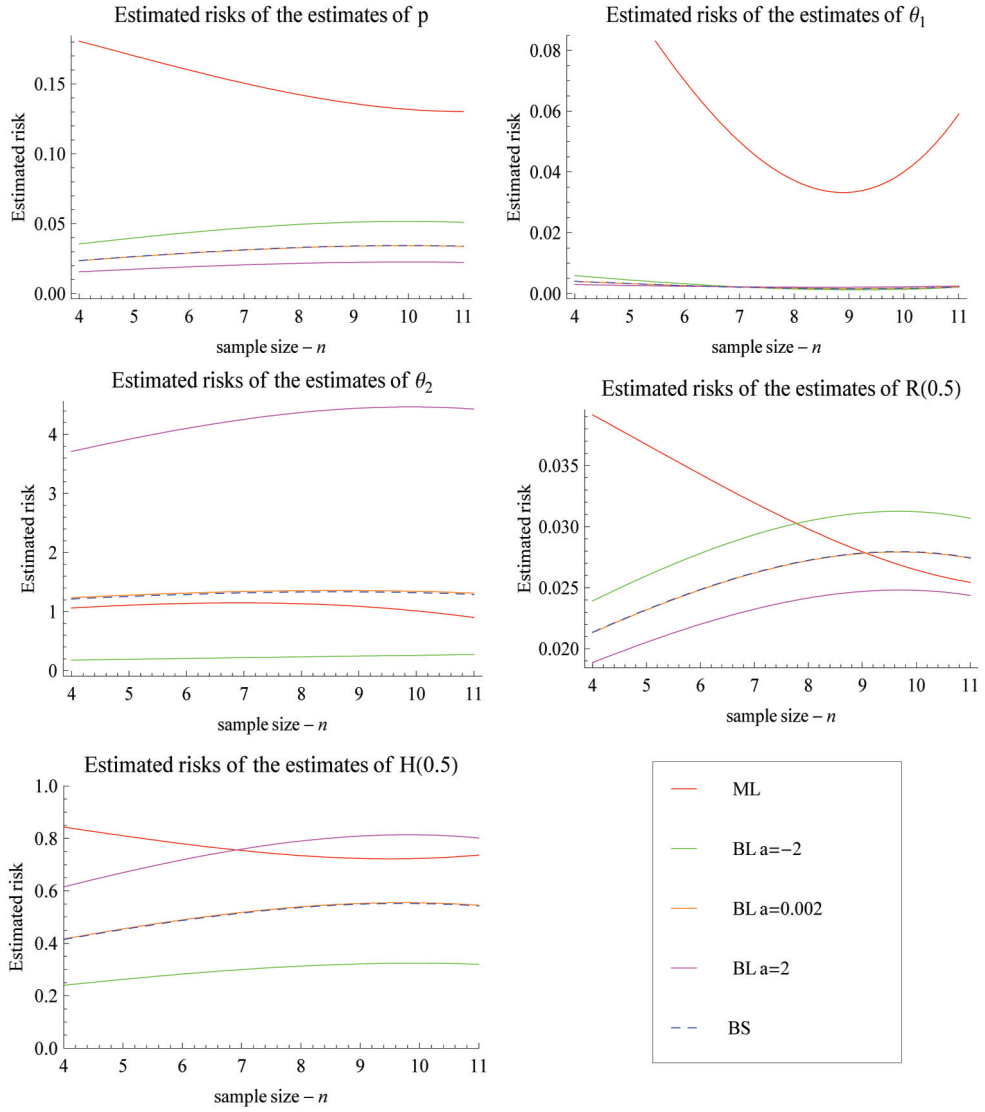

Figure 1. Estimated Risks (ER) of the estimates of $\theta = (p, \theta_1, \theta_2)$, $R(t)$ and $H(t)$ based on upper record values

1) For given values of p, θ_1 and θ_2, we generate $n = 5, 8, 10$ upper record values from the $MTRD$.
2) The ML estimates of the parameters p, θ_1 and θ_2 are obtained by solving the nonlinear equations (8), with $m_i = -1, k = 1$, numerically.
3) Based on squared error and $LINEX$ loss functions the Bayes estimates of the parameters, reliability and Hazard rate functions are computed, from (26) and (27), according to the above $MCMC$ method. The above steps are repeated 1000 times. The estimated risks (ER) are computed by averaging the squared deviations over the 1000

repetitions.

The computational (our) results were computed by using Mathematica 6.0. Under conjugate prior the prior parameters chosen as $b_1 = 1.2$, $b_2 = 2.3$, $\alpha_1 = 2$, $\beta_1 = 0.3$, $\alpha_2 = 2$, $\beta_2 = 3$ which yield the generated values of $p = 0.391789, \theta_1 = 0.307317$ and $\theta_2 = 3.33166$ (as the true values). The true values of $R(t)$ and $H(t)$ when $t = 0.5$, are computed to be $R(0.5) = 0.627253$ and $H(0.5) = 1.58232$. While, under non-informative prior values of (p, θ_1, θ_2) chosen as $(0.4, 0.3, 3)$. The true values of $(R(0.5), H(0.5))$ are computed to be $(0.654517, 1.46916)$. The value of the shape parameter a of the *LINEX* loss function is $a = (-2, 0.02, 2)$. The estimated risks (ER) are displayed in Tables 1-2-3-4. Figures 1 and 2 represents the estimated risks of the estimates in the case of upper record values.

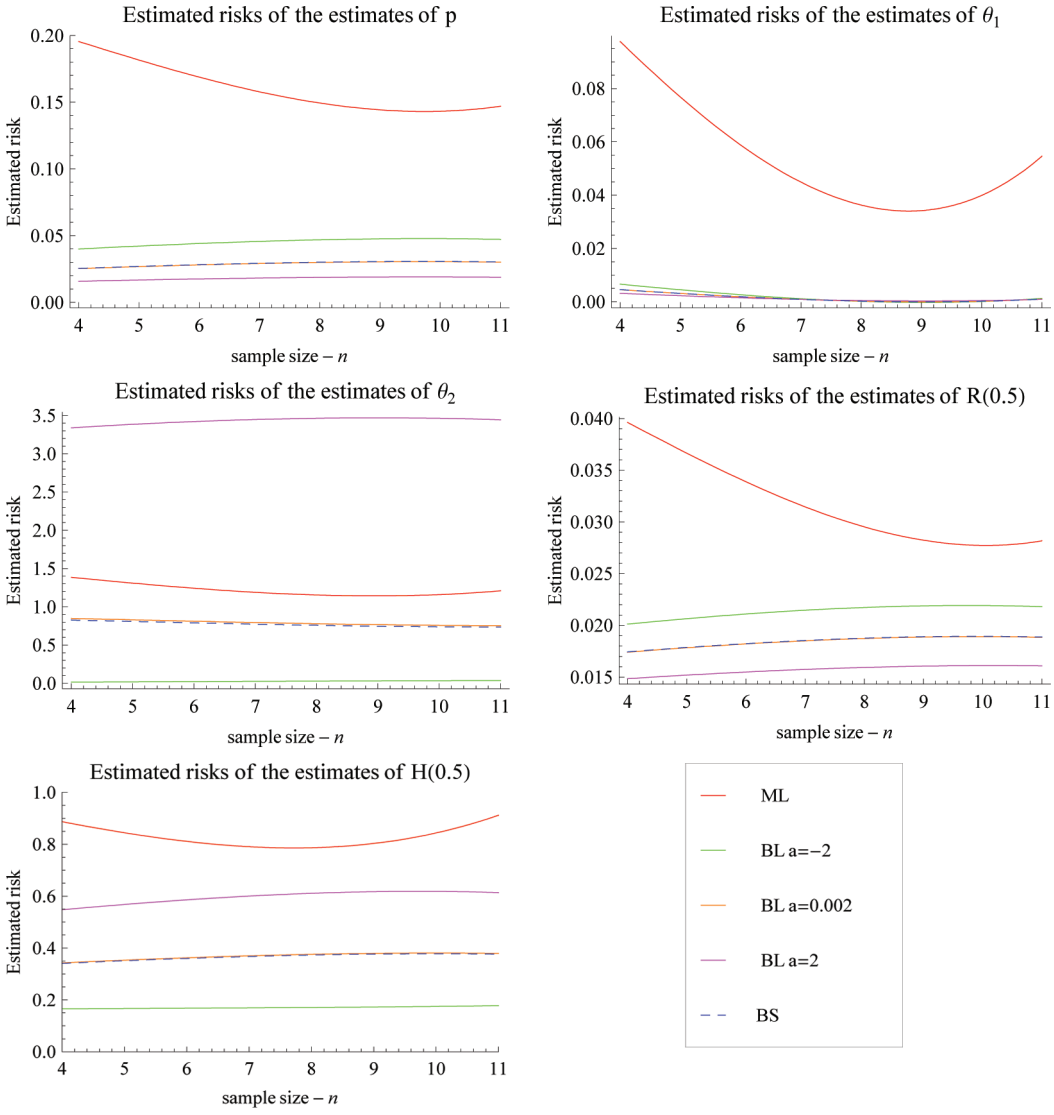

Figure 2. Estimated Risks (ER) of the estimates of $\theta = (p, \theta_1, \theta_2)$, $R(t)$ and $H(t)$ based on upper record values

5. Concluding Remarks

Based on *GOS* model, this paper proposes Bayesian and non-Bayesian approach to estimate the unknown parameters for the mixture of two-component Rayleigh model. Samples from heterogeneous populations that can be represented by a finite mixture of two-component Rayleigh model are subjected to Type II right progressively censoring and upper record values cases. The *ML* and some Bayes methods are used to estimate the parameters, reliability and hazard rate functions. From the results of the simulation study we observe that:

Table 1. (Progressively censored samples) *ER* of the *ML* estimates and the Bayes (*BS*, *BL*) estimates assuming the conjugate prior of p, θ_1, θ_2, $R(0.5)$ and $H(0.5)$ for different m, n and schemes

n	m	Scheme	*Parameters*	*ML*	*BS*	*BL*		
						$a = -2$	$a = 0.002$	$a = 2$
		$(25, 24^{*0})$	P	0.003245	0.005342	0.003536	0.005365	0.008055
			θ_1	0.67785	0.54289	0.8398	0.54036	0.3354
			θ_2	0.43825	0.11166	0.12969	0.431641	0.65218
			$R(t = 0.5)$	0.09767	0.05947	0.05757	0.05953	0.06114
			$H(t = 0.5)$	0.839173	0.40463	0.524707	0.40454	0.50510
50	25	$(24^{*0}, 25)$	P	0.003137	0.005140	0.003397	0.005161	0.007764
			θ_1	0.66734	0.54715	0.84843	0.54457	0.33613
			θ_2	0.44965	0.05612	0.13305	0.43447	0.6539
			$R(t = 0.5)$	0.09160	0.05778	0.056155	0.05784	0.05889
			$H(t = 0.5)$	0.888689	0.402153	0.52249	0.40206	0.50192
		$(12^{*0}, 25, 12^{*0})$	P	0.003495	0.005146	0.003405	0.005167	0.007768
			θ_1	0.71011	0.55162	0.085252	0.54905	0.34024
			θ_2	0.443812	0.05585	0.12879	0.42776	0.64839
			$R(t = 0.5)$	0.075615	0.04808	0.04649	0.048142	0.05914
			$H(t = 0.5)$	0.86252	0.41210	0.533473	0.412899	0.502635
		$(15, 34^{*0})$	P	0.000224	0.005202	0.003435	0.0052234	0.007861
			θ_1	0.40911	0.47685	0.3435	0.5223	0.7861
			θ_2	0.44669	0.041131	0.037051	0.043567	0.065858
			$R(t = 0.5)$	0.04352	0.036785	0.03525	0.03685	0.04780
			$H(t = 0.5)$	0.39355	0.106689	0.11706	0.106590	0.116211
50	35	$(34^{*0}, 15)$	P	0.018655	0.005111	0.0033806	0.0051322	0.007716
			θ_1	0.7927	0.048203	0.077453	0.047956	0.382002
			θ_2	0.415377	0.031153	0.013247	0.043822	0.066359
			$R(t = 0.5)$	0.064502	0.03607	0.03477	0.03613	0.05674
			$H(t = 0.5)$	0.938207	0.105817	0.116416	0.105716	0.104960
		$(17^{*0}, 15, 17^{*0})$	P	0.003343	0.0051459	0.0034011	0.0051673	0.00777
			θ_1	0.61314	0.04841	0.07767	0.048163	0.38382
			θ_2	0.43851	0.031173	0.011847	0.042113	0.06491
			$R(t = 0.5)$	0.09116	0.036801	0.035594	0.036859	0.05752
			$H(t = 0.5)$	0.46447	0.10739	0.11846	0.10729	0.10644
		$(50, 49^{*0})$	P	0.000099	0.0048152	0.0029527	0.0049369	0.007266
			θ_1	0.31932	0.050734	0.081869	0.050475	0.030445
			θ_2	0.41668	0.010226	0.01062	0.038718	0.059884
			$R(t = 0.5)$	0.007391	0.004301	0.004435	0.0047075	0.005429
			$H(t = 0.5)$	0.302130	0.105334	0.10585	0.105236	0.105243
100	50	$(49^{*0}, 50)$	P	0.000214	0.005003	0.0032048	0.005125	0.007610
			θ_1	0.385417	0.050729	0.082766	0.053472	0.032663
			θ_2	0.42668	0.010134	0.0115348	0.038483	0.058634
			$R(t = 0.5)$	0.007874	0.004659	0.004571	0.004606	0.005314
			$H(t = 0.5)$	0.355113	0.105033	0.105358	0.105139	0.102095
		$(20^{*0}, 50, 29^{*0})$	P	0.000371	0.005176	0.003086	0.0055098	0.007265
			θ_1	0.320971	0.053405	0.081019	0.052142	0.034789
			θ_2	0.42669	0.010182	0.011466	0.040447	0.06167
			$R(t = 0.5)$	0.006237	0.004372	0.004597	0.00468	0.005720
			$H(t = 0.5)$	0.31214	0.10685	0.10756	0.10675	0.10678
		$(20, 79^{*0})$	P	0.000014	0.004085	0.002361	0.0041057	0.006683
			θ_1	0.277822	0.045999	0.075848	0.045744	0.024999
			θ_2	0.346678	0.001115	0.003173	0.033384	0.055403
			$R(t = 0.5)$	0.006750	0.003761	0.003604	0.003767	0.005002
			$H(t = 0.5)$	0.26320	0.104754	0.115234	0.090465	0.094745
100	80	$(79^{*0}, 20)$	P	0.000149	0.004237	0.002465	0.004259	0.006902
			θ_1	0.28209	0.04592	0.075811	0.045668	0.02492
			θ_2	0.346678	0.001565	0.002703	0.032736	0.054899
			$R(t = 0.5)$	0.006451	0.003902	0.003727	0.003908	0.005041
			$H(t = 0.5)$	0.241003	0.104896	0.125183	0.094801	0.095045
		$(40^{*0}, 20, 39^{*0})$	P	0.000149	0.004085	0.002361	0.004106	0.006683
			θ_1	0.277822	0.045999	0.075847	0.045744	0.024999
			θ_2	0.346678	0.0011146	0.003172	0.033384	0.055403
			$R(t = 0.5)$	0.006750	0.003761	0.003604	0.003767	0.004866
			$H(t = 0.5)$	0.255123	0.112615	0.112844	0.092519	0.092512

BL ≡ Bayes (*LINEX* loss function); *BS* ≡ Bayes (squared error loss function).

Table 2. (Progressively censored samples) ER of the ML estimates and the Bayes (BS, BL) estimates assuming the noninformative prior of p, θ_1, θ_2, $R(0.5)$ and $H(0.5)$ for different m, n and schemes

n	m	Scheme	*Parameters*	ML	BS	BL		
						$a = -2$	$a = 0.002$	$a = 2$
		$(25, 24^{*0})$	P	0.000275	0.004360	0.00633	0.009191	0.006308
			θ_1	0.91462	0.08854	0.05759	0.03607	0.05785
			θ_2	1	0.045487	0.014611	0.005531	0.011142
			$R(t = 0.5)$	0.018702	0.009628	0.00990	0.011494	0.00989
			$H(t = 0.5)$	0.867288	0.21134	0.184647	0.184727	0.184773
50	25	$(24^{*0}, 25)$	P	0.000269	0.004403	0.006372	0.0092145	0.006348
			θ_1	0.975792	0.090199	0.05867	0.036548	0.058939
			θ_2	1	0.048138	0.015518	0.005860	0.00554
			$R(t = 0.5)$	0.019622	0.009806	0.010031	0.011569	0.010023
			$H(t = 0.5)$	0.925593	0.215921	0.18774	0.186478	0.187879
		$(12^{*0}, 25, 12^{*0})$	P	0.000515	0.004465	0.006469	0.009359	0.006444
			θ_1	0.93515	0.08980	0.058226	0.036283	0.058495
			θ_2	1	0.047101	0.015102	0.005692	0.005514
			$R(t = 0.5)$	0.018367	0.009844	0.010099	0.011683	0.010090
			$H(t = 0.5)$	0.853737	0.216007	0.188208	0.187661	0.188343
		$(15, 34^{*0})$	P	0.000276	0.004458	0.006457	0.009343	0.006433
			θ_1	0.922268	0.091183	0.059727	0.037789	0.059996
			θ_2	1	0.047677	0.015509	0.005901	0.005562
			$R(t = 0.5)$	0.018958	0.009883	0.010152	0.011745	0.010143
			$H(t = 0.5)$	0.882936	0.21764	0.190233	0.189985	0.19037
50	35	$(34^{*0}, 15)$	P	0.000474	0.004481	0.006509	0.009444	0.006485
			θ_1	0.93531	0.090395	0.059027	0.037214	0.059295
			θ_2	1	0.046569	0.015072	0.005726	0.005638
			$R(t = 0.5)$	0.01830	0.009859	0.0101645	0.011806	0.010155
			$H(t = 0.5)$	0.84993	0.216491	0.189719	0.190196	0.189845
		$(17^{*0}, 15, 17^{*0})$	P	0.000401	0.004443	0.006431	0.009301	0.006407
			θ_1	0.92685	0.090563	0.058979	0.036915	0.059251
			θ_2	1	0.048117	0.015634	0.005935	0.005563
			$R(t = 0.5)$	0.018387	0.009863	0.01011	0.011674	0.010101
			$H(t = 0.5)$	0.85276	0.217056	0.189157	0.188294	0.189295
		$(50, 49^{*0})$	P	0.000227	0.004266	0.006066	0.008953	0.006142
			θ_1	0.90859	0.089084	0.057725	0.034868	0.057993
			θ_2	1	0.044692	0.013094	0.005427	0.00523
			$R(t = 0.5)$	0.017199	0.0096387	0.009712	0.010704	0.010103
			$H(t = 0.5)$	0.80632	0.210004	0.18875	0.181498	0.18888
100	50	$(49^{*0}, 50)$	P	0.000155	0.004133	0.006185	0.009111	0.006061
			θ_1	0.91742	0.08977	0.057582	0.035839	0.057848
			θ_2	1	0.044688	0.013437	0.005256	0.005364
			$R(t = 0.5)$	0.017731	0.009612	0.009754	0.011512	0.00995
			$H(t = 0.5)$	0.80107	0.21064	0.18720	0.18678	0.18733
		$(20^{*0}, 50, 29^{*0})$	P	0.000224	0.004239	0.006158	0.009153	0.006234
			θ_1	0.88808	0.089878	0.057612	0.035851	0.057879
			θ_2	1	0.044694	0.013433	0.00535	0.005299
			$R(t = 0.5)$	0.016517	0.009669	0.010438	0.011751	0.010134
			$H(t = 0.5)$	0.801348	0.210222	0.185989	0.187079	0.188118
		$(20, 79^{*0})$	P	0.000187	0.0043384	0.006149	0.0091904	0.006126
			θ_1	0.90281	0.088638	0.059406	0.035764	0.054673
			θ_2	1	0.045035	0.013406	0.037764	0.005473
			$R(t = 0.5)$	0.0176969	0.009685	0.010029	0.0115935	0.0100201
			$H(t = 0.5)$	0.804349	0.210819	0.188241	0.1879	0.188375
100	80	$(79^{*0}, 20)$	P	0.000273	0.004381	0.006298	0.009207	0.006270
			θ_1	0.912554	0.088357	0.059200	0.035603	0.005566
			θ_2	1	0.046335	0.013266	0.005369	0.011089
			$R(t = 0.5)$	0.017935	0.009684	0.010169	0.011788	0.010159
			$H(t = 0.5)$	0.80684	0.210059	0.190006	0.190344	0.190134
		$(40^{*0}, 20, 39^{*0})$	P	0.000257	0.004345	0.006243	0.009233	0.006219
			θ_1	0.894361	0.088981	0.059850	0.036156	0.055114
			θ_2	1	0.045609	0.013513	0.005398	0.00138
			$R(t = 0.5)$	0.017017	0.009658	0.010140	0.011747	0.010131
			$H(t = 0.5)$	0.80785	0.210085	0.190130	0.190330	0.190262

$BL \equiv$ Bayes ($LINEX$ loss function); $BS \equiv$ Bayes (squared error loss function).

1) All of the results obtained in this article in case of Type II right progressively censoring can be specialized to

both the complete sample case by taking ($m = n, r_i = 0, i = 1, 2, 3, 4, ...m$), and the Type-II right censored sample for ($r_i = 0, i = 1, 2, 3, 4, ...m - 1, r_m = n - m$).

2) From Table 1, we see that the Bayes estimators relative to asymmetric loss functions ($LINEX$), are sensitive to the value of the shape parameters a, for small a the Bayes estimates under the $LINEX$ loss function (BL), are very close to the Bayes estimates under the squared error loss function. The Bayes estimates of RF under the $LINEX$ loss function (BL), when $a = -2$ have the smallest $ER's$ as compared with their corresponding estimates. While, the Bayes estimates of θ_1, θ_2 and HRF have the smallest $ER's$ as compared with their corresponding ML estimates. In most of the considered cases, it is observed that the ML estimates of p have the smallest $ER's$ as compared with their corresponding Bayes estimates.

3) Also from Table 1, as the proportion m/n increases, the $ER's$ reduce significantly. We observed that for large sample size, the symmetric and asymmetric Bayes estimates are better than the ML estimates.

4) Table 2 shows that the Bayes estimates of θ_1, θ_2 and HRF under the $LINEX$ loss function have the smallest $ER's$ as compared with their corresponding estimates. While, the Bayes estimates of RF under the squared error loss function have the smallest $ER's$ as compared with their corresponding estimates. In most of the considered cases, Table 2 shows that the ML estimates of p have the smallest $ER's$ as compared with their corresponding Bayes estimates.

Table 3. (Upper record values) ER of the ML estimates and the Bayes (BS, BL) estimates assuming the conjugate prior of p, θ_1, θ_2, $R(0.5)$ and $H(0.5)$ for different sample size

n	*Parameters*	*ML*	*BS*	*BL*		
				$a = -2$	$a = 0.002$	$a = 2$
5	P	0.170122	0.0264868	0.0397904	0.0263757	0.0174207
	θ_1	0.0956744	0.00328384	0.00447608	0.00327487	0.00266551
	θ_2	1.109	1.2566	0.19321	1.27885	3.91814
	$R(t = 0.5)$	0.0367232	0.0232053	0.0259773	0.023178	0.020526
	$H(t = 0.5)$	0.810279	0.45288	0.262485	0.455084	0.669544
8	P	0.142407	0.0329758	0.0494921	0.0328373	0.0216422
	θ_1	0.0371982	0.00174507	0.00150609	0.00174812	0.00212429
	θ_2	1.13347	1.33192	0.234972	1.35535	4.37182
	$R(t = 0.5)$	0.0298045	0.027248	0.0304673	0.0272163	0.0241562
	$H(t = 0.5)$	0.734561	0.536832	0.313168	0.539407	0.790005
10	P	0.131772	0.0344254	0.0516252	0.0342812	0.0226098
	θ_1	0.0400329	0.00174828	0.00139018	0.00175243	0.00223124
	θ_2	1.01259	1.32074	0.261058	1.3437	4.46605
	$R(t = 0.5)$	0.0264525	0.0279305	0.0312235	0.0278982	0.0247848
	$H(t = 0.5)$	0.723828	0.551947	0.32409	0.554589	0.813456

$BL \equiv$ Bayes ($LINEX$ loss function); $BS \equiv$ Bayes (squared error loss function).

Table 4. (Upper record values) ER of the ML estimates and the Bayes (BS, BL) estimates assuming the non-informative prior of p, θ_1, θ_2, $R(0.5)$ and $H(0.5)$ for different sample size

n	*Parameters*	*ML*	*BS*	*BL*		
				$a = -2$	$a = 0.002$	$a = 2$
5	P	0.181571	0.0269282	0.0421724	0.0268018	0.0167192
	θ_1	0.0768179	0.00311924	0.00447972	0.0031081	0.0022672
	θ_2	1.30989	0.808531	0.0191031	0.830045	3.385
	$R(t = 0.5)$	0.036651	0.0178594	0.0206431	0.0178319	0.0151835
	$H(t = 0.5)$	0.844474	0.350973	0.166689	0.35319	0.567843
8	P	0.149364	0.0300759	0.0468695	0.0299361	0.0187382
	θ_1	0.0362808	0.000218835	0.000147729	0.000220181	0.00043849
	θ_2	1.15463	0.758679	0.0296448	0.778903	3.46421
	$R(t = 0.5)$	0.0295189	0.0187535	0.021728	0.0187244	0.0159262
	$H(t = 0.5)$	0.786627	0.373189	0.170778	0.375601	0.610974
10	P	0.143208	0.0306175	0.0477189	0.0304752	0.0190788
	θ_1	0.0399092	0.000177435	0.0000962643	0.000178922	0.00041641
	θ_2	1.15942	0.738898	0.0342423	0.758505	3.4634
	$R(t = 0.5)$	0.0277292	0.0189317	0.0219134	0.0189025	0.0161039
	$H(t = 0.5)$	0.84348	0.378136	0.174893	0.380563	0.618301

$BL \equiv$ Bayes ($LINEX$ loss function); $BS \equiv$ Bayes (squared error loss function).

5) From Tables 3 and 4, we see that in most of the considered cases, the Bayes estimates of p, θ_1, RF and HRF have the smallest $ER's$ as compared with their corresponding ML estimates. While, the Bayes estimates of θ_2 under the $LINEX$ loss function (BL) when $a = -2$ have the smallest $ER's$ as compared with their corresponding estimates. Also, it is observed that the Bayes estimates of p and RF under the $LINEX$ loss function perform best when the value of the shape parameters a is large. While, the Bayes estimates of θ_1, θ_2 and HRF under the $LINEX$ loss function perform best when the value of the shape parameters a is small.

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