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Comparison of Test Statistics for Testing the Regression Coefficients in the Ridge, Liu and Kibria-Lukman Logistic Regression Models: Simulation and Application

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Abstract

Ridge, Liu and Kibria- Lukman regression are methods that have been proposed to solve the multicollinearity problem for both linear and non-linear regression models. This paper studies different Ridge, Liu and Kibria-Lukman regression z-type tests of the individual coefficients for logistic regression model. A simulation study was conducted to evaluate and compare the performance of the test statistics with respect to their empirical sizes and powers under different simulation conditions. Our simulations allowed us to identify among the proposed tests, which ones maintain type I error rates close to the 5% nominal level, while at the at same time showing considerable gain in statistical power over the standard Wald z-test commonly used in logistic regression model. Our paper is the first of its kind in comparing the z-type tests for these different shrinkage approaches to estimation in logistic regression. The results will be of value for applied statisticians and researchers in the area of regression models.

Keywords: empirical power, Kibria-Lukman regression, logistic regression, Liu regression, ridge regression, simulation study, type I error rate

1. Introduction

Multicollinearity occurs when there are high inter-correlations among independent variables within a multiple regression model. In such cases, the use of ordinary least square (OLS) estimation methods can lead to unstable and unreliable regression coefficient estimates. Several methods have been proposed in literature to address the instability of coefficients caused by multicollinearity. Among the most popular are Ridge, Liu, and Kibria-Lukman regression methods.

Ridge regression was pioneered by Hoerl and Kennard (1970). They found that there is a nonzero value of k (ridge or shrinkage parameter) for which mean square error (MSE) for the ridge regression estimator is smaller than the variance of the ordinary least squares (OLS) estimator. Since ridge regression estimator is a non-linear function of ridge parameter k, Liu (1993) proposed another class of biased estimators for the regression coefficients that also had a smaller MSE than the OLS estimator for certain values of the shrinkage parameter d. More recently, Kibria and Lukman (2021) proposed another biased estimator with similar good properties. These three estimators have been extended beyond linear regression to generalized linear models, see for examples: Schaefer, et al. (1984), Le Cessie(1992) ,Vago and Kemmeny (2006), Kibria, Manson and Shukur (2012), among others.

Liu, Ridge and KL estimators depend on a tuning/shrinkage parameter, whose estimation is a vital issue. Some researchers at different period of times have worked in this area of research and proposed different estimators for the tuning/shrinkage parameter, for example, in the case of ridge regression Hoerl and Kennard (1970), Kibria (2003), Khalaf and Shukur (2005), Muniz et al. (2012), Mansson et al. (2010) Saleh et al. (2019) among many others. In the case of Liu regression Liu (1993), Shukur, Månsson, and Sjolander (2014), and Qasim, Amin and Omer (2020), among others.

Another important aspect of statistical inference is hypothesis testing. In the context of a regression model, we focus on testing the statistical significance of the coefficients of the model. This testing allows us to discriminate which variables are significant predictors of model and which do not, in other words, it facilitates variable selection.

In literature the work on the test statistics for testing the regression coefficients in ridge linear regression models is based on the foundational paper of Halawa and El-Bassiouni (2000). Halawa and El-Bassiouni (2000) proposed non exact t-tests for the regression coefficients under ridge regression estimation and compared empirical sizes and powers of the tests, based on the estimator of k proposed by Hoerl and Kennard (1970) and Hoerl, Kennard and Baldwin (1975). Their results evidenced that for models with large standard errors, the ridge-based t-tests have correct sizes with considerable gain in powers over those of the least squares t-test. After them other researchers have ventured into this area, for example: Gokpinar and Ebegil (2016), Kibria and Banik (2019) and Kibria and Perez-Melo (2020). In all cases the tests were found to be reliable for certain rules of choosing the shrinkage parameter, showing correct Type I error and gains in power over the ordinary least squares-based tests. Particularly, Kibria and Perez-Melo (2020) studied forty different ridge regression t-type tests of the individual coefficients of a linear regression model. Among forty tests, seven performed better than the rest in terms of achieving higher power gains while maintaining a 5% nominal size.

As for Liu linear regression, there is, to our knowledge, only one article that deals with the issue of testing of hypothesis for coefficients, namely: Immadulah, Aslam and Altaf (2017), in which they propose to use the same non-exact t-statistics originally defined by Halawa and El-Bassiouni (2000). Immadulah, Aslam and Altaf (2017) do not go into any mathematical proofs nor provide any simulation for the validity of the tests beyond simply proposing them.

For Kibria-Lukman estimator, there is no literature on testing hypotheses for the model's coefficients.

On the other hand, the area of testing of coefficients for shrinkage generalized linear models remains practically unexplored, except for one paper by Cule et al. (2011) that extended the non-exact t-type tests of Hallawa and El-Bassiouni (2000) to ridge logistic regression with satisfactory results in Type I error and power over the regular Wald tests based on unpenalized MLE estimators, for various values of the shrinkage parameter. The study by Cule et al. (2011) however does not compare the performance of the tests under different rules of estimation of the shrinkage parameter. No other paper has been published on this topic since, and no other generalized linear model beyond logistic regression has been considered.

Clearly much work needs to be done for the logistic regression model, which is vital for applications. Therefore, our goal in this paper will be to further extend the research into non-exact t-type tests of significance for coefficients to logistic regression models, under ridge, Liu and KL estimators.

The rest of the paper is organized as follows. The proposed test statistics for the logistic regression model are described in Section 2. To compare the performance of the test statistics, a Monte Carlo simulation study has been conducted in Section 3. A real-life dataset is analyzed in Section 4. Finally, some concluding remarks are provided in Section 5.

2. Test Statistics for Logistic Regression Coefficients

The term generalized linear model (GLM) refers to a larger class of models popularized by McCullagh and Nelder (1982, 2nd edition 1989). In these models, the response variable Y_i is assumed to follow an exponential family distribution with mean μ_i, which is assumed to be some (often nonlinear) function of $x_i^T\beta$, where β is (q x 1) dimensional unknown coefficient vector.

There are three components to any GLM:

- Random Component – refers to the probability distribution of the response variable (Y); e.g. normal distribution for *Y* in the linear regression, or Bernoulli distribution for *Y* in the logistic regression
- Systematic Component - specifies the explanatory variables $X_1, X_2 \ldots X_q$ in the model, more specifically their linear combination in creating the so called *linear predictor*; e.g., $X^T\beta$
- *Link Function* g(μ) - specifies the link between random and systematic components. It says how the expected value of the response relates to the linear predictor of explanatory variables; e.g., $\eta = g(\mu) = X^T\beta$

In the case of logistic regression $Y_i \sim Bernoulli\ (\pi_i)$

$$\eta = g(\mu) = logit(\mu) = log\left(\frac{\pi}{1-\pi}\right) = X^T\beta$$

Hence,

$$\pi_i = \frac{e^{x_i\beta}}{1+e^{x_i\beta}}$$

where x_i is the i-th row of X matrix which is an n ×(q+1) data matrix with q explanatory (or independent) variables and β is a (q+1) vector of coefficients.

The coefficients of the model are estimated from the data through maximum likelihood estimation via numerical methods (iteratively weighted least squares), resulting in,

$$\hat{\beta}_{ML} = \left(X^T\widehat{W}X\right)^{-1}X^T\widehat{W}\hat{z},$$

where $\widehat{W} = \widehat{\pi_i}(1-\widehat{\pi_i})$ and $\hat{z}$ is a vector where the i-th element equals,

$$\widehat{z_i} = \log(\widehat{\pi_i}) + \frac{y_i-\widehat{\pi_i}}{\widehat{\pi_i}(1-\widehat{\pi_i})}.$$

To test whether the i-th component of the parameter vector β is equal to zero, the Wald test is used based on the MLE estimator:

$$H_0: \beta_i = 0 \; versus \; H_1: \beta_i \neq 0$$

$$t = \frac{\widehat{\beta_i}}{S(\widehat{\beta_i})}$$

where $\widehat{\beta_i}$ is the i-th component of $\hat{\beta}_{ML}$ and $S(\widehat{\beta_i})$ is the square root of the ith diagonal element of $Var\left(\hat{\beta}_{ML}\right)$, the estimated variance-covariance matrix of the last step of the iteratively weighted least squares algorithm. For large sample size and under the null hypothesis, this test statistic is distributed as N(0,1). However, when X^TX is ill conditioned due to multicollinearity, the maximum likelihood estimator in (2.2) produces unstable estimators with unduly large sampling variance.

2.1 Logistic Ridge Regression

In the case of linear regression, adding a constant *k* to the diagonal elements of X^TX improves the ill conditioned situation, which is known ridge regression method. Ridge estimation has been adapted to logistic regression by Schaeffer et all (1984) and the ridge regression of the parameter vector β is obtained as,

$$\hat{\beta}_{(k)} = \left(X^T\widehat{W}X + kI_n\right)^{-1}X^T\widehat{W}X\,\widehat{\beta_{ML}}$$

where *k* > 0 is the ridge, biasing or shrinkage parameter.

The variance-covariance matrix of $\hat{\beta}_{(k)}$ is obtained as follows:

$$Var\left(\widehat{\beta_{(k)}}\right) = F_k Var(\widehat{\beta_{ML}})F_k^T$$

where $F_k = \left(X^T\widehat{W}X + kI_n\right)^{-1}X^T\widehat{W}X$

To test whether the *i*-th component of the parameter vector β is equal to zero, Halawa and Bassouni (2000) proposed the following t-test statistic based on the ridge estimator of the parameter vector:

$$t_k = \frac{\hat{\beta}_{i(k)}}{S(\hat{\beta}_{i(k)})} \quad (2.8)$$

where $\hat{\beta}_{i(k)}$ is the ith element of $\hat{\beta}_{(k)}$ and $S(\hat{\beta}_{i(k)})$ is the square root of the ith diagonal element of $Var\left(\widehat{\beta_{(k)}}\right)$.

Under the null hypothesis, the test statistic (2.8) is assumed to be asymptotically N(0,1) . For more details on this topic, we refer our readers to Halawa and Bassiouni (2000) and Cule et all (2011) among others.

Since the shrinkage parameter k is unknown, it needs to be estimated from observed data. There are many methods that exist in literature to estimate the shrinkage parameter k. However, this section presents forty different ridge shrinkage parameter regression estimators considered in our simulation study for the test statistic defined in (2.8). Table 2.1 below shows the estimators. For the detailed derivation of the estimators, we refer readers to the corresponding original papers listed in the references.

Table 2.1. Ridge shrinkage parameter estimators

Authors	Ridge estimator formula (k)
Hoerl and Kennard (1970)	$k_1 = \widehat{\sigma^2} \Big/ \max(\hat{\alpha}^2)$
Schaffer et all (1984)	$k_2 = 1/\max(\hat{\alpha}^2)$
Kibria (2003)	$k_3 = Geometric\ Mean\ (\widehat{\sigma^2}/\widehat{\alpha}_j^2)$
Kibria (2003)	$k_4 = Median(\widehat{\sigma^2}/\widehat{\alpha}_j^2)$
Muniz and Kibria (2009)	$k_5 = \max\left(1/\left(\widehat{\sigma^2}/\widehat{\alpha}_j^2\right)^{\frac{1}{2}}\right)$
Muniz and Kibria (2009)	$k_6 = \max\left(\left(\widehat{\sigma^2}/\widehat{\alpha}_j^2\right)^{\frac{1}{2}}\right)$
Muniz and Kibria (2009)	$k_7 = Geometric\ Mean(1/\left(\widehat{\sigma^2}/\widehat{\alpha}_j^2\right)^{\frac{1}{2}})$
Muniz and Kibria (2009)	$k_8 = Geometric\ Mean(\left(\widehat{\sigma^2}/\widehat{\alpha}_j^2\right)^{\frac{1}{2}})$
Muniz and Kibria (2009)	$k_9 = Median(1/\left(\widehat{\sigma^2}/\widehat{\alpha}_j^2\right)^{\frac{1}{2}})$
Muniz and Kibria (2009)	$k_9 = Median(\left(\widehat{\sigma^2}/\widehat{\alpha}_j^2\right)^{\frac{1}{2}})$
Muniz et all (2012)	$k_{11} = \max\left(1/g_j\right)$, $where\ g_j = (\lambda_{max}\widehat{\sigma^2})/((n-q)\widehat{\sigma^2} + \lambda_{max}\,\widehat{\alpha}_j^2)$
Muniz et all (2012)	$k_{12} = \max(g_j)$
Muniz et all (2012)	$k_{13} = Geometric\ Mean(1/g_j)$
Muniz et all (2012)	$k_{14} = \text{Geometric Mean}\ (g_j)$
Muniz et all (2012)	$k_{15} = \text{median}(1/g_j)$
Muniz et all (2012)	$k_{16} = \max(g_j)$
Suggested by a colleague	$k_{17} = 0.1$
Asar et all (2014)	$k_{18} = (q^2\widehat{\sigma^2})/(\lambda_{max}^2 \sum_{j=1}^{q} \widehat{\alpha}_j^2)$
Asar et all (2014)	$k_{19} = (q^3\widehat{\sigma^2})/(\lambda_{max}^3 \sum_{j=1}^{q} \widehat{\alpha}_j^2)$
Asar et all (2014)	$k_{20} = (q\widehat{\sigma^2})/(\lambda_{max}^{1/3} \sum_{j=1}^{q} \widehat{\alpha}_j^2)$
Asar et all (2014)	$k_{21} = (q\widehat{\sigma^2}) \Big/ \{\left(\sum_{j=1}^{q} \sqrt{\lambda_j}\right)^{\frac{1}{3}} \sum_{j=1}^{q} \widehat{\alpha}_j^2\}$

Asar et all (2014)	$k_{22} = (2q\widehat{\sigma^2})/(\lambda_{max}^{1/2}\sum_{j=1}^{q}\widehat{\alpha}_j^2)$
Alkhamisi and Shukur (2007)	$k_{23} = \max(\widehat{\sigma^2}/\widehat{\alpha}_j^2 + 1/\lambda_j)$
Nomura (1988)	$k_{24} = q\widehat{\sigma^2} \Big/ \sum_{j=1}^{q}\left[\widehat{\alpha}_j^2 \Big/ \left\{1 + \left(1 + \lambda_j\left(\left(\widehat{\alpha}_j^2/\widehat{\sigma^2}\right)\right)^{\frac{1}{2}}\right)\right\}\right]$
Goktas and Sevinc (2016)	$k_{25} = \sqrt{Median((\widehat{\sigma^2}/\widehat{\alpha}_j^2)^{\frac{1}{2}})}$
Goktas and Sevinc (2016)	$k_{26} = \widehat{\sigma^2} \Big/ \left(Median\left((\widehat{\sigma^2}/\widehat{\alpha}_j^2)^{\frac{1}{2}}\right)\right)^2$
Khalaf (2012)	$k_{27} = k_1 + 2/(\lambda_{max} + \lambda_{min})$
Dorugade (2014)	$k_{28} = ArithmeticMean((2\widehat{\sigma^2})/(\lambda_{max}\ \widehat{\alpha}_j^2))$
Dorugade (2014)	$k_{29} = Median\ ((2\widehat{\sigma^2})/(\lambda_{max}\ \widehat{\alpha}_j^2))$
Dorugade (2014)	$k_{30} = Harmonic\ Mean((2\widehat{\sigma^2})/(\lambda_{max}\ \widehat{\alpha}_j^2))$
Dorugade (2014)	$k_{31} = Geometric\ Mean((2\widehat{\sigma^2})/(\lambda_{max}\ \widehat{\alpha}_j^2))$

Here, $\hat{\alpha}$ is defined as:

$$\hat{\alpha} = P^T\hat{\beta}$$

where P is an orthonormal matrix that satisfies $P^TX^T\widehat{W}XP = \Lambda$, and Λ is a diagonal matrix of eigenvalues (λ_j j = 1, 2,…, q) of $X^T\widehat{W}X$, and $\widehat{\sigma^2}$ is the residual variance of the raw residuals divided by n-q-1 degrees of freedom . For details see Mansson (2011)

2.2 Liu Logistic Regression

The solution proposed by Liu (1993) to the issue of multicollinearity in linear regression was extended to logistic regression by Mansson, Kibria and Shukur (2011) as follows:

$$\hat{\beta}_{(d)} = \left(X^T\widehat{W}X + I_q\right)^{-1}\left(X^T\widehat{W}X + dI_q\right)\widehat{\beta_{ML}} \quad (2.9)$$

where 0≤d≤1 is the tuning or shrinkage parameter. The proposed estimator has been shown in the literature to achieve a smaller mean square error than the regular MLE estimator under collinearity conditions, see Manson, Kibria and Shukur (2011)

The variance-covariance matrix of $\hat{\beta}_{(d)}$is:

$$Var\left(\hat{\beta}_{(d)}\right) = F_dVar(\widehat{\beta_{ML}})F_d^T$$
$$where\ F_d = \left(X^T\widehat{W}X + I_q\right)^{-1}\left(X^T\widehat{W}X + dI_q\right)$$

To test whether the i-th component of the parameter vector β is equal to zero, Immadulah, Aslam and Altaf (2017) proposed to follow Halawa and Bassouni (2000). The approach of Halawa and Bassouni (2000) is to use the following t-test statistic based on the shrinkage (in this case Liu type) estimator of the parameter vector:

$$t_d = \frac{\widehat{\beta}_{i(d)}}{S(\widehat{\beta}_{i(d)})} \quad (2.13)$$

where $\hat{\beta}_{i(d)}$ is the ith element of $\hat{\beta}_{(d)}$, and $S(\hat{\beta}_{i(d)})$ is the square root of the i-th diagonal element of $Var\left(\widehat{\beta_{(d)}}\right)$. Under the null hypothesis, the test statistic (2.13) was shown to be asymptotically distributed as N(0,1). For more details on this topic, see Halawa and Bassiouni (2000) and Cule et all (2011).

Since the shrinkage parameter d is unknown, it needs to be estimated from observed data. This section gives the formulas for the fifteen different Liu shrinkage parameter regression estimators considered in our simulation study for the test statistic defined in (2.13). Table 2.2 below shows the estimators. For details on how the estimators were derived, we refer the readers to the corresponding original papers that are available in the list of references.

Table 2.2. Liu shrinkage parameter estimators

Authors	**Tuning Parameter Estimator Formula (d)**
Mansson, Kibria and Shukur (2011)	$D1 = max\left(0, \frac{\hat{\alpha}_{max}^2 - 1}{\frac{1}{\lambda_{max}} + \hat{\alpha}_{max}^2}\right)$
Mansson, Kibria and Shukur (2011)	$D2 = max\left(0, median\left(\frac{\hat{\alpha}_j^2 - 1}{\frac{1}{\lambda_j} + \hat{\alpha}_j^2}\right)\right)$
Mansson, Kibria and Shukur (2011)	$D3 = max\left(0, mean\left(\frac{\hat{\alpha}_j^2 - 1}{\frac{1}{\lambda_j} + \hat{\alpha}_j^2}\right)\right)$
Mansson, Kibria and Shukur (2011)	$D4 = max\left(0, max\left(\frac{\hat{\alpha}_j^2 - 1}{\frac{1}{\lambda_j} + \hat{\alpha}_j^2}\right)\right)$
Mansson, Kibria and Shukur (2011)	$D5 = max\left(0, min\left(\frac{\hat{\alpha}_j^2 - 1}{\frac{1}{\lambda_j} + \hat{\alpha}_j^2}\right)\right)$

Here, $\hat{\alpha}$ is defined as:

$$\hat{\alpha} = P^T\hat{\beta}$$

where P is an orthonormal matrix that satisfies $P^T X^T \widehat{W} X P = \Lambda$, and Λ is a diagonal matrix of eigenvalues (λ_j j = 1, 2,…, q) of $X^T\widehat{W}X$

2.3 Kibria-Lukman Logistic Regression

The solution proposed by Kibria and Lukman (2020) to the issue of multicollinearity in linear regression was extended to logistic regression in Lukman et all (2023) as such:

$$\hat{\beta}_{(k)} = \left(X^T\widehat{W}X + kI_n\right)^{-1}\left(X^T\widehat{W}X - kI_n\right)\hat{\beta} = W(k)M(k)\widehat{\beta_{ML}} \quad (2.14)$$

where $k > 0$ is the shrinkage parameter and $\widehat{\beta_{ML}}$ is the MLE estimator.

The variance-covariance matrix expression of $\hat{\beta}_{(k)}$ is given as follows:

$$Var\left(\widehat{\beta_{(k)}}\right) = W(k)M(k)Var(\widehat{\beta_{ML}})M^{-1}(k)W^{-1}(k) \quad (2.16)$$

To test whether the *i*-th component of the parameter vector β is equal to zero, we propose, following Halawa and Bassouni (2000), the following t-test statistic based on the Kibria-Lukman estimator of the parameter vector:

$$t_k = \frac{\hat{\beta}_{i(k)}}{S(\hat{\beta}_{i(k)})} \quad (2.18)$$

where $\hat{\beta}_{i(k)}$ is the *i*-th element of $\hat{\beta}_{(k)}$ and $S(\hat{\beta}_{i(k)})$ is the square root of the *i*-th diagonal element of $Var\left(\widehat{\beta_{(k)}}\right)$.

Under the null hypothesis, the test statistic (2.18) has been shown to be asymptotically distributed as a N(0,1). For more details on this topic, see Halawa and Bassiouni (2000) and Cule et all (2011).

Since the shrinkage parameter k is unknown, it needs to be estimated from the observed data. This section gives the formulas for two different Kibria-Lukman shrinkage parameter regression estimators considered in our simulation study for the test statistic defined in (2.18). Table 2.3 below shows the estimators. For details on how the estimators were derived, we refer the readers to the corresponding original papers that are available in the list of references.

Table 2.3. Kibria-Lukman shrinkage parameter estimators

Authors	**Tuning Parameter Estimator Formula (k)**
Lukman et all (2023)	$K1 = \min(\frac{1}{\widehat{\alpha_i^2}})$
Lukman et all (2023)	$K2 = min\left[\frac{\lambda_i}{1+2\lambda_i\widehat{\alpha_i^2}}\right]$

Here, $\hat{\alpha}$ is defined as:

$$\hat{\alpha} = P^T\hat{\beta}$$

where P is an orthonormal matrix that satisfies $P^TX^T\widehat{W}XP = \Lambda$, and Λ is a diagonal matrix of eigenvalues (λ_j j = 1, 2,…, q) of $X^T\widehat{W}X$

3. Simulation Study

To calculate the value of test statistics, we considered each of the shrinkage rules in the previous tables and thus obtained 39 different values of the test statistics. Since a theoretical assessment among the test statistics was not possible, a simulation study will be conducted to evaluate the performances of the suggested tests in the following section.

Our simulation study has two parts. First, we analyzed the empirical Type I error of the tests. The test statistics that achieved the nominal size of close to 5% will be kept, and the ones that deviated significantly from the 5% size will be discarded. The second part of the simulation study compared the tests statistics that achieved 5% nominal size, in regard to statistical power gains over the Wald test.

3.1 Type I Error Rates Simulation Procedure

R version 4.1.2 was used for all calculations and simulations. For the empirical Type I error simulation and the power of the test, we considered sample sizes n = 30, 50, 100 and 300 the number of regressors q = 4, 6 and 10. To see the effects of multicollinearity we assumed a compound symmetry , or exchangeable, correlation matrix among the regressors, with ρ = 0.6, 0.8 and 0.95. The data $n \times q$ matrix X was created as H $\Lambda^{0.5}$ G^T, where H is any ($n \times q$) matrix whose columns are orthogonal, Λ is the diagonal matrix of eigenvalues of the correlation matrix, and G is the matrix of normalized eigenvectors of the correlation matrix, respectively.

Halawa and Bassiouni (2020), based their simulations on both the most favorable (MF) and least favorable (LF) direction of β for model (1). The MF orientation of β corresponds to the normalized eigenvector of the largest eigenvalue of the matrix X^TX, which is a vector of the form $(1/\sqrt{q})1_q$.The LF orientation is a unit vector orthogonal to $(1/\sqrt{q})1_q$. For a detailed explanation of MF and LF directions of β and other details of the simulation procedure, please see the paper by Halawa and Bassiouni (2020). They determined theoretically and empirically, and this has been confirmed by the other papers investigating these tests, that the LF orientation produces tests which keep the Type I nominal level no matter what estimator is used, see for example, Perez-Melo and Kibria (2020). On the other hand, some tests present a higher Type I error than they are supposed to, under the MF orientation of β. Therefore, from a practical standpoint the MF orientation is the one that allows us to discard tests that are overly liberal in rejecting the null hypothesis, therefore our simulation will be based on the MF orientation of β.

To estimate the 5% nominal size (α = 0.05) for testing $H_0: \beta_i = 0$ $versus$ $H_1: \beta_i \neq 0$ under different conditions, 3000 pseudo random vectors of n observations are obtained from the $Bernoulli\ (\pi_i)$ distribution, where

$$\pi_i = \frac{e^{x_i\beta}}{1+e^{x_i\beta}}$$

where x_i is the i-th row of X which is the n ×(q+1) data matrix with q explanatory (or independent) variables and β is a (q+1) vector of coefficients , as above.

Without loss of any generality, we let zero intercept for the model. Under the null model, substituting the i-th element of the considered MF β by zero. The estimated sizes were computed as the percentage of times the absolute values of all selected test statistics were greater than the critical value of $Z_{0.025}$ = 1.96.

3.2 Type I Error Rates Simulation Results

In Tables 3.1 , 3.2 and 3,3, we recorded the empirical sizes of the tests for the MF orientation for correlation levels of 0.60, 0.80 and 0.95, respectively.

Table 3.1. Simulated Type I errors for ρ = 0.60 and α=0.05, MF orientation

	q=4				q=6				q=10				rho=0.60
Statistics	n=30	n=50	n=100	n=300	n=30	n=50	n=100	n=300	n=30	n=50	n=100	n=300	Average
tK0	0.038	0.042	0.033	0.045	0.039	0.036	0.039	0.05	0.029	0.055	0.063	0.05	4.3%
tK1	0.044	0.047	0.053	0.064	0.046	0.042	0.054	0.069	0.03	0.064	0.077	0.047	5.3%
tK2	0.046	0.047	0.051	0.064	0.045	0.039	0.053	0.068	0.025	0.054	0.077	0.048	5.1%
tK3	0.04	0.048	0.048	0.056	0.043	0.042	0.059	0.071	0.037	0.069	0.078	0.064	5.5%
tK4	0.039	0.045	0.049	0.056	0.04	0.037	0.057	0.069	0.033	0.062	0.078	0.06	5.2%
tK5	0.045	0.06	0.057	0.063	0.062	0.06	0.091	0.08	0.07	0.121	0.112	0.083	7.5%
tK6	0.044	0.047	0.056	0.062	0.048	0.053	0.068	0.074	0.057	0.104	0.115	0.079	6.7%
tK7	0.04	0.053	0.054	0.056	0.055	0.047	0.065	0.071	0.046	0.084	0.082	0.064	6.0%
tK8	0.035	0.04	0.044	0.056	0.037	0.033	0.048	0.062	0.023	0.051	0.069	0.047	4.5%
tK9	0.041	0.052	0.056	0.057	0.056	0.051	0.072	0.07	0.056	0.091	0.081	0.064	6.2%
tK10	0.039	0.039	0.042	0.052	0.038	0.033	0.049	0.06	0.023	0.049	0.072	0.046	4.5%
tK11	0.052	0.066	0.058	0.068	0.064	0.075	0.097	0.088	0.069	0.134	0.136	0.096	8.4%
tK12	0.036	0.04	0.031	0.045	0.041	0.027	0.038	0.05	0.037	0.067	0.067	0.045	4.4%
tK13	0.047	0.065	0.058	0.068	0.049	0.071	0.096	0.088	0.045	0.101	0.134	0.096	7.7%
tK14	0.033	0.038	0.032	0.045	0.027	0.029	0.038	0.05	0.018	0.031	0.065	0.046	3.8%
tK15	0.05	0.066	0.058	0.068	0.059	0.073	0.096	0.088	0.047	0.119	0.134	0.096	8.0%
tK16	0.037	0.04	0.032	0.045	0.038	0.032	0.038	0.05	0.026	0.051	0.065	0.046	4.2%
tK17	0.035	0.04	0.033	0.043	0.039	0.033	0.039	0.05	0.022	0.048	0.066	0.044	4.1%
tK18	0.037	0.042	0.031	0.044	0.039	0.035	0.04	0.05	0.028	0.055	0.065	0.047	4.3%
tK19	0.037	0.043	0.034	0.045	0.04	0.034	0.04	0.052	0.028	0.054	0.064	0.047	4.3%
tK20	0.037	0.043	0.034	0.044	0.04	0.034	0.039	0.053	0.029	0.054	0.065	0.043	4.3%
tK21	0.037	0.042	0.032	0.044	0.04	0.034	0.04	0.052	0.029	0.054	0.064	0.045	4.3%
tK22	0.037	0.043	0.034	0.044	0.039	0.035	0.04	0.051	0.028	0.054	0.065	0.046	4.3%
tK23	0.049	0.059	0.058	0.064	0.058	0.078	0.091	0.086	0.076	0.130	0.133	0.089	8.1%
tK24	0.041	0.041	0.042	0.057	0.038	0.033	0.048	0.063	0.024	0.052	0.071	0.044	4.6%
tK25	0.037	0.041	0.045	0.054	0.038	0.033	0.052	0.06	0.025	0.053	0.071	0.048	4.6%
tK26	0.042	0.053	0.051	0.057	0.041	0.053	0.06	0.069	0.043	0.080	0.109	0.068	6.1%
tK27	0.042	0.05	0.062	0.059	0.049	0.045	0.066	0.072	0.034	0.071	0.084	0.055	5.7%
tK28	0.038	0.045	0.047	0.057	0.052	0.04	0.057	0.066	0.037	0.075	0.09	0.066	5.6%
tK29	0.038	0.041	0.042	0.046	0.039	0.036	0.043	0.056	0.027	0.053	0.065	0.049	4.5%
tK30	0.037	0.043	0.034	0.044	0.039	0.035	0.04	0.051	0.028	0.054	0.065	0.046	4.3%
tK31	0.037	0.039	0.035	0.051	0.041	0.034	0.045	0.057	0.027	0.054	0.064	0.046	4.4%
tD1	0.038	0.041	0.033	0.045	0.039	0.036	0.039	0.05	0.029	0.055	0.064	0.05	4.3%
tD2	0.038	0.041	0.034	0.047	0.04	0.036	0.044	0.051	0.028	0.058	0.068	0.053	4.5%
tD3	0.037	0.044	0.034	0.05	0.041	0.031	0.042	0.057	0.026	0.059	0.068	0.049	4.5%
tD4	0.038	0.041	0.033	0.045	0.04	0.036	0.04	0.05	0.029	0.056	0.064	0.051	4.4%
tD5	0.036	0.041	0.044	0.054	0.048	0.035	0.056	0.068	0.028	0.065	0.076	0.055	5.1%
tKL1	0.039	0.045	0.035	0.066	0.04	0.035	0.042	0.064	0.028	0.054	0.064	0.046	4.7%
tKL2	0.037	0.042	0.031	0.049	0.039	0.035	0.04	0.051	0.028	0.055	0.064	0.046	4.3%

Table 3.2. Simulated Type I errors for ρ = 0.80 and α=0.05, MF orientation

	q=4				q=6				q=10				rho=0.80
Statistics	n=30	n=50	n=100	n=300	n=30	n=50	n=100	n=300	n=30	n=50	n=100	n=300	Average
tK0	0.046	0.049	0.055	0.05	0.043	0.04	0.049	0.046	0.028	0.062	0.044	0.054	4.7%
tK1	0.06	0.069	0.075	0.08	0.052	0.05	0.05	0.07	0.034	0.054	0.056	0.067	6.0%
tK2	0.058	0.067	0.074	0.084	0.049	0.052	0.055	0.07	0.034	0.056	0.056	0.067	6.0%
tK3	0.047	0.065	0.065	0.075	0.055	0.056	0.066	0.08	0.042	0.078	0.074	0.092	6.6%
tK4	0.043	0.058	0.067	0.066	0.05	0.046	0.064	0.073	0.04	0.064	0.07	0.083	6.0%
tK5	0.055	0.076	0.078	0.073	0.098	0.102	0.108	0.112	0.092	0.190	0.178	0.167	11.1%
tK6	0.049	0.071	0.068	0.078	0.076	0.07	0.09	0.097	0.062	0.126	0.118	0.145	8.8%
tK7	0.045	0.075	0.07	0.066	0.081	0.074	0.088	0.099	0.072	0.128	0.132	0.131	8.8%
tK8	0.045	0.061	0.063	0.076	0.044	0.044	0.057	0.069	0.032	0.056	0.064	0.074	5.7%
tK9	0.046	0.075	0.072	0.069	0.085	0.084	0.095	0.1	0.074	0.138	0.134	0.133	9.2%
tK10	0.044	0.06	0.061	0.076	0.042	0.044	0.055	0.066	0.032	0.056	0.06	0.071	5.6%
tK11	0.058	0.083	0.088	0.076	0.098	0.1	0.126	0.117	0.092	0.200	0.196	0.182	11.8%
tK12	0.047	0.042	0.052	0.05	0.065	0.056	0.046	0.048	0.046	0.110	0.054	0.054	5.6%
tK13	0.046	0.081	0.086	0.076	0.077	0.083	0.125	0.117	0.064	0.156	0.182	0.181	10.6%
tK14	0.019	0.047	0.053	0.05	0.023	0.027	0.047	0.048	0.028	0.063	0.038	0.053	4.1%
tK15	0.054	0.082	0.088	0.076	0.094	0.098	0.125	0.117	0.06	0.154	0.186	0.181	11.0%
tK16	0.045	0.047	0.053	0.05	0.04	0.038	0.047	0.048	0.034	0.054	0.044	0.053	4.6%
tK17	0.044	0.047	0.054	0.06	0.037	0.042	0.046	0.047	0.036	0.048	0.046	0.053	4.7%
tK18	0.046	0.049	0.055	0.05	0.042	0.038	0.051	0.047	0.028	0.060	0.046	0.053	4.7%
tK19	0.046	0.049	0.054	0.05	0.042	0.038	0.051	0.047	0.028	0.060	0.046	0.053	4.7%
tK20	0.046	0.05	0.053	0.052	0.04	0.036	0.05	0.048	0.028	0.060	0.046	0.053	4.7%
tK21	0.046	0.05	0.053	0.052	0.04	0.036	0.05	0.048	0.028	0.060	0.046	0.055	4.7%
tK22	0.046	0.05	0.053	0.052	0.042	0.038	0.05	0.047	0.028	0.060	0.046	0.055	4.7%
tK23	0.059	0.083	0.083	0.075	0.098	0.102	0.117	0.118	0.1	0.194	0.192	0.174	11.6%
tK24	0.047	0.054	0.059	0.075	0.04	0.04	0.05	0.064	0.036	0.050	0.048	0.055	5.2%
tK25	0.044	0.063	0.065	0.077	0.047	0.048	0.064	0.074	0.032	0.064	0.072	0.089	6.2%
tK26	0.056	0.063	0.067	0.071	0.055	0.058	0.07	0.073	0.048	0.080	0.102	0.124	7.2%
tK27	0.054	0.077	0.071	0.08	0.062	0.056	0.071	0.081	0.038	0.070	0.068	0.084	6.8%
tK28	0.044	0.067	0.065	0.076	0.051	0.048	0.075	0.072	0.046	0.098	0.066	0.056	6.4%
tK29	0.046	0.051	0.06	0.062	0.039	0.038	0.047	0.049	0.028	0.060	0.046	0.056	4.9%
tK30	0.046	0.05	0.053	0.052	0.042	0.038	0.05	0.047	0.028	0.060	0.046	0.053	4.7%
tK31	0.047	0.05	0.059	0.062	0.038	0.038	0.05	0.048	0.028	0.056	0.046	0.053	4.8%
tD1	0.046	0.049	0.055	0.05	0.043	0.04	0.05	0.046	0.028	0.062	0.046	0.054	4.7%
tD2	0.048	0.047	0.056	0.052	0.041	0.038	0.05	0.048	0.028	0.060	0.046	0.06	4.8%
tD3	0.048	0.049	0.057	0.054	0.041	0.036	0.054	0.05	0.032	0.062	0.05	0.061	5.0%
tD4	0.046	0.049	0.055	0.05	0.043	0.04	0.05	0.047	0.028	0.062	0.046	0.054	4.8%
tD5	0.047	0.07	0.065	0.072	0.05	0.036	0.069	0.092	0.044	0.074	0.074	0.107	6.7%
tKL1	0.046	0.054	0.06	0.073	0.039	0.038	0.052	0.068	0.028	0.060	0.046	0.059	5.2%
tKL2	0.046	0.05	0.054	0.059	0.042	0.038	0.051	0.049	0.028	0.060	0.046	0.052	4.8%

Table 3.3. Simulated Type I errors for ρ = 0.95 and α=0.05, MF orientation

	q=4				q=6				q=10				rho=0.95
Statistics	n=30	n=50	n=100	n=300	n=30	n=50	n=100	n=300	n=30	n=50	n=100	n=300	Average
tK0	0.042	0.052	0.05	0.05	0.06	0.038	0.042	0.052	0.022	0.062	0.056	0.049	4.8%
tK1	0.062	0.072	0.066	0.078	0.056	0.046	0.046	0.055	0.018	0.080	0.053	0.047	5.7%
tK2	0.056	0.068	0.068	0.076	0.06	0.046	0.048	0.055	0.016	0.082	0.053	0.047	5.6%
tK3	0.052	0.082	0.06	0.078	0.06	0.058	0.062	0.079	0.032	0.094	0.069	0.078	6.7%
tK4	0.056	0.076	0.066	0.091	0.06	0.062	0.06	0.084	0.024	0.086	0.063	0.071	6.7%
tK5	0.084	0.104	0.09	0.102	0.124	0.162	0.136	0.132	0.158	0.256	0.263	0.259	15.6%
tK6	0.066	0.088	0.07	0.107	0.096	0.106	0.098	0.119	0.076	0.156	0.161	0.193	11.1%
tK7	0.082	0.104	0.086	0.098	0.13	0.148	0.12	0.127	0.15	0.236	0.244	0.239	14.7%
tK8	0.06	0.076	0.062	0.079	0.06	0.05	0.062	0.072	0.02	0.082	0.06	0.07	6.3%
tK9	0.082	0.102	0.086	0.1	0.124	0.152	0.128	0.128	0.156	0.242	0.245	0.246	14.9%
tK10	0.062	0.074	0.062	0.079	0.056	0.054	0.054	0.071	0.02	0.082	0.061	0.064	6.2%
tK11	0.084	0.104	0.092	0.102	0.124	0.162	0.138	0.131	0.156	0.256	0.265	0.257	15.6%
tK12	0.066	0.07	0.04	0.048	0.072	0.066	0.056	0.041	0.062	0.134	0.096	0.041	6.6%
tK13	0.0625	0.107	0.093	0.102	0.148	0.158	0.143	0.131	0.138	0.244	0.251	0.259	15.3%
tK14	0.039	0.035	0.042	0.049	0.068	0.042	0.026	0.044	0.031	0.092	0.047	0.041	4.6%
tK15	0.078	0.1	0.092	0.102	0.13	0.152	0.134	0.131	0.128	0.196	0.259	0.259	14.7%
tK16	0.058	0.044	0.046	0.049	0.06	0.048	0.04	0.044	0.028	0.076	0.055	0.04	4.9%
tK17	0.05	0.056	0.052	0.053	0.052	0.04	0.044	0.046	0.016	0.076	0.054	0.045	4.9%
tK18	0.042	0.052	0.05	0.05	0.062	0.038	0.042	0.05	0.022	0.062	0.056	0.048	4.8%
tK19	0.042	0.052	0.05	0.05	0.062	0.038	0.042	0.049	0.022	0.062	0.056	0.048	4.8%
tK20	0.042	0.052	0.05	0.051	0.06	0.038	0.042	0.048	0.022	0.064	0.055	0.046	4.8%
tK21	0.042	0.052	0.05	0.051	0.06	0.038	0.042	0.047	0.022	0.066	0.056	0.046	4.8%
tK22	0.042	0.052	0.05	0.05	0.062	0.038	0.042	0.048	0.022	0.062	0.055	0.047	4.8%
tK23	0.084	0.106	0.092	0.102	0.124	0.164	0.14	0.132	0.162	0.256	0.268	0.26	15.8%
tK24	0.042	0.062	0.046	0.063	0.058	0.04	0.046	0.05	0.02	0.066	0.054	0.043	4.9%
tK25	0.07	0.082	0.07	0.094	0.076	0.074	0.066	0.1	0.04	0.100	0.085	0.104	8.0%
tK26	0.052	0.08	0.058	0.077	0.078	0.07	0.068	0.087	0.04	0.104	0.098	0.103	7.6%
tK27	0.086	0.092	0.066	0.098	0.088	0.074	0.07	0.097	0.032	0.094	0.074	0.079	7.9%
tK28	0.06	0.07	0.06	0.075	0.064	0.056	0.066	0.064	0.032	0.096	0.08	0.09	6.8%
tK29	0.044	0.05	0.05	0.053	0.06	0.042	0.042	0.047	0.022	0.062	0.053	0.046	4.8%
tK30	0.042	0.052	0.05	0.05	0.062	0.038	0.042	0.048	0.022	0.062	0.055	0.047	4.8%
tK31	0.046	0.056	0.05	0.054	0.06	0.038	0.042	0.047	0.02	0.064	0.055	0.045	4.8%
tD1	0.042	0.052	0.05	0.05	0.06	0.038	0.042	0.052	0.022	0.062	0.056	0.049	4.8%
tD2	0.042	0.052	0.052	0.053	0.062	0.038	0.042	0.052	0.022	0.062	0.056	0.049	4.9%
tD3	0.048	0.06	0.05	0.055	0.062	0.04	0.046	0.055	0.024	0.066	0.06	0.055	5.2%
tD4	0.042	0.052	0.05	0.05	0.06	0.038	0.042	0.052	0.022	0.062	0.056	0.049	4.8%
tD5	0.062	0.078	0.056	0.078	0.08	0.07	0.072	0.083	0.046	0.108	0.113	0.122	8.1%
tKL1	0.042	0.054	0.056	0.068	0.062	0.038	0.044	0.051	0.022	0.066	0.056	0.046	5.0%
tKL2	0.042	0.054	0.052	0.054	0.062	0.04	0.042	0.05	0.022	0.062	0.055	0.046	4.8%

If the true Type I error rate is 5%, then, for a simulation based on 3000 runs, the observed Type I error will be in the following interval 99.99% of the times $0.05 \pm 6\sqrt{\frac{0.05\times0.95}{3000}} \approx (3\ \%, 7\ \%)$. Based on that we discarded the tests with an observed average Type I error exceeding 7%.

From Tables 4 to 6, we observed the following:

i) The tests based on the estimators K5 to K7, K9, K11, K13, K15 , K23 , K25 to K27 and D5 have Type I errors above the 5% nominal size in the case of the MF orientation and therefore cannot be recommended

The rest of the tests (including the test based on the MLE) were, on average, very close to or below the nominal size of 5% for different sample sizes, number of variables, and levels of correlation analyzed. These tests are the ones that will be compared in terms of statistical power below. We also carried out simulations for nominal sizes of 10% and 1%, and the behavior of the tests was consistent with what was observed for a nominal size of 5%.

3.3 Statistical Power Simulation Procedure

After discarding the tests which exceed the nominal size of 5% significantly, the remaining test statistics will be compared in terms of power. Following the paper by Gokpinar and Ebegil (2016), we will replace the *i*-th component of the β vector by $J\,w(0)\beta_i$, where *J* is a whole positive number, and $w^2(0) = (1 + (q-2)\rho)/[(1-\rho)(1+(q-1)\rho)]$. We will pick *J* such that for each combination of correlation level and number of predictors at the maximum power achieved by the most powerful of the tests is 100%. This will allow for a better comparison.

Based on 3000 simulation runs, the powers of the tests will be computed by the proportion of times the absolute value of the test statistic exceeded the critical value $Z_{0.025.}$ All combinations of sample sizes of *n* = 30, 50, 100, 300 and number of regressors *q* = 4, 6, 10 will be considered under correlation levels of 0.60, 0.80 and 0.95, respectively. The tests that show highest power gains will then be recommended to statistical practitioners.

3.4 Statistical Power Simulation Results

We recorded the empirical statistical power of the tests for the MF orientation for correlation levels of 0.60 ,0.80 and 0.95 in Tables 3.4, 3.5 and 3.6, respectively. We used only the MF orientation for the power comparisons since the LF orientation is of no use to discard any tests, according to previous papers, see Perez-Melo and Kibria (2020).

Table 3.4. Powers of tests for ρ = 0.60 and α =0.05

	q=4				q=6				q=10			
Statistics	n=30	n=50	n=100	n=300	n=30	n=50	n=100	n=300	n=30	n=50	n=100	n=300
tK0	0.651	0.885	0.934	0.964	0.435	0.872	0.925	0.961	0.074	0.827	0.950	0.973
tK1	0.639	0.977	0.997	1	0.357	0.924	0.999	1	0.053	0.636	0.997	1
tK2	0.633	0.977	0.997	1	0.356	0.923	0.999	1	0.051	0.629	0.994	1
tK3	0.636	0.975	0.996	0.999	0.351	0.919	0.999	1	0.05	0.633	0.998	1
tK4	0.633	0.976	0.997	0.999	0.344	0.915	0.997	0.998	0.046	0.620	0.997	1
tK8	0.635	0.968	0.996	0.999	0.339	0.914	0.999	0.998	0.044	0.619	0.995	1
tK10	0.634	0.968	0.995	0.999	0.334	0.915	0.996	0.997	0.044	0.615	0.993	0.999
tK12	0.626	0.934	0.955	0.968	0.336	0.878	0.957	0.966	0.05	0.623	0.972	0.976
tK14	0.646	0.92	0.953	0.968	0.324	0.853	0.952	0.966	0.041	0.597	0.955	0.976
tK16	0.608	0.918	0.954	0.968	0.331	0.865	0.949	0.966	0.039	0.605	0.958	0.976
tK17	0.616	0.941	0.98	0.993	0.309	0.861	0.974	0.987	0.036	0.583	0.964	0.987
tK18	0.67	0.914	0.949	0.98	0.429	0.876	0.944	0.972	0.062	0.795	0.947	0.974
tK19	0.671	0.921	0.959	0.986	0.426	0.878	0.948	0.978	0.057	0.784	0.948	0.977
tK20	0.671	0.921	0.957	0.985	0.427	0.879	0.95	0.981	0.055	0.768	0.951	0.979
tK21	0.669	0.919	0.954	0.984	0.425	0.878	0.949	0.978	0.055	0.771	0.950	0.979
tK22	0.671	0.921	0.958	0.985	0.429	0.877	0.946	0.976	0.061	0.790	0.948	0.975

tK24	0.652	0.967	0.995	0.999	0.349	0.919	0.995	0.997	0.042	0.622	0.990	0.999
tK28	0.663	0.958	0.988	0.997	0.378	0.908	0.986	0.993	0.046	0.673	0.988	0.998
tK29	0.67	0.941	0.974	0.993	0.411	0.892	0.961	0.983	0.051	0.748	0.955	0.984
tK30	0.671	0.921	0.958	0.985	0.429	0.877	0.946	0.976	0.061	0.790	0.948	0.975
tK31	0.677	0.944	0.979	0.994	0.41	0.895	0.971	0.987	0.047	0.736	0.961	0.986
tD1	0.651	0.887	0.936	0.964	0.435	0.873	0.925	0.961	0.074	0.827	0.951	0.973
tD2	0.656	0.895	0.941	0.975	0.437	0.882	0.937	0.974	0.075	0.831	0.955	0.98
tD3	0.666	0.903	0.949	0.979	0.443	0.892	0.944	0.977	0.078	0.833	0.959	0.985
tD4	0.652	0.888	0.937	0.964	0.435	0.873	0.927	0.962	0.074	0.828	0.951	0.973
tKL1	0.669	0.94	0.98	0.993	0.36	0.879	0.964	0.984	0.029	0.693	0.949	0.981
tKL2	0.664	0.919	0.952	0.978	0.411	0.878	0.943	0.971	0.057	0.772	0.947	0.975

Table 3.5. Powers of tests for ρ = 0.80 and α =0.05

Statistics	q=4				q=6				q=10			
	n=30	n=50	n=100	n=300	n=30	n=50	n=100	n=300	n=30	n=50	n=100	n=300
tK0	0.355	0.592	0.712	0.797	0.173	0.62	0.734	0.8	0.018	0.553	0.810	0.884
tK1	0.664	0.973	0.997	1	0.31	0.898	0.994	0.996	0.019	0.478	0.977	0.997
tK2	0.661	0.971	0.997	1	0.312	0.895	0.993	0.995	0.019	0.480	0.974	0.997
tK3	0.643	0.959	0.996	0.998	0.303	0.885	0.997	0.999	0.021	0.503	0.995	0.999
tK4	0.636	0.959	0.995	0.997	0.288	0.866	0.995	0.996	0.02	0.479	0.985	0.997
tK8	0.629	0.961	0.996	0.999	0.274	0.863	0.995	1	0.016	0.438	0.981	0.999
tK10	0.628	0.956	0.995	1	0.259	0.85	0.994	0.999	0.014	0.423	0.973	1
tK12	0.612	0.914	0.886	0.855	0.29	0.864	0.926	0.861	0.021	0.519	0.986	0.923
tK14	0.586	0.84	0.864	0.852	0.268	0.77	0.877	0.854	0.018	0.448	0.897	0.913
tK16	0.512	0.833	0.865	0.851	0.2	0.752	0.877	0.855	0.011	0.353	0.897	0.913
tK17	0.533	0.868	0.955	0.975	0.197	0.749	0.941	0.961	0.007	0.349	0.907	0.97
tK18	0.379	0.638	0.765	0.853	0.176	0.628	0.76	0.833	0.017	0.523	0.807	0.889
tK19	0.387	0.647	0.77	0.862	0.173	0.628	0.763	0.843	0.017	0.520	0.808	0.89
tK20	0.4	0.692	0.825	0.897	0.171	0.648	0.791	0.879	0.011	0.475	0.815	0.904
tK21	0.402	0.695	0.29	0.9	0.172	0.65	0.792	0.88	0.011	0.471	0.816	0.904
tK22	0.395	0.677	0.813	0.881	0.174	0.63	0.777	0.855	0.015	0.507	0.808	0.894
tK24	0.573	0.919	0.986	0.994	0.215	0.795	0.963	0.983	0.011	0.364	0.922	0.988
tK28	0.547	0.888	0.942	0.977	0.247	0.792	0.936	0.96	0.013	0.440	0.934	0.983
tK29	0.476	0.794	0.898	0.948	0.188	0.677	0.84	0.902	0.011	0.460	0.818	0.911
tK30	0.395	0.677	0.813	0.881	0.174	0.63	0.777	0.855	0.015	0.507	0.808	0.894
tK31	0.473	0.805	0.903	0.962	0.191	0.69	0.854	0.912	0.011	0.441	0.827	0.923
tD1	0.355	0.592	0.713	0.798	0.173	0.62	0.734	0.8	0.018	0.553	0.810	0.884
tD2	0.365	0.611	0.736	0.834	0.178	0.629	0.752	0.845	0.018	0.553	0.817	0.9
tD3	0.385	0.637	0.763	0.855	0.19	0.646	0.779	0.862	0.019	0.559	0.829	0.912
tD4	0.356	0.596	0.714	0.801	0.174	0.62	0.735	0.806	0.018	0.553	0.810	0.886
tKL1	0.462	0.814	0.922	0.957	0.141	0.654	0.847	0.917	0.007	0.348	0.814	0.918
tKL2	0.403	0.69	0.805	0.88	0.165	0.642	0.782	0.858	0.012	0.468	0.809	0.898

Table 3.6. Powers of tests for ρ = 0.95 and α =0.05

	q=4				q=6				q=10			
Statistics	n=30	n=50	n=100	n=300	n=30	n=50	n=100	n=300	n=30	n=50	n=100	n=300
tK0	0.051	0.275	0.375	0.404	0.015	0.263	0.44	0.625	0.002	0.110	0.455	0.619
tK1	0.569	0.957	0.997	0.985	0.155	0.703	0.988	0.994	0.009	0.259	0.920	0.997
tK2	0.576	0.958	0.997	0.985	0.155	0.722	0.988	0.995	0.008	0.266	0.932	0.975
tK3	0.545	0.93	0.984	0.979	0.151	0.721	0.992	0.998	0.007	0.287	0.967	0.995
tK4	0.537	0.932	0.982	0.98	0.151	0.712	0.989	0.995	0.007	0.263	0.944	0.99
tK8	0.535	0.952	0.997	1	0.143	0.698	0.996	1	0.006	0.232	0.956	1
tK10	0.533	0.95	0.996	1	0.136	0.682	0.995	1	0.005	0.216	0.929	0.999
tK12	0.541	0.946	0.963	0.691	0.151	0.707	0.996	0.907	0.008	0.297	0.981	0.973
tK14	0.512	0.914	0.907	0.65	0.158	0.712	0.983	0.864	0.002	0.209	0.940	0.881
tK16	0.374	0.794	0.898	0.646	0.081	0.474	0.914	0.865	0.003	0.112	0.732	0.881
tK17	0.5	0.938	0.994	0.998	0.136	0.714	0.999	1	0.007	0.232	0.966	0.996
tK18	0.05	0.281	0.401	0.456	0.014	0.26	0.451	0.643	0.002	0.104	0.446	0.626
tK19	0.05	0.283	0.401	0.457	0.014	0.26	0.454	0.645	0.002	0.104	0.446	0.626
tK20	0.055	0.326	0.489	0.571	0.017	0.263	0.497	0.709	0.002	0.085	0.441	0.652
tK21	0.061	0.335	0.517	0.604	0.018	0.271	0.506	0.722	0.002	0.082	0.438	0.66
tK22	0.053	0.311	0.441	0.527	0.016	0.261	0.466	0.688	0.002	0.094	0.441	0.634
tK24	0.266	0.716	0.884	0.909	0.043	0.39	0.813	0.935	0.002	0.073	0.570	0.849
tK28	0.288	0.649	0.81	0.859	0.073	0.457	0.774	0.912	0.004	0.138	0.657	0.868
tK29	0.132	0.476	0.642	0.707	0.024	0.296	0.554	0.758	0.002	0.082	0.444	0.656
tK30	0.053	0.311	0.441	0.527	0.016	0.261	0.466	0.668	0.002	0.094	0.441	0.634
tK31	0.134	0.478	0.664	0.733	0.025	0.299	0.583	0.778	0.002	0.072	0.443	0.671
tD1	0.051	0.275	0.375	0.404	0.015	0.263	0.44	0.626	0.002	0.110	0.455	0.619
tD2	0.051	0.277	0.393	0.432	0.016	0.265	0.445	0.645	0.002	0.110	0.457	0.635
tD3	0.068	0.295	0.419	0.457	0.018	0.275	0.462	0.665	0.003	0.118	0.468	0.65
tD4	0.051	0.275	0.375	0.408	0.015	0.263	0.44	0.626	0.002	0.110	0.455	0.621
tKL1	0.13	0.51	0.694	0.773	0.016	0.254	0.618	0.793	0.002	0.047	0.418	0.692
tKL2	0.084	0.383	0.552	0.596	0.017	0.269	0.482	0.692	0.002	0.087	0.433	0.639

The following Table 3.7 provides the average gains in power for the tests under investigation compared to the Wald test for the simulated levels of correlations, namely 0.60, 0.80 and 0.95.

Table 3.7. Average gain in power over the Wald test for α = 0.05, for correlation levels of 0.60, 0.80 and 0.95

	rho=0.60	rho=0.80	rho=0.95
tK1	1.07%	18.79%	40.83%
tK2	0.90%	18.72%	41.03%
tK3	0.87%	18.75%	41.02%
tK4	0.59%	18.04%	40.40%
tK8	0.46%	17.53%	40.68%
tK10	0.32%	17.03%	40.06%
tK12	-1.75%	13.41%	37.73%
tK14	-2.50%	9.49%	34.15%
tK16	-2.62%	7.26%	26.17%
tK17	-1.83%	11.37%	40.38%
tK18	0.51%	1.83%	0.83%
tK19	0.68%	2.17%	0.90%
tK20	0.61%	3.83%	3.94%
tK21	0.50%	-0.54%	4.85%
tK22	0.72%	3.15%	2.50%
tK24	0.62%	13.88%	23.47%
tK28	1.04%	13.43%	23.79%
tK29	0.93%	7.29%	9.49%
tK30	0.72%	3.15%	2.33%
tK31	1.13%	7.87%	10.40%
tD1	0.05%	0.02%	0.01%
tD2	0.72%	1.58%	0.78%
tD3	1.31%	3.23%	2.20%
tD4	0.11%	0.17%	0.06%
tKL1	-0.25%	6.27%	10.94%
tKL2	0.13%	3.03%	5.02%

For a better visualization, we provide the average gain in power over the Wald's z test for α = 0.05 in Figure 3.1.

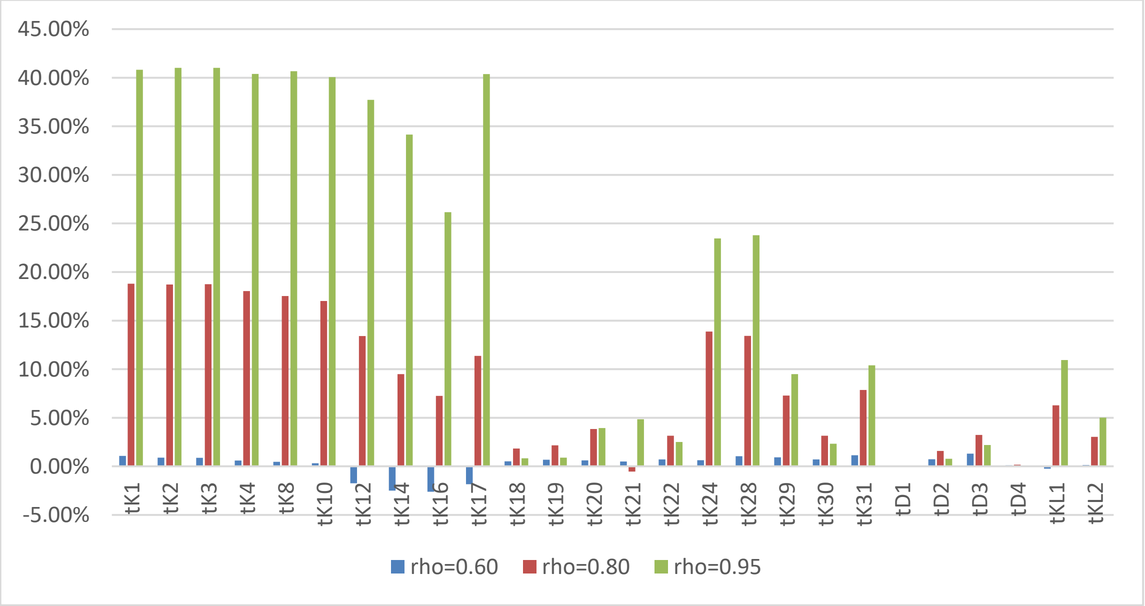

Figure 3.1. Average gain in power over the Wald's z test for α = 0.05 for correlation levels of 0.60, 0.80 and 0.95

Based on Tables 3.4, 3.5 and 3.6, we observed the following:

i) As the sample size increases, while keeping the number of variables constant, power increases for most tests considered.

ii) Most of the tests were more powerful than the Wald z test on average for correlation levels of 0.8 and 0.95, while for 0.6 they had about the same power on average.

iii) Among the Ridge tests the most powerful ones compared to the Wald z test were tK1, tK2, tK3, tK4, tK8, tK10 and tK17 with maximum average gain of around 40% respectively at correlation level of 0.95.

iv) The Liu tests were comparable to the Wald z test on power, not showing a significant gain on power

vi) Among the two Kibria-Lukman tests considered tKL1 displayed the highest average gain in power over the Wald z test, with maximum average gain of 10.94 % at correlation level of 0.95.

vii) The Ridge tests overall achieved a much better performance in test of power than the Liu and Kibria- Lukman tests, for the simulated conditions. This result is consistent with what was found in the case of linear regression.

4. Application Example

In our application, we consider the dataset "myopia", featured in the textbook, Applied Logistic Regression, by Hosmer, Lemeshow and Sturdivant. It is available in the R package "aplore3". It consists of measurements for n=618 subjects. The outcome variable for our model is the presence myopia that takes value 1 if the patient is myopic or 0 if not.

The explanatory variables in the logistic regression model include:

- **age**: The age of subjects at their first visit, providing insights into how age correlates with the likelihood of myopia.
- **spheq (Spherical Equivalent Refraction)**: This metric measures the overall refractive error of the eye, indicating its impact on myopia odds.
- **acd (Anterior Chamber Depth)**: A measure of the depth of the anterior chamber of the eye, offering insights into its role in influencing myopia odds.
- **lt (Lens Thickness)**: Representing the thickness of the lens in the eye, lt's coefficient in the model provides understanding regarding its impact on myopia development.
- **vcd (Vitreous Chamber Depth)**: Reflecting the depth of the vitreous chamber of the eye, vcd is another relevant metric influencing myopia odds.
- **sporthr (Hours of sports/outdoor activities per week)**: The number of hours per week a subject spends engaging in sports or outdoor activities, contributing to the exploration of outdoor activities' potential protective effect against myopia.
- **readhr (Hours of reading for pleasure per week)**: The hours spent reading for pleasure per week, included to examine its association with myopia.
- **comphr (Hours of playing video/computer games per week)**: Accounting for the time spent playing video or computer games, its coefficient in the model provides insights into the relationship between screen time and myopia.
- **studyhr (Hours of studying/reading for school assignments per week)**: Measures the time spent on academic activities outside of school, shedding light on the impact of academic workload on myopia.
- **tvhr (Hours of watching television per week)**: This variable represents the number of hours per week a subject spends watching television, adding an important dimension to our analysis of lifestyle factors affecting myopia.
- **al (Axial Length)**: Representing the axial length of the eye, al is a crucial metric for understanding the anatomical factors associated with myopia.

There is evidence of multicollinearity in the data, as measured by four of the variance inflation factors (VIFs) being greater than 10 (Ozkale and Kacıranlar, 2007), which is in practice considered the threshold for multicollinearity (Table 4.1). Also the Condition number (CN) which is defined as $CN = \sqrt{\frac{largest\ eigenvalue\ (X^TX)}{smallest\ eigenvalue(X^TX)}} = 147$ is greater than 30 indicating high dependency between the explanatory variables.

Table 4.1. VIFs of the regressors

Variable	VIF
age	1.3
spheq	1.1
acd	3481.8
lt	1569.1
vcd	28997.7
sporthr	1.1
readhr	1.1
comphr	1.0
studyhr	1.3
tyhr	1.1
al	30375.6

Since the problem of multicollinearity exists in the data, Ridge, Liu and Kibria-Lukman logistic regression estimation are preferable to unconstrained MLE estimation for this model. We contrasted the results of the unconstrained MLE method with some of the shrinkage parameter estimators that showed higher power in our simulations, and the analyses are given in the following Table 4.2. The R package "ridge" was used to fit the models (https://cran.r-project.org/web/packages/ridge/index.html)

Table 4.2. Regression Analysis Myopia Data

	Unconstrained MLE (K=0)			Ridge (K3 =0.3369)			Ridge (K4 = 0.3456)		
	coef	z	p-value	coef	z	p-value	coef	z	p-value
intercept	11.326	NA	NA	8.176	NA	NA	8.121	NA	NA
age	-0.197	-0.891	0.373	-0.151	-0.731	0.465	-0.149	-0.727	0.467
spheq	-4.225	-9.092	<0.001	-3.708	-9.888	<0.001	-3.698	-9.903	<0.001
acd	19.333	0.511	0.609	0.856	2.054	0.040	0.851	2.053	0.040
lt	17.083	0.451	0.651	-0.679	-1.350	0.177	-0.673	-1.350	0.177
vcd	17.905	0.473	0.636	-0.287	-1.018	0.308	-0.287	-1.024	0.306
sporthr	-0.052	-2.540	0.011	-0.050	-2.503	0.012	-0.050	-2.502	0.012
readhr	0.081	1.698	0.089	0.076	1.662	0.096	0.077	1.662	0.096
comphr	0.024	0.573	0.566	0.021	0.489	0.625	0.021	0.488	0.625
studyhr	-0.174	-1.800	0.071	-0.158	-1.734	0.083	-0.158	-1.733	0.083
tyhr	-0.015	-0.578	0.564	-0.013	-0.521	0.602	-0.013	-0.520	0.603
al	-18.41	-0.487	0.626	-0.126	-0.443	0.658	-0.125	-0.440	0.660

In the case of the unconstrained logistic regression, spheq and sporthr are significant predictors at 5% level, while readhr and studyhr are marginally significant (at 10% level). In both logistic and ridge logistic regression models both spheq and sporthr are still significant predictors at 5% level, while acd is now significant as well (p-value = 0.04 for both shrinkage parameters used). Also, readhr and studyhr are still marginally significant (at 10% level). It is worth noting that the coefficient for acd shrunk significantly while becoming statistically significant from the unconstrained to the ridge models, which indicates that the shrinkage is working. Other variables that were not significant, particularly lt and vcdhad, had a direction change in the coefficient and approach zero, another indication of the shrinkage effect.

5. Some Concluding Remarks

In this paper, we investigated thirty-nine different test statistics based on the MLE, Ridge, Liu and Kibria-Lukman estimators gathered from the existent literature to find test statistics for testing the regression coefficients of the logistic regression model in the presence of multicollinearity. A simulation study under different conditions has been conducted to make the empirical comparison among the tests with different shrinkage parameters. We compared the performance of the test statistics based on the empirical size and the power of the test. It is observed from our simulations that the tK1, tK2, tK3, tK4, tK8, tK10 and tK17 tests are the best in terms of achieving higher power gains consistently with respect to the Wald tests based in the MLE estimator, while maintaining a 5% nominal size. We therefore recommend those tests to statistical practitioners for the purpose of testing logistic regression coefficients when multicollinearity is present.

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Conflicts of Interest

The authors declared no conflict of interest

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