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Marshall-Olkin Extended Generalized Exponential Distribution: Properties, Inference and Application to Traffic Data

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Abstract

This paper aims to develop a three-parameter distribution called the Marshall–Olkin Extended Generalized Exponential ($MOEGE$) distribution, which can be used in analyzing both reliability and survival data. Some statistical properties of the new distribution have been studied, which include, moments, incomplete moments, Renyl entropy, stochastic ordering, order statistics, and the moment generating function. The $MOEGE$ distribution has submodels such as the Marshall–Olkin Extended Exponential ($MOEE$), the Generalized Exponential (GE), and the Exponential (E) distribution. The maximum likelihood estimation technique is used to obtain the parameters estimate of the $MOEGE$ distribution, also, we constructed a 95% asymptotic confidence interval for the parameters. The performances of the estimators have been studied using Monte Carlo simulation, and finally, to demonstrate the applicability of the $MOEGE$ distribution, a traffic data set has been used.

Keywords: Asymptotic confidence interval, estimator, Monte Carlo, Marshall–Olkin Extended Generalized exponential

1. Introduction

In survival/reliability experiment, exponential distribution is the most popular distribution for analyzing life-time data. But the application of this distribution is limited due to constant hazard rate because in many practical situation it is not realizable. Therefore, several generalization based on this distribution have been developed to address the problem of monotone failure rate behaviour see, Barlow and Proschan (1975), Gupta and Kundu (1999; 2001; 2002; 2007), Rajwant et al. (2016) etc. The Generalized Exponential (GE) distribution was proposed and studied by Gupta and Kundu (1999), developed to address the problem of monotone failure rate in Exponential (E) distribution. Further, the MOEE is also another generalization of the exponential distribution developed by Rajwant et al. (2016) to further increase the scope of applications of E distribution in modeling real life data. In this study we developed the Marshall-Olkin Extended Generalized Exponential distribution using a generalized family of distributions introduced by Marshall and Olkin (1997) called Marshall and Olkin family of distribution. This generalization have been used by many authors which includes: Alice and Jose (2003), Garcia et al. Ghitany et al. (2007), Ristic et al. (2007), Ristic and Kundu (2015), Okasha and Kayid (2016), and, Ogunde et al. (2020) among many others. The probability density function (PDF), cumulative distribution function (CDF) and the hazard function (hf) of this distribution are given as

$$f(x) = \frac{\alpha g(x)}{[1-\bar{\alpha}\bar{G}(x)]^2}, \tag{1}$$

$$F(x) = \frac{G(x)}{1-\bar{\alpha}\bar{G}(x)}, \tag{2}$$

and

$$h(x) = \frac{g(x)}{G(x)[1-\bar{\alpha}\bar{G}(x)]} \tag{3}$$

respectively and $\bar{\alpha} = 1 - \alpha. \alpha$ is called the tilt parameter, the $G(x)$ and the $g(x)$ are respectively the baseline distribution function and the baseline density function.

The chief motivation for this study is to develop an extended form of the generalized exponential distribution which will be more applicable in several areas of reliability and survival studies with a more flexible modeling characteristics.

The new distribution (MOEGED)

The Probability density function (PDF) and survival function sf of the generalized exponential distribution, introduced and studied Ristic and Kundu (1999), for $x > 0$ and, $\rho > 0$, are given by

$$g(x) = \eta\rho e^{-\rho x}(1 - e^{-\rho x})^{\eta-1}, \tag{4}$$

$$S(x) = 1 - (1 - e^{-\rho x})^{\eta}. \tag{5}$$

Plugging (4) and (5) in (1) to obtain the PDF $MOEGE$ distribution as

$$f(x) = \frac{\alpha\eta\rho e^{-\rho x}(1 - e^{-\rho x})^{\eta-1}}{[1 - \bar{\alpha}(1 - [1 - e^{-\rho x}]^{\eta})]^{2}}; \qquad x, \alpha, \eta, \rho > 0, \tag{6}$$

The CDF and hazard function associated to (6) is respectively, given by

$$F(x) = \frac{(1 - e^{-\rho x})^{\eta}}{[1 - \bar{\alpha}(1 - [1 - e^{-\rho x}]^{\eta})]} \qquad ; x, \alpha, \eta, \rho > 0, \tag{7}$$

and

$$h(x) = \frac{\alpha\eta\rho e^{-\rho x}(1 - e^{-\rho x})^{\eta-1}}{[1 - \bar{\alpha}(1 - e^{-\rho x})^{\eta}] - \{[1 - \bar{\alpha}(1 - e^{-\rho x})^{\eta}] - (1 - e^{-\rho x})^{\eta}\}} \qquad ; x, \alpha, \eta, \rho > 0. \tag{8}$$

where $\bar{\alpha} = 1 - \alpha$. The $MOEGE$ distribution can be applied in modeling a real life experiment and it can be explored as an alternative to the Weibull, gamma and other exponentiated family of distributions see Singh et al. (2013). Another beauty of this model is the density function (1) has increasing failure rate for $\alpha, \eta > 1$, decreasing failure rate for $\alpha, \eta < 1$ and constant failure rate for $\alpha = \eta = 1$.

The graph of the CDF, PDF and hazard function are given if figures 1 and 2 drawn below.

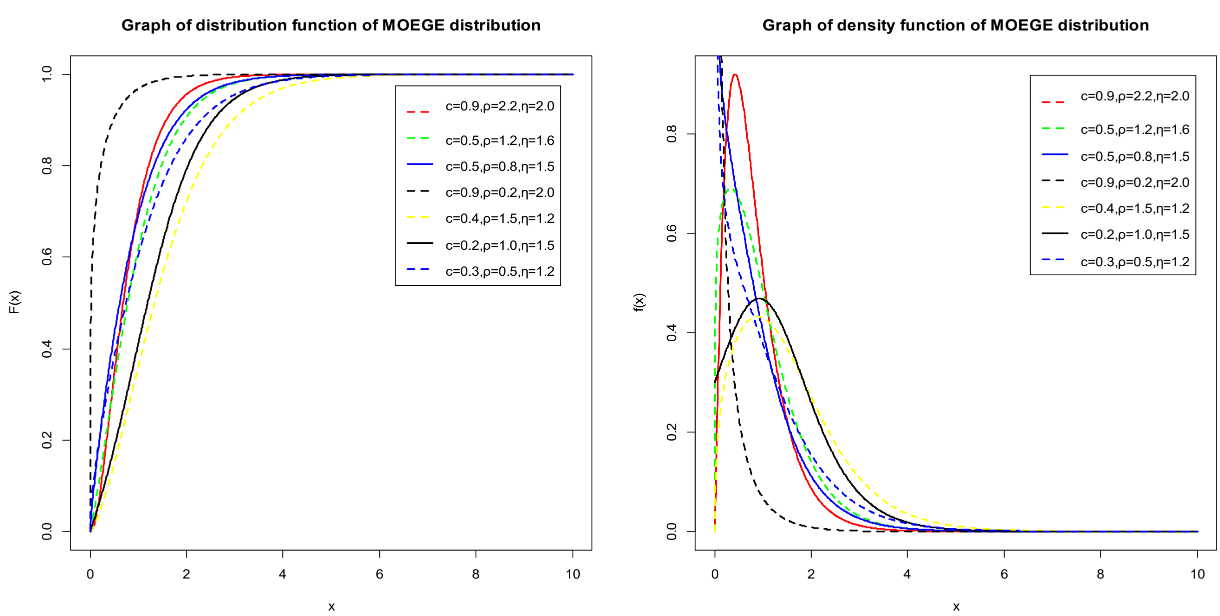

Figure 1. Plots of the CDFs and PDFs of $MOEGE$ distribution.

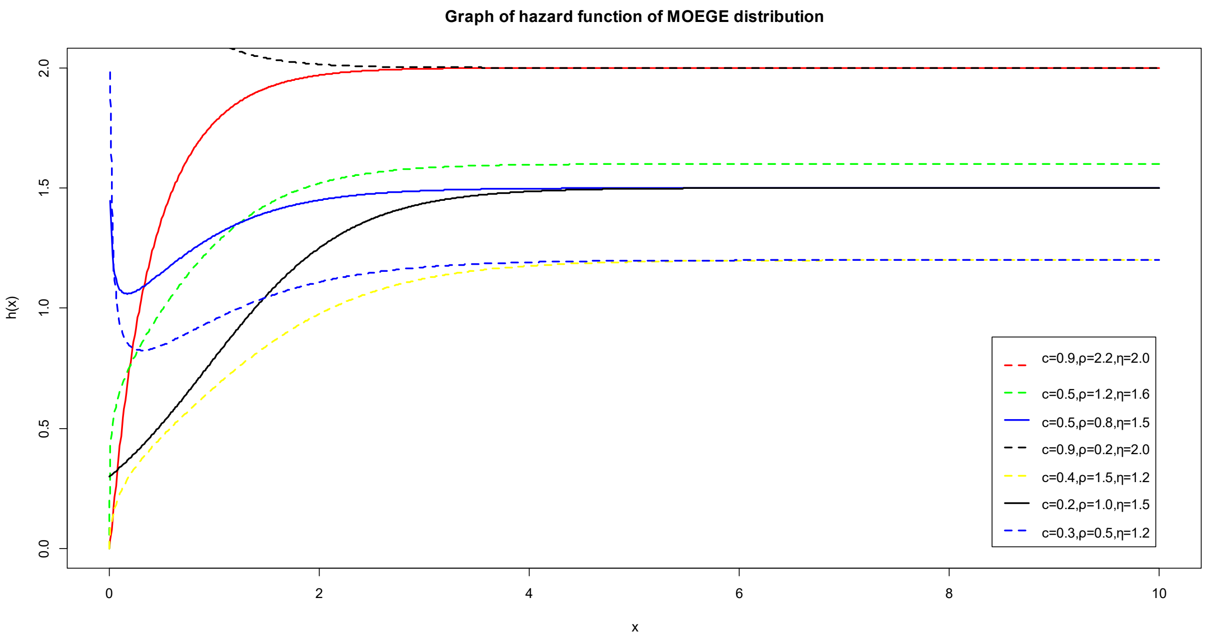

Figure 2. Plot of the hazard function of $MOEGE$ distribution

Figure 2 shows that the MOEGE distribution can be increasing, decreasing, constant, and bathtub shape which clearly indicates the MOEGE distribution can be used to model data exhibiting any shape of the hazard function.

3. Mixture Representation

The mixture representation of the density is always employed when deriving the statistical properties of generalized distributions. In this section, the mixture representation of the $MOEGE$ density function is derived. Considering the series representation

$$(1-z)^{-k} = \sum_{i=0}^{\infty} \frac{\Gamma(k+i)}{i!\,\Gamma(k)} z^i \tag{9}$$

Using (9) in (6), we have

$$[1-\bar{\alpha}(1-[1-e^{-\rho x}]^{\eta})]^2 = \sum_{i=0}^{\infty} \frac{\Gamma(2+i)}{i!} \bar{\alpha}^i (1-[1-e^{-\rho x}]^{\eta})^i \tag{10}$$

The PDF of MOEGE distribution given in (6) can be re-written as

$$f(x) = \alpha\eta\rho e^{-\rho x} \sum_{i,j=0}^{\infty} \frac{\Gamma(2+i)}{i!} (-1)^j \binom{j}{i} \bar{\alpha}^i (1-e^{-\rho x})^{\eta(j+1)-1} \tag{10}$$

Thus, the expression given in (10) is the exponentiated exponential distribution with shape parameter $\eta(j+1)$ and scale parameter ρ

Further simplification of (10), gives

$$f(x) = \alpha\eta\rho \sum_{i,j,k=0}^{\infty} \binom{\eta(j+1)-1}{k} \binom{j}{i} (-1)^{j+k} \frac{\Gamma(2+i)}{i!} \bar{\alpha}^i e^{-\rho(k+1)x} \tag{11}$$

3.1 Quantile Function

The quantile function plays a useful role when simulating random variates from a statistical distribution. The quantile function of the MOEGE distribution, say $x = Q(u)$ is given by:

$$Q(u) = -\frac{1}{\rho}log\left\{1-\left[1-\frac{1-u}{\alpha+\bar{\alpha}(1-u)}\right]^{1/\eta}\right\}, \quad 0<u<1. \tag{12}$$

The median (q_2)and the upper quartile (q_3) is obtained by substituting p = 0.5 and 0.75 respectively, into the quantile function. Hence, the median and the upper quartile are respectively,

$$q_2 = -\frac{1}{\rho}log\left\{1-\left[1-\frac{0.5}{\alpha+0.5\bar{\alpha}}\right]^{1/\eta}\right\}, \tag{13}$$

and

$$q_3 = -\frac{1}{\rho}log\left\{1-\left[1-\frac{0.25}{\alpha+0.25\bar{\alpha}}\right]^{1/\eta}\right\}. \tag{14}$$

In many heavy tailed distributions, the classical measures of skewness and kurtosis cannot be obtained due to non-convergence of their higher moments. In such situations, the quantile can be used to estimate such measures. The Bowley's coefficient of skewness which is developed using quartiles can be used to estimate the coefficient of skewness. It is given by

$$B = \frac{Q_{0.75}-2Q_{0.5}+Q_{0.25}}{Q_{0.75}-Q_{0.25}}, \tag{15}$$

Consequently, the coefficient of kurtosis can be calculated using the Moors' coefficient of kurtosis which is measured based on the octiles as

$$M = \frac{Q_{0.875}-Q_{0.625}-Q_{0.375}+Q_{0.125}}{Q_{0.75}-Q_{0.25}}, \tag{16}$$

Table 1 shows the Bowley's coefficient of skewness and Moors' coefficient of kurtosis for fixed value of $\rho = 0.5$ and varying the values of α and η.

Table 1. Values for Bowley's coefficient of skewness and Moors' coefficient of kurtosis

Parameters	0.125	0.25	0.325	0.5	0.625	0.75	0.875	B	M
$\alpha = 0.3$ $\eta = 1.5$	0.1450	0.3189	0.5340	0.8119	1.1945	1.7800	2.8948	0.1059	3.8494
$\alpha = 0.5$ $\eta = 1.5$	0.1810	0.3973	0.6632	1.0028	1.4621	2.1459	3.3917	0.1029	3.5361
$\alpha = 0.5$ $\eta = 3.5$	0.2302	0.4991	0.8214	1.2219	1.7475	2.5040	3.8301	0.0903	3.1321
$\alpha = 0.8$ $\eta = 3.5$	0.2526	0.5461	0.8955	1.3261	1.8853	2.6792	4.0489	0.0869	3.0003
$\alpha = 0.8$ $\eta = 5.5$	0.2579	0.5567	0.9117	1.3480	1.9131	2.7135	4.0889	0.0860	2.9680
$\alpha = 0.8$ $\eta = 8.5$	0.2611	0.5633	0.9217	1.9642	1.9302	2.7343	4.1136	0.0851	2.0365

From Table 1 it can be deduced that the MOEGE distribution can be positively skewed, mesokurtic, platykurtic and leptokurtic.

3.2 Moments

The moment of a random variable can be used in computing measures of central tendencies, dispersions and shapes. The r^{th} non-central moment of the $MOEGE$ random variable is:

$$E(X^r) = \mu_r' = \int_0^\infty x^r dF_{MOEGE}(x) \tag{17}$$

Putting (11) in (17), we obtain

$$\mu_r' = \alpha\eta\rho \sum_{i,j,k=0}^{\infty} \binom{\eta(j+1)-1}{k}\binom{j}{i}(-1)^{j+k}\frac{\Gamma(2+i)}{i!}\bar{\alpha}^i \int_0^\infty x^r e^{-\rho(k+1)x}dx \tag{18}$$

By letting $m = \rho(k+1)x, dm = \rho(k+1)dx$, plugging it into (18), finally we have

$$\mu_r' = \alpha\eta \sum_{i,j,k=0}^{\infty} \binom{\eta(j+1)-1}{k}\binom{j}{i}(-1)^{j+k}\frac{\Gamma(2+i)}{i!}\bar{\alpha}^i\rho^{-r}(k+1)^{-(r+1)}\Gamma(r+1) \tag{19}$$

for $r = 1,2,3,\ldots$, where $\Gamma(\cdot)$ is the gamma function. Table 2 displays the first six moments, variance (σ^2), Coefficient of Variation (CV), Coefficient of Skewness (CSK) and Coefficient of Kurtosis (CKU). The values for σ^2, CV, CSK and CKU for $\rho = 2.5$ are respectively given by

$$\sigma^2 = (\mu_2' - \mu^2)^{1/2}, \quad CV = \frac{\sigma}{\mu}, \quad \text{CSK} = \frac{\mu_3' - 3\mu\mu_2' - 2\mu^3}{\sqrt[3/2]{\mu_2' - \mu^2}} \; and$$

$$\text{CKU} = \frac{\mu_4' - 4\mu\mu_3' + 6\mu^2\mu_2' - 2\mu^4}{(\mu_2' - \mu^2)^2}$$

Table 2. First six moments, σ^2, CV, CSK and CKU

μ_r'	$\alpha = 0.3$, $\eta = 1.5$	$\alpha = 0.5$, $\eta = 5.5$	$\alpha = 0.8$, $\eta = 3.5$
μ_1'	0.8222	1.1405	1.3061
μ_2'	0.9644	1.5933	2.0064
μ_3'	1.4408	2.6508	3.5711
μ_4'	2.6167	5.1578	7.2877
μ_5'	5.6264	11.5900	16.9248
μ_6'	14.0629	29.7766	44.4520
σ^2	0.2883	0.2929	0.30050
CV	0.6531	0.4743	0.2732
CSK	1.1212	1.0510	1.0051
CKU	5.0368	4.9525	4.8378

From Table 2, it can be concluded that the moments of MOEGE distribution increases when the values of the parameters increases but the coefficient of variation, coefficient of kurtosis, and coefficient of skewness decreases.

3.3 Moment-Generating Function

The moment generating function of random variable X that follows the $MOEGE$ distribution, if it exist, is represented by

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx} f(x)dx = \sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r' \tag{20}$$

$$= \alpha\eta \sum_{i,j,k=0}^{\infty} \frac{t^r}{r!}\binom{\eta(j+1)-1}{k}\binom{j}{i}(-1)^{j+k}\frac{\Gamma(2+i)}{i!}\bar{\alpha}^i\rho^{-r}(j+1)^{-(r+1)}\Gamma(r+1)$$

3.4 Incomplete Moment

The incomplete moment is a very useful applications in different fields of study. The first incomplete moment is employed in estimation of the Bonferroni and Lorenz curves which are useful in reliability, insurance, demography, medicine, and economics. The r^{th} incomplete moment of the $MOEGE$ random variable is:

$$\varphi_r(t) = \int_0^t x^r dF_{MOEGE}(x) \tag{21}$$

$$= \alpha\eta\rho \sum_{i,j,k=0}^{\infty} \binom{\eta(j+1)-1}{k}\binom{j}{i}(-1)^{j+k}\frac{\Gamma(2+i)}{i!}\bar{\alpha}^i \int_0^t x^r e^{-\rho(k+1)x}dx \tag{22}$$

By using the complementary incomplete gamma function in (22), it yields:

$$\varphi_r(t) = \alpha\eta \sum_{i,j,k=0}^{\infty} \binom{\eta(j+1)-1}{k}\binom{j}{i}(-1)^{j+k}\frac{\Gamma(2+i)}{i!}\bar{\alpha}^i\rho^{-r}(k+1)^{-(r+1)}\Gamma\{(r+1),\rho(j+1)t\} \tag{23}$$

where $\Gamma\,(v,z) = \int_z^{\infty} w^{v-1}e^{-w}dw$ is the complementary incomplete gamma function.

3.5 Entropy

Entropy has been applied in many areas of engineering sciences and information theory as measures of uncertainty. The Renyi entropy of a random variable X having the $MOEGE$ distribution is given as:

$$I_R(v) = \frac{1}{1-v} log\left[\int_0^{\infty} f_{MOEGE}^{v}(x)dx\right], \tag{24}$$

Plugging (11) into (24) followed by simple algebraic manipulation, we have

$$I_R(v) = \frac{1}{1-v} log\left[(\alpha\eta\rho)^v \sum_{k,l}^{\infty}(-1)^l\binom{\eta(k+1)-1}{l}\frac{\Gamma(2v+k)}{\Gamma(2v)k!}(\bar{\alpha})^k \int_0^{\infty} e^{-\rho(l+1)x}\,dx\right] \tag{25}$$

Letting, $y = \rho(l+1)x, dy = \rho(l+1)dx$ and plugging it in (25), yields and expression for the Entropy of $MOEGE$ distribution given as

$$I_R(v) = \frac{1}{1-v} log\left[\alpha^v\eta^v\rho^{v-1} \sum_{k,l}^{\infty}(-1)^l\binom{\eta(k+1)-1}{l}\frac{\Gamma(2v+k)}{\Gamma(2v)k!}(\bar{\alpha})^k(l+1)^{-1}\right] \tag{26}$$

3.6 Stochastic Ordering

Stochastic ordering is the frequent way of expressing ordering mechanism in lifetime distributions. Let $X_1 \sim MOEGE(\alpha_1,\eta,\rho)$ and $X_2 \sim MOEGE(\alpha_2,\eta,\rho)$. The random variable X_2 is stochastically significant than X_1 in the:

Stochastic order $(X_1 \leq_s X_2)$ if the corresponding CDFs satisfies $F_{X_1} \leq F_{X_2}$ for all values x

Hazard rate order $(X_1 \leq_h X_2)$ if the corresponding CDFs satisfies $h_{X_1} \leq h_{X_2}$ for all values x

Likelihood ratio order $(X_1 \leq_l X_2)$ if $\frac{f_{X_1}}{f_{X_2}}$ is a decreasing function

Given the PDFs of X_1 and X_2,

$$\frac{f_{X_1}}{f_{X_2}} = \left(\frac{\alpha_1}{\alpha_2}\right)\left[\frac{1-\bar{\alpha}_1(1-[1-e^{-\rho x}]^{\eta})}{1-\bar{\alpha}_2(1-[1-e^{-\rho x}]^{\eta})}\right]^2$$

Taking the logarithm of both sides of the equation and differentiating the ratio of the densities gives

$$\frac{d}{dx}log\frac{f_{X_1}}{f_{X_2}} = 2\rho\eta e^{-\rho x}(1-e^{-\rho x})^{\eta-1}\left[\frac{\bar{\alpha}_1}{1-\bar{\alpha}_1(1-[1-e^{-\rho x}]^{\eta})} - \frac{\bar{\alpha}_2}{1-\bar{\alpha}_2(1-[1-e^{-\rho x}]^{\eta})}\right] < 0$$

If $\alpha_1 < \alpha_2$ for all values of x greater than zero. Then it follows from the implications of stochastic ordering that:

$$X_1 \leq_l X_2 \Rightarrow X_1 \leq_h X_2 \Rightarrow X_1 \leq_s X_2$$

3.8 Order Statistics

Suppose $x_{1:n} < x_{2:n} < \cdots < x_{n:n}$ represents order statistics obtained from the $MOEGE$ distribution. Then the PDF, $f_{s:n}(x)$, of the s^{th} order statistic $x_{s:n}$ is:

$$f_{s:n}(x) = \frac{1}{B(s,n-s+1)}[F(x)]^{s-1}[1-F(x)]^{n-s}f(x), \tag{27}$$

Reducing () using binomial series expansion gives

$$f_{s:n}(x) = \frac{1}{B(s,n-s+1)}\sum_{i=1}^{n}\binom{n-s}{i}[F(x)]^{s+i-1}f(x), \tag{28}$$

Where $F(x)$ and $f(x)$ are the CDF and PDF of the $MOEGE$ distribution respectively, and $B(.,.)$ is the beta function. Plugging the CDF and the PDF of the $MOEGE$ distribution in (28) follow by algebraic manipulation using (9) gives:

$$f_{s:n}(x) = \frac{\alpha\eta\rho e^{-\rho x}}{B(s,n-s+1)}\sum_{i=1}^{n}\sum_{j=1}^{s+i+1}\sum_{k=1}^{j}(-1)^{i+k}\binom{n-s}{i}\frac{\Gamma(s+i+j+1)}{\Gamma(s+i+1)}\bar{\alpha}^{j}(1-e^{-\rho x})^{\eta(k+1)-1} \tag{29}$$

Consequently, the PDFs of the smallest and the largest order statistics are respectively given by:

$$f_{X_1}(x) = \frac{\alpha\eta\rho e^{-\rho x}}{B(s,n-s+1)}\sum_{i=1}^{n}\sum_{j=1}^{s+i+1}\sum_{k=1}^{j}(-1)^{i+k}\binom{n-s}{i}\frac{\Gamma(s+i+j+1)}{\Gamma(s+i+1)}\bar{\alpha}^{j}(1-e^{-\rho x})^{\eta(k+1)-1}$$

and

$$f_{X_n}(x) = \frac{\alpha\eta\rho e^{-\rho x}}{B(s,n-s+1)}\sum_{i=1}^{n}\sum_{j=1}^{s+i+1}\sum_{k=1}^{j}(-1)^{i+k}\binom{n-s}{i}\frac{\Gamma(s+i+j+1)}{\Gamma(s+i+1)}\bar{\alpha}^{j}(1-e^{-\rho x})^{\eta(k+1)-1}$$

4. Parameter Estimation

In this section, the parameters of the $MOEGE$ distribution are estimated using the maximum-likelihood estimation method. Given a random sample $x_1, x_2, x_3, \ldots, x_n$of size n from the $MOEGE$ distribution with parameter vector $\psi = (\alpha, \rho, \eta)'$, then the log-likelihood function is given by:

$$l = \log(\alpha\eta\rho) - \rho\sum_{i=1}^{n}x_i + (\eta-1)\sum_{i=1}^{n}log(1-e^{-\rho x}) - 2\sum_{i=1}^{n}log[1-\bar{\alpha}_2(1-[1-e^{-\rho x}]^{\eta})] \tag{27}$$

By taking the partial derivatives of the log-likelihood function with respect to the parameters gives the component score vector $V_\psi = (V_\alpha, V_\rho, V_\eta)'$ as:

$$V_\alpha = \frac{\partial l}{\partial\alpha} = \frac{n}{\alpha} + 2\sum_{i=1}^{n}\frac{(1-e^{-\rho x_i})^{\eta}}{[1-\bar{\alpha}(1-[1-e^{-\rho x}]^{\eta})]} \tag{28}$$

$$V_\rho = \frac{\partial l}{\partial\rho} = \frac{n}{\rho} - \sum_{i=1}^{n}x_i + (\eta-1)\sum_{i=1}^{n}\left[\frac{xe^{-\rho x_i}}{(1-e^{-\rho x_i})}\right] + 2\sum_{i=1}^{n}\frac{\bar{\alpha}x_i e^{-\rho x_i}(1-e^{-\rho x_i})^{\eta}}{[1-\bar{\alpha}(1-[1-e^{-\rho x}]^{\eta})]} \tag{29}$$

$$V_\eta = \frac{\partial l}{\partial\eta} = \frac{n}{\eta} + \sum_{i=1}^{n}(1-e^{-\rho x_i}) + 2\sum_{i=1}^{n}\frac{\bar{\alpha}(1-e^{-\rho x_i})^{\eta}log(1-e^{-\rho x_i})}{[1-\bar{\alpha}(1-[1-e^{-\rho x}]^{\eta})]} \tag{30}$$

Setting $V_\psi = 0$ and solving then simultaneously gives the MLEs of $\hat{\alpha}$, $\hat{\rho}$, and $\hat{\eta}$

To estimate an approximate confidence intervals (CIs) of the parameters of $MOEGE$ distribution, it is important to obtain an estimated values of the elements of variance covariance matrix F of the MLEs. The variance-covariance matrix F is estimated by the observed information matrix $\hat{F}$,

Where

$$\hat{F} = -\begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}, \quad (31)$$

where J_{ij}, i, j = 1, 2, 3, are the second partial derivatives of (27) with respect to α, η, and ρ. They are the values of Fisher's information matrix analogous to α, η, and ρ, respectively. The diagonal element of the matrix in (31) gives the variances of the MLEs of α, η, and ρ, respectively. An approximate 100(1 − c)% confidence interval for θ_P as

$$\hat{\theta}_p \pm Z_{\frac{c}{2}}\sqrt{\widehat{var}(\hat{\theta}_p)},$$

Where $\hat{\theta}_p = (\hat{\alpha}, \hat{\eta}, \hat{\rho})$, $Z_{\frac{c}{2}}$ is the upper $(\frac{c}{2})$ 100^{th} percentile of SN distribution. We can use the likelihood ratio (LR) test to compare the fit of the $MOEGE$ distribution with its submodels for a given data set. For example, to test γ = 0, the LR statistic is $H = 2[ln((\hat{\alpha}, \hat{\eta}, \hat{\rho})) - ln(L(0, \tilde{\eta}, \tilde{\rho}))]$, where $\hat{\alpha}, \hat{\eta},$ and $\hat{\rho}$ are the unrestricted estimates and $\tilde{\eta}, \tilde{\rho}$ are the restricted estimates.

The LR test rejects the null hypothesis if $H > \chi^2_{\epsilon}$, where χ^2_{ϵ} denotes the upper 100 ∈ % point of the χ^2-distribution with 1 degree of freedom.

4.1 Simulation Study of $\boldsymbol{MOEGE}$ *Distribution*

The validity of the method of estimation used in obtaining the estimate of the parameters of the $MOEGE$ distribution can be ascertained via a simulation study. The following steps can be followed:

(1) By using (12), 2,000 samples of size n are obtained.

The variates of the $MOEGE$ distribution are developed using

$$X = -\frac{1}{\rho} log\left\{1 - \left[1 - \frac{1-u}{\alpha + \bar{\alpha}(1-u)}\right]^{1/\eta}\right\}, \quad 0 < u < 1 \quad (32)$$

(2) The MLEs are computed for the samples, say $\hat{\theta}_p = (\hat{\alpha}_p, \hat{\eta}_p, \hat{\rho}_p)$ for $p = 1, 2, \ldots, 1{,}000$.

(3)The mean square errors (MSEs) are calculated for every parameter.

The above steps were repeated for n = 50, 100, 200, 300, and 400 with $\alpha = 0.6$, $\eta = 0.6$, $\rho = 1.2$. Table 3 shows the absolute bias and standard error (SE) and the mean square error (MSEs) of α, η, and ρ. It can be deduced through the table that MSEs for individual parameters diminish to zero when sample size increases.

Table 3. AB, SE, and MSE for the $MOEGE$ parameters

Parameters	*Sample sizes*	*AB*	*SE*	*MSE*
ρ	50	0.8653	1.8784	4.2771
	100	0.8293	1.2250	2.1884
	200	0.7735	0.5678	0.9207
	300	0.5510	0.6038	0.6682
	400	0.2323	0.4202	0.2305
α	50	0.1180	0.1931	0.0512
	100	0.1796	0.1151	0.0455
	200	0.0768	0.0901	0.0140
	300	0.1063	0.0752	0.0170
	400	0.0648	0.0661	0.0086
η	50	0.6452	0.3098	0.5123
	100	0.5281	0.1950	0.3169
	200	0.4922	0.1438	0.2629
	300	0.4760	0.1133	0.2394
	400	0.4352	0.1004	0.1995

4.2 Application of $\boldsymbol{MOEGE}$ Distribution to Real Life Data Set

In this section, we compare the results of fitting the $MOEGE$, Marshall-Olkin Extended Exponential (MOEE), Generalized Exponential (GE) and Exponential distributions to one real data set. All the computations done using the R programming language (R Development Core Team, First, we consider the data set consisting of the length of intervals times at which vehicles pass a point on a road. The data are initially provided by Jorgensen (1982) and has been used by Lemonte et al. (2013). The Exploratory data analysis for the traffic data is given in Table 4 which shows that the data is positively skewed with excess kurtosis of -1.08 meaning that the data is platykurtic, since the value of the mean is less than the variance we can conclude that the data is over-dispersed. The Kernel density plot and the Total Time on Test (TTT) plot is given in Figure 3. The Maximum Likelihood Estimates (MLEs) of the model parameters with errors in parentheses, confidence interval in curly brackets and the values of the AICr (Akaike Information Criterion), CAICr (Consistent Akaike Information Criterion), BICr (Bayesian Information Criterion) and HQICr (Hannan-Quinin Information Criterion) are given in Table 5. From the values of these $MOEGE$ model is better than the $MOEE$, GE and E models.

To further compare the $MOEGE$ distribution with its sub-models, a Likelihood Ratio Test (LRT) is performed. The LRT results shown in Table 6 reveal that the MOEGE provides a better parametric fit to the Traffic data than its sub-models. The estimated variance-covariance matrix for the parameters of the $MOEGE$ distribution for the Traffic data is also obtained.

Table 4. Exploratory data analysis for the traffic data

Min.	q_1	q_2	q_3	Max.	Mean	Var.	skewness	Kurtosis
2.5	5.93	12.10	21.60	119.80	21.60	574.55	1.92	6.58

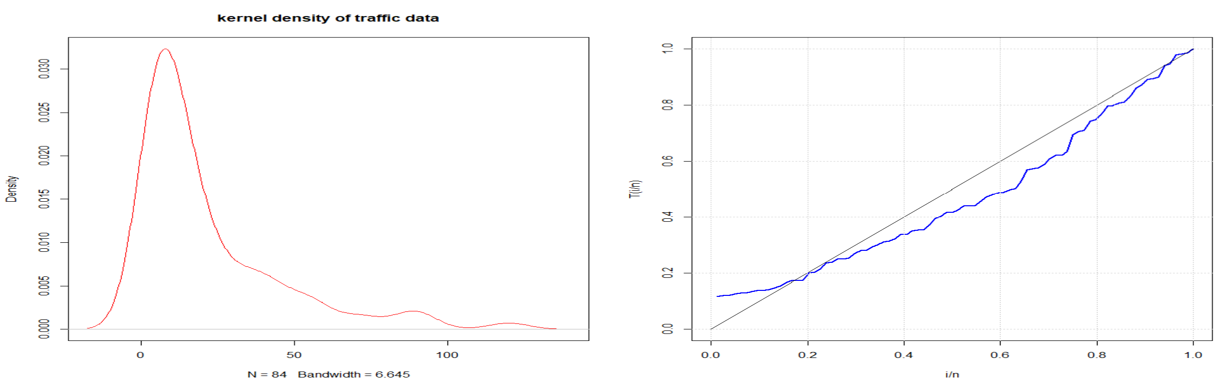

Figure 3. Plot of the kernel density and TTT plot for the Traffic data

Table 5. MLEs of the model parameters; traffic data

$\boldsymbol{Model}$	Estimates			Statistics				
	ρ	η	α	$-l$	$AICr$	$CAICr$	$HQICr$	$BICr$
$\boldsymbol{MOEGE}$	$6.95(5.38)$ $\{-3.59,17.49\}$	$0.03(0.01)$ $\{0.01,0.05\}$	$1.65(0.21)$ $\{1.24,2.06\}$	337.2	680.4	680.7	683.3	687.7
$\boldsymbol{MOEE}$	$1.48(0.56)$ $\{0.38,2.58\}$	$-(-)$ $\{-\}$	$0.04(0.01)$ $\{0.02,0.06\}$	341.5	687.0	687.2	689.0	691.9
$\boldsymbol{GE}$	$0.05(0.01)$ $\{0.03,0.07\}$	$1.12(0.17)$ $\{0.79,1.45\}$	$-(-)$ $\{-\}$	341.8	687.7	687.8	689.6	692.5
$\boldsymbol{E}$	$0.04(0.01)$ $\{0.02,0.06\}$	$-(-)$ $\{-\}$	$-(-)$ $\{-\}$	342.1	686.2	686.3	687.2	688.7

To further compare the performance of MOEGE distribution in modeling lifetime data with its sub-models, a Likelihood Ratio Test (LRT) is carried out. The LRT results given in Table 6 reveals that the MOEGE distribution provides a better parametric fit to the data than its sub-models which is also visible in Figure 4.

Table 6. LRT Statistics

$Models$	$Hypothesis$	$LRT\ Statistics$	$p-value$
GE	$H_0: \alpha = 1\ vs\ H_1 = H_0\ is\ false$	9.2	0.010
$MOEE$	$H_0: \eta = 1\ vs\ H_1 = H_0\ is\ false$	8.6	0.014
E	$H_0: \alpha = \eta = \theta = 1\ vs\ H_1 = H_0\ is\ false$	9.8	0.007

The variance-covariance matrix for the parameters of the MOEGE model for the traffic data set is:

$$F^{-1} = \begin{bmatrix} 0.41018 & 0.00252 & -0.18986 \\ 0.00252 & 0.00008 & -0.00017 \\ -0.18986 & 0.00002 & 0.13795 \end{bmatrix}$$

The empirical and fitted densities plots for the estimated CDFs and PDFs for the data are given in Figure

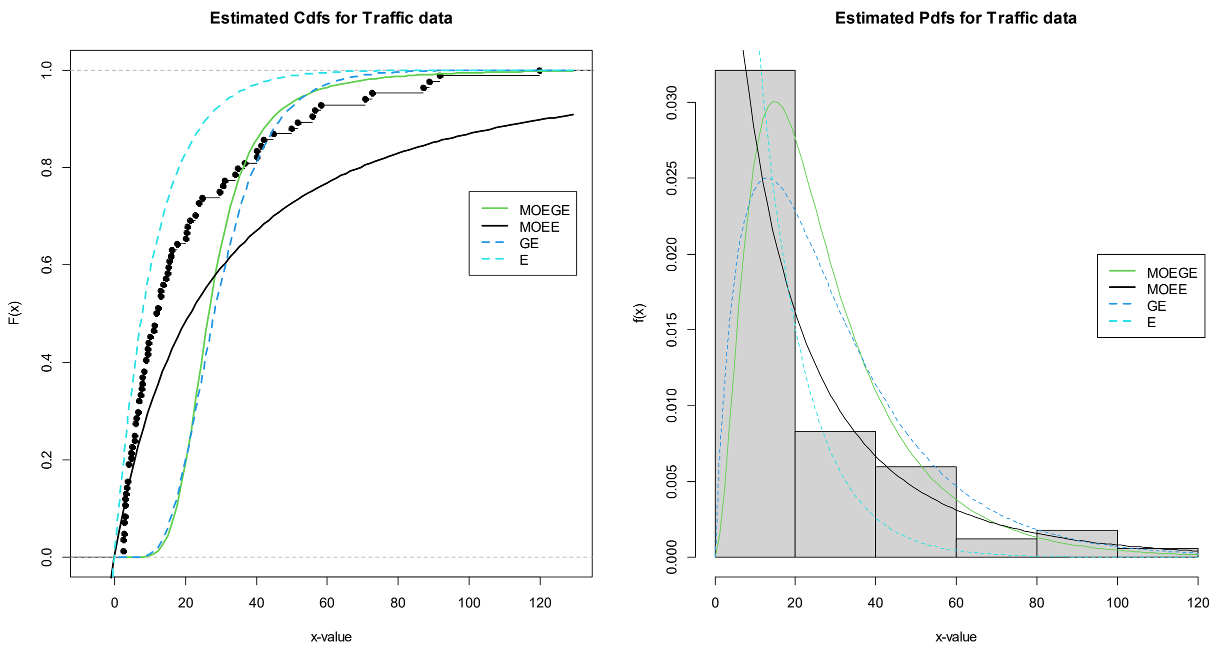

Figure 4. Plots of the estimated CDFs and PDFs for the Traffic data

5. Concluding Remarks

We have introduced a three parameter Marshall-Olkin Extended Generalized Exponential distribution as a suitable distribution in modeling lifetime data. Standard statistical properties of the new model were discuss which includes moments, incomplete moments, moment generating function, Stochastic ordering, Renyi entropy, and order statistics. Maximum likelihood estimation of the parameters are obtain and Monte Carlo simulation are performed to validate the properties of the estimator. Also, the observed information matrix for the model is obtained. Application of the Marshall-Olkin Extended Generalized Exponential distribution to a traffic data set shows that this distribution can produce a better fit than some known models. We expect that this generalization will attract wider applications in reliability and lifetime data analysis.

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