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A Comparison of Copulas Based on the Range Distribution and Its Application

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Abstract

This paper compares some Archimedean Copulas based on the range distribution and its application. We show that the Clayton Copula is better than the other Copula models considered in this paper. Also, Clayton Copula performed better than the regular one-way ANOVA when there were no outliers. However, when the data had outliers, the Clayton Copula performed almost at the same level (power-wise) as the regular one-way ANOVA.

Keywords: copula, range, power

1. Introduction

Here, we compare the performance of some Archimedean Copulas such as the Clayton Copula, Gumbel Copula, and Frank Copula in the context of the range distribution and its application. First, we use the mean and median of the range distribution for the comparison. Here, we study the effect of the sample size on the range distribution. Next, we construct a range based simultaneous test based on these Copulas in order to compare the level means of a single factor. This approach can serve as an alternative to the one-way ANOVA when the number of observations at each level are equal. As per the literature, the ANOVA test was formulated by Sir Ronald Fisher in 1918 in the context of comparing farm yields. The ANOVA test decomposes the variation among the observations into some factor variations that are mainly due to the effect of orthogonal components. In other words, this is the best application of the Pythagoras Theorem. In 1952, William Kruskal and Alan Wallis developed the nonparametric equivalent of ANOVA. But, the main issue with the nonparametric tests are that these tests are less powerful compared to the parametric tests. In this paper, we present an alternative test for simultaneous testing of the means as an application. The test that we propose uses the copulas and the range to conduct the simultaneous testing. We show that this test is more powerful than the ANOVA test.

The use of Copulas in statistical analysis became popular after Sklar’s Theorem in 1959. All these different types of Archimedean Copula models were developed as a result of this theorem. Clayton, Frank, and Gumbel developed their own versions of the Archimedean Copulas. The Copulas have wide applications in Economics, Finance, Engineering, and Education. As for the research done within the last seven years, Kahadawala (2018) looked at the multivariate dependence using Clayton Copula. Hofert et al (2018) used R software to model the dependence structure by using Copulas. Munhammar and Widen (2017) used the Copula to model an autocorrelated time series. Young et al (2021) used Copula to model the quality control limits for an auto-correlated AR (1) time series. We refer the interested readers to Roger Nelsen (2006) for the literature review about the Copulas.

This paper combines the concept of Copulas and Order Statistics. Order Statistics has a lot of applications. For example, reliability assessment and survival distributions involve order statistics especially in situations where there is censoring. Recently, Das and Bhattacharya (2020) looked at the application of lognormal distribution to represent the first-order statistics of wireless channels. The interested readers are referred to David and Nagaraj (2003) for the literature review about the Order Statistics.

We divide this paper as follows. We present the methodology in section 2, the numerical results in section 3, application in section 4, and conclusion and discussion in section 5.

2. Methodology

Let $x_1, x_2, ..., x_n$ follow a distribution with cumulative probability function $F(x)$.

In other words, if X is a random observation then $F(x) = P(X \le x)$. Moreover, let the order-statistics be

$x_{(1)} \le x_{(2)} \le \ldots\ldots \le x_{(n)}$.

Then,

$$P\left(x_{(1)} \le x, x_{(n)} \le y\right) = \begin{cases} \left(F(y)\right)^n - \left(F(y) - F(x)\right)^n & , x \le y \\ \left(F(y)\right)^n & , x > y \end{cases} \tag{1}$$

In this case, the marginal distributions of the order statistics are as follows,

$$u = P\left(x_{(1)} \le x\right) = 1 - \left(1 - F(x)\right)^n \tag{2}$$

$$v = P\left(x_{(n)} \le y\right) = \left(F(y)\right)^n \tag{3}$$

Let the range, $R = x_{(n)} - x_{(1)}$

Also, let $Y = x_{(n)}$ and $X = x_{(1)}$

Case 1: Uniform distribution between 0 and 1.

Then, $P\left(R \le r\right) = P\left(Y - X \le r\right) = P\left(Y \le X + r\right) = P\left(X \le Y \le X + r\right)$

$$= \int_0^{1-r} \int_x^{x+r} n(n-1)(y-x)^{(n-2)}\, dy\, dx + \int_{1-r}^{1} \int_x^{1} n(n-1)(y-x)^{(n-2)} dy\, dx$$

$$= n r^{n-1}(1-r) + r^n \qquad , \quad 0 \le r \le 1 \tag{4}$$

The density function for the range in the case of standard uniform distribution is as follows,

$$f_R(r) = n(n-1) r^{(n-2)} (1-r) \qquad , \quad 0 \le r \le 1 \tag{5}$$

Remark: The statistical properties are listed below.

$$E(R) = \frac{(n-1)}{(n+1)} \tag{6}$$

$$\text{Median of } R \approx \frac{\left(n - \frac{4}{3}\right)}{\left(n + \frac{1}{3}\right)} \tag{7}$$

$$Var(R) = \frac{2(n-1)}{(n+1)^2 (n+2)} \tag{8}$$

Case 2: Exponential case where $F(x) = 1 - e^{-\lambda x}$

$$P\left(R \le r\right) = \int_0^{\infty} \int_x^{x+r} n(n-1)\left(e^{-\lambda x} - e^{-\lambda y}\right)^{n-2} \lambda e^{-\lambda x}\, \lambda e^{-\lambda y} dx\, dy$$

$$= \left(1 - e^{-\lambda r}\right)^n \tag{9}$$

$$E(R)=\int_0^{\infty} n\left(1-e^{-\lambda r}\right)^{n-1}\lambda r\, e^{-\lambda r}dr$$

$$=n\sum_{i=0}^{n-1}\binom{n-1}{i}(-1)^i\int_0^{\infty} e^{-\lambda r(i+1)}\lambda r\,dr$$

$$=n\sum_{i=0}^{n-1}\binom{n-1}{i}(-1)^i\left(\frac{1}{i+1}\right)^2\frac{1}{\lambda} \tag{10}$$

$$\text{Moreover, Median of } R=\frac{-\ln\left(1-(0.5)^{\frac{1}{n}}\right)}{\lambda} \tag{11}$$

Case 3: General case where the population distribution is $F(x)$.

$$P(R\le r)=\int_{-\infty}^{\infty}\int_x^{x+r} n(n-1)\left(F(y)-F(x)\right)^{(n-2)} f(x)f(y)\,dy\,dx \tag{12}$$

$$\Rightarrow$$

$$E(R)=\int_0^{\infty}\left(1-P(R\le r)\right)dr \tag{13}$$

Special Case:

$F(x)=\Phi(x)$, where $\Phi(x)$ means the standard normal distribution function.

When the sample size $n=25$,

$$E(R)\approx 3.8624 \ \&\ \text{Median of } R\approx 3.864 \tag{14}$$

Clayton Copula Density Function:

$$f_c(x,y)=n^2(1+\alpha)\left[\left[1-\left(1-F(x)\right)^n\right]^{-\alpha}+\left(F(y)\right)^{-n\alpha}-1\right]^{-\left(\frac{1}{\alpha}\right)-2}\left[1-\left(1-F(x)\right)^n\right]^{-\alpha-1}$$

$$.\left(1-F(x)\right)^{n-1}\left[F(y)\right]^{-n\alpha-1} f(x)\,f(y) \tag{15}$$

Gumbel Copula Density Function:

$$f_g(x,y)=e^{-\left\{\left(-\ln\left[1-\left(1-F(x)\right)^n\right]\right)^{\alpha}+\left(-\ln\left[F(y)\right]^n\right)^{\alpha}\right\}^{\frac{1}{\alpha}}}.\left[\left(-\ln\left[1-\left(1-F(x)\right)^n\right]\right)^{\alpha}+\left(-\ln\left[F(y)\right]^n\right)^{\alpha}\right]^{\frac{1}{\alpha}-2}$$

$$\left\{(\alpha-1)+\left[\left(-\ln\left[1-\left(1-F(x)\right)^n\right]\right)^{\alpha}+\left(-\ln\left[F(y)\right]^n\right)^{\alpha}\right]^{\frac{1}{\alpha}}\right\}.\frac{1}{\left[1-\left(1-F(x)\right)^n\right]}.\frac{1}{\left[F(y)\right]^n}$$

$$.\,n^2.\left[1-F(x)\right]^{n-1}.\left[F(y)\right]^{n-1}.f(x).f(y) \tag{16}$$

Frank Copula Density Function:

$$f_F(x,y)=\frac{\alpha\left(e^{\alpha}-1\right)e^{\left(1-(1-F(x))^{n}\right)}e^{(F(y))^{n}}n\left(1-F(x)\right)^{n-1}n\left(F(y)\right)^{n-1}f(x)f(y)}{\left[\left(e^{\alpha}-1\right)+\left(e^{\alpha\left(1-(1-F(x))^{n}\right)}-1\right)\left(e^{\alpha(F(y))^{n}}-1\right)\right]^{2}} \tag{16}$$

Remark: We will compare the performance of these Copulas based on the mean and median of the Range distribution for different sample sizes.

3. Numerical Study

Here, we discuss the numerical results. We study the effect of the underlying population distribution for different Copulas such as Clayton, Gumbel, and Frank based on different sample sizes. This is in the context of the median and mean of the Range distribution for the samples selected at random from the underlying distributions discussed in this section.

Numerical Results (for Range Median):

- Standard Uniform

Sample Size	Theory	Clayton	Gumbel	Frank
3	0.5000	0.5215	0.5210	0.5716
4	0.6154	0.6235	0.6234	0.6442
5	0.6875	0.6910	0.6904	0.7031
6	0.7368	0.7385	0.7382	0.7467
7	0.7727	0.7738	0.7734	0.7899
8	0.8000	0.8012	0.8005	0.8057
9	0.8214	0.8223	0.8220	0.8262
10	0.8387	0.8392	0.8392	0.8426
11	0.8529	0.8539	0.8535	0.8564
12	0.8649	0.8657	0.8652	0.8679
13	0.8750	0.8757	0.8755	0.8779
14	0.8837	0.8845	0.8842	0.8863
15	0.8913	0.8920	0.8920	0.8937
16	0.8979	0.8987	0.8986	0.9001
17	0.9038	0.9047	0.9045	0.9059
18	0.9091	0.9099	0.9099	0.9111
19	0.9138	0.9147	0.9146	0.9157
20	0.9180	0.9189	0.9187	0.9199
21	0.9219	0.9227	0.9227	0.9237
22	0.9254	0.9263	0.9263	0.9271
23	0.9286	0.9295	0.9295	0.9303
24	0.9315	0.9324	0.9324	0.9331
25	0.9342	0.9351	0.9351	0.9358

- Standard Exponential $\lambda = 1$

Sample Size	Theory	Clayton	Gumbel	Frank
3	1.5784	1.3207	1.3547	1.4216
4	1.8382	1.6404	1.6542	1.6549
5	2.0444	1.8898	1.8931	1.8798
6	2.2154	2.0877	2.1273	2.0795
7	2.3615	2.2789	2.2652	2.2487
8	2.4889	2.4536	2.4092	2.3988
9	2.6020	2.5617	2.5396	2.5291
10	2.7036	2.6772	2.6572	2.6431
11	2.7957	2.7738	2.7641	2.7514
12	2.8802	2.8794	2.8584	2.8463
13	2.9580	2.9598	2.9454	2.9365
14	3.0302	3.0322	3.0252	3.0172
15	3.0976	3.1092	3.1022	3.0932
16	3.1607	3.1794	3.1782	3.1672
17	3.2201	3.2475	3.2432	3.2356
18	3.2761	3.3042	3.3087	3.2977
19	3.3291	3.3719	3.3649	3.3596
20	3.3795	3.4289	3.4242	3.4176
21	3.4275	3.4852	3.4786	3.4707
22	3.4733	3.5379	3.5314	3.5244
23	3.5170	3.5883	3.5811	3.5731
24	3.5590	3.6378	3.6317	3.6242
25	3.5992	3.6829	3.6801	3.6718

- Standard Normal

Sample Size	Theory	Clayton	Gumbel	Frank
3	1.5677	1.6977	1.7215	1.7995
4	1.9773	2.0535	2.0541	2.0673
5	2.2686	2.3206	2.3104	2.3056
6	2.4724	2.5113	2.5115	2.4996
7	2.6465	2.6760	2.6789	2.6695
8	2.7913	2.8173	2.8174	2.8112
9	2.9253	2.9373	2.9411	2.9313
10	3.0206	3.0434	3.0444	3.0404
11	3.1167	3.1397	3.1397	3.1327
12	3.2101	3.2231	3.2231	3.2161
13	3.2846	3.3017	3.3017	3.2937
14	3.3567	3.3693	3.3693	3.3646
15	3.4244	3.4354	3.4354	3.4301
16	3.4807	3.4929	3.4930	3.4898
17	3.5345	3.5496	3.5497	3.5447
18	3.5884	3.5994	3.6024	3.5972
19	3.6431	3.6491	3.6495	3.6469
20	3.6821	3.6945	3.6947	3.6923
21	3.7250	3.7379	3.7381	3.7341
22	3.7700	3.7726	3.7794	3.7760
23	3.8119	3.8191	3.8192	3.8148
24	3.8480	3.8544	3.8546	3.8514
25	3.8644	3.8891	3.8895	3.8879

Numerical Results (for Range Mean):

- Standard Uniform

Sample Size	Theory	Clayton	Gumbel	Frank
3	0.5000	0.5046	0.4908	0.5332
4	0.6000	0.5851	0.5811	0.5924
5	0.6667	0.6463	0.6448	0.6476
6	0.7143	0.6921	0.6916	0.6921
7	0.7500	0.7272	0.7270	0.7269
8	0.7778	0.7548	0.7547	0.7545
9	0.8000	0.7769	0.7768	0.7766
10	0.8182	0.7950	0.7950	0.7948
11	0.8334	0.8101	0.8101	0.8099
12	0.8462	0.8228	0.8228	0.8226
13	0.8571	0.8338	0.8338	0.8336
14	0.8667	0.8432	0.8433	0.8431
15	0.8750	0.8515	0.8516	0.8514
16	0.8823	0.8588	0.8589	0.8587
17	0.8889	0.8653	0.8654	0.8652
18	0.8947	0.8711	0.8711	0.8709
19	0.9000	0.8768	0.8763	0.8762
20	0.9048	0.8810	0.8810	0.8809
21	0.9091	0.8852	0.8853	0.8851
22	0.9130	0.8892	0.8892	0.8890
23	0.9167	0.8927	0.8928	0.8926
24	0.9200	0.8960	0.8961	0.8959
25	0.9231	0.8990	0.8991	0.8989

- Standard Exponential $\lambda = 1$

Sample Size	Theory	Clayton	Gumbel	Frank
3	1.5046	1.6470	1.5037	1.7653
4	1.8357	1.7435	1.7423	1.8346
5	2.0820	1.9825	2.1110	1.9946
6	2.2835	2.1935	2.1841	2.1840
7	2.4477	2.3782	2.3678	2.3622
8	2.5945	2.6742	2.5290	2.5223
9	2.7183	2.6831	2.6735	2.6658
10	2.8267	2.8131	2.8026	2.7956
11	2.9355	2.9303	2.9210	2.9140
12	3.0177	3.0394	3.0304	3.0230
13	3.1029	3.1400	3.1316	3.1242
14	3.1732	3.2336	3.2268	3.2187
15	3.2525	3.3209	3.3146	3.3075
16	3.3182	3.4068	3.3943	3.3913
17	3.3851	3.4851	3.4777	3.4707
18	3.4355	3.5609	3.5538	3.5462
19	3.5059	3.6324	3.6258	3.6183
20	3.5507	3.7051	3.6948	3.6872
21	3.5887	3.7691	3.7602	3.7533
22	3.6481	3.8323	3.8247	3.8184
23	3.6912	3.8920	3.8848	3.8782
24	3.7337	3.9495	3.9452	3.9372
25	3.7724	4.0065	4.0019	3.9944

- Standard Normal

Sample Size	Theory	Clayton	Gumbel	Frank
3	1.6940	1.6377	1.6441	1.9107
4	2.0552	1.8821	1.8675	1.9847
5	2.3250	2.1093	2.1074	2.1365
6	2.5314	2.3051	2.3058	2.3112
7	27039	2.4758	2.4708	2.4709
8	2.8496	2.6118	2.6119	2.6106
9	2.9681	2.7340	2.7340	2.7324
10	3.0823	2.8411	2.8414	2.8396
11	3.1742	2.9366	2.9367	2.9350
12	3.2575	3.0224	3.0226	3.0207
13	3.3362	3.1002	3.1002	3.0986
14	3.4069	3.1714	3.1715	3.1698
15	3.4693	3.2367	3.2368	3.2352
16	3.5341	3.2975	3.2976	3.2959
17	3.5849	3.3538	3.3539	3.3523
18	3.6396	3.4066	3.4069	34049
19	3.6910	3.4561	3.4563	3.4546
20	3.7354	3.5026	3.5028	3.5012
21	3.7807	3.5469	3.5657	3.5453
22	3.8210	3.5790	3.5889	3.5770
23	3.8606	3.6283	3.6283	3.6267
24	3.8968	3.6590	3.6593	3.6575
25	3.9298	3.7020	3.7020	3.7005

Note: It appears based on the above numerical results, there is not much of a difference between these Archimedean Copulas.

Next, we will focus on an application.

4. Applications

Consider a one-way ANOVA situation where the response

$$y_{ij} = \mu + \tau_i + \varepsilon_{ij}$$

Where μ is the overall average response, τ_i is the level i effect, and ε_{ij} is the error effect.

Moreover, level i average response $\mu_i = \mu + \tau_i$

The Copula based construction of the range distribution is very useful in the context of ANOVA type simultaneous testing of the means. We can use this method in place of ANOVA when the sample sizes are equal (balanced design).

Let us suppose that there are k number of levels.

Level 1: Mean $= \mu_1$, Sample Size $= n_1$

Level 2: Mean $= \mu_2$, Sample Size $= n_2$

Level k: Mean $= \mu_k$, Sample Size $= n_k$

We want to test the hypothesis,

$H_0 : \mu_1 = \mu_2 = \ldots\ldots = \mu_k$

$H_1 :$ At least one of the means is different

$$\text{Note that, } Max_{1\le i,\, j\le k}\left|\mu_i - \mu_j\right| = r \tag{18}$$

$$\Rightarrow$$

$$\text{P value} = P\left(Max_{1\le i,\, j\le k}\left|\mu_1 - \mu_j\right| > x\right) = P\left(r > x\right) \le \alpha \tag{19}$$

So, reject H_0 when $P\left(r > x\right) \le \alpha$

Example

In an analytical chemistry experiment, the finger prints measurements based on five different acids (Alanine. Glycine2, leucine, Serine, Pro-line) was obtained from 50 different individuals over a five week period. Two data points were deleted due to incomplete entries. The objective here is to compare the means obtained from the finger prints.

Acid	**Mean**	**Standard deviation**
Alanine	4.4921978	7.3047429
Glycine2	8.1878385	9.9768738
Leucine	7.6662499	12.3445249
Serine	3.5428281	5.9022167
Pro-line	5.4121423	9.5018675

Remark 1: For the above data, the **Levene's test** for the equality of variances yielded a P-Value = 0.226 suggesting that the variability is the same for all the acids involved in this study.

Moreover, since the sample size is 48 (reasonably large), we can treat the sample means as following the normal distribution.

Note that

$$z_i = \frac{\overline{X}_i - \mu_i}{\left(\sigma/\sqrt{n}\right)} \quad , \quad z_j = \frac{\overline{X}_j - \mu_j}{\left(\sigma/\sqrt{n}\right)}$$

$$\Rightarrow$$

$$z_i - z_j = \frac{\overline{X}_i - \overline{X}_j - \left(\mu_i - \mu_j\right)}{\left(\sigma/\sqrt{n}\right)}$$

Under the assumption that the null hypothesis is true, we will evaluate the probability

$$P\left(r > observed\ \max\left|z_i - z_j\right|\right) = P\left(r > \frac{observed\ \max\left|\overline{X_i} - \overline{X_j}\right|}{\left(\sigma / \sqrt{n}\right)} \right)$$

$$= P\left(r > \frac{\left(8.1878385 - 3.5428281\right)}{\left(9.278 / \sqrt{48}\right)} \right)$$

$$= P\left(r > 3.469\right)$$

For Clayton Copula, $P\left(r > 3.469\right) = 0.086420$

For Gumbel Copula, $P\left(r > 3.469\right) = 0.085867$

For Frank Copula, $P\left(r > 3.469\right) = 0.12260$

Remark 2:

For the same data, regular one-way ANOVA analysis gave the following result.

Source	DF	SS	MS	F	p-value
Acid	4	773.9	193.475	2.248	0.064654
Error	235	20229.1	86.081		

Note: All the tests indicate that the means are equal at 5% significance level.

Next, we compare the copula power based on a simulation study.

Power Comparison

Parameters	ANOVA	Clayton	Gumbel	Frank
$\tau_1 = 0.5, \tau_2 = 0.7, \tau_3 = -0.3, \tau_4 = 0.1, \tau_5 = -1$	0.07	0.10	0.10	0.10
$\tau_1 = 1, \tau_2 = 2, \tau_3 = -2, \tau_4 = -1, \tau_5 = 0$	0.43	0.61	0.55	0.53
$\tau_1 = 1, \tau_2 = 2, \tau_3 = -3, \tau_4 = 0, \tau_5 = 0$	0.55	0.72	0.71	0.59
$\tau_1 = 1, \tau_2 = 2, \tau_3 = -3, \tau_4 = 1, \tau_5 = -1$	0.70	0.77	0.75	0.71
$\tau_1 = 1, \tau_2 = 1.5, \tau_3 = -2.5, \tau_4 = 2.5, \tau_5 = -2.5$	0.92	0.96	0.95	0.74
$\tau_1 = 3, \tau_2 = -3, \tau_3 = 4, \tau_4 = -2, \tau_5 = -2$	0.96	0.97	0.96	0.95

4. Discussion and Conclusion

This paper presents a Copula approach based on the range distribution to see which Copula is better. In this context, the mean and the median were used. The mean and median were evaluated numerically for the Archimedean Copulas; Clayton, Gumbel, and Frank at different sample sizes. As for the mean and median of the range distribution, the Archimedean Copulas such as Clayton, Gumbel, and Frank were about the same regardless of the sample size. However in the context of the simultaneous comparison of the means, Clayton Copula seemed to perform better than Gumbel Copula and Frank Copula. This is evident from the power calculation presented above. Moreover, Clayton and Gumbel Copulas seem to possess higher power compared to the regular one-way ANOVA. This pattern was observed when there were no outliers in the data. The power computation shows that Clayton Copula is the best of the three Archimedean Copulas presented in this paper. However, in the context of outliers and negative association, the regular one-way ANOVA and the test based on the Clayton Copula performed at the same level when it comes to power. Furthermore out of the three Archimedean Copulas studied, Clayton Copula slightly performed better than Gumbel

Copula. Gumbel Copula performed better than the Frank Copula.

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