International Journal of Statistics and Probability; Vol. 12, No. 4; July 2023
ISSN 1927-7032 E-ISSN 1927-7040
Published by Canadian Center of Science and Education

# A Copula Based Investigation of Reliability for the Multivariate Exponential Family of Distributions

A. Nanthakumar

Correspondence: State University of New York at Oswego, Oswego NY 13126, USA

Received: May 21, 2023 Accepted: July 21, 2023 Online Published: July 30, 2023

doi:10.5539/ijsp.v12n4p64 URL: https://doi.org/10.5539/ijsp.v12n4p64

**Abstract**

This paper explores the possibility for using the copulas in the context of evaluating the reliability for the exponential family of distributions.

**Keywords:** exponential family, copula models, reliability

## 1. Introduction:

The exponential family of distributions have many applications in our lives. For example, our life span follows the exponential pattern. The life-time of electronic gadgets follow the exponential pattern. The exponential family of distributions are used in both statistical and engineering quality control analysis. These distributions are used in reliability evaluations. There has been steady interest in the reliability evaluations since the time of Marshall and Olkin Failure Model (1966). Bemis (1971) has chronicled the literature with regard to the exponential distribution prior to 1971. Ristic and Kundu (2015) summarize the literature about the research work done after 1971. Recently, the focus is on the generalized exponential family of distributions. However, there is very little known work related to copulas in the context of exponential distributions. This article somewhat seeks to fill the void in the context of exponential distributions.

The idea of using copulas to approximate the joint probability distributions originated after Sklar's Theorem (1959). Gumbel (1960), Clayton (1978) and Joe (1993) developed Copula models bearing their names. There were a sudden influx of research papers beginning from the 1980's in Economics, Finance, and Engineering. Frees and Valdez (1998) discuss in detail the copula construction and the applications in topics such as quantile regression, actuarial science and stochastic ordering. The interested readers are referred to Nelson (1999) for the literature review.

In this paper, we investigate the reliability in the context of exponential families by using the Copula models. We are interested in identifying the Copula model which yields a better approximation for the joint distribution. In this regard, we begin our investigation first with the Bivariate Exponential distribution. Next, we extend it to the Tri-variate situation. Overall, the Clayton Copula model seemed to perform better while approximating the Bivariate and Tri-variate Exponential distributions. Moreover, there is no difference whether we use the arithmetic mean or geometric mean or harmonic mean of the dependence parameter based on the pairwise copulas for the dependence parameter in the three variable situation.

## 2. Methodology

Here, we will use the Archimedean Copulas such as Clayton, Gumbel, and Frank and the non-Archimedean Copula such as the Gaussian Copula for modeling the joint distribution.

### Bivariate Exponential Distribution

First, we will look to see the how the bivariate-exponential arises in nature. Let us consider the situation where there are three independent exponential variables such that $U \sim \exp(\lambda_1)$, $V \sim \exp(\lambda_2)$, and $W \sim \exp(\lambda)$.

Let $X = \min(U,W)$, $Y = \min(V,W)$

$\Rightarrow$

$$F_X(x) = P(X \le x) = 1 - P(X > x)$$
$$= 1 - P(\min(U,W) > x) = 1 - P(U > x, V > x)$$

$$= 1 - P(U > x)P(W > x)$$

$$= 1 - e^{-\lambda_1 x} e^{-\lambda x}$$

$$= 1 - e^{-(\lambda_1+\lambda)x}$$

Similarly, $F_Y(y) = 1 - e^{-(\lambda_2+\lambda)y}$

Next, we will derive the joint distribution of $X$ and $Y$.

Let $H(x,y) = P(X \le x, Y \le y)$

$$= P(\min(U,W) \le x, \min(V,W) \le y)$$

Consider,

$$\tilde{H}(x,y) = P(\min(U,W) > x, \min(V,W) > y)$$

$$= P(U > x, V > y, W > \max(x,y))$$

$$= P(U > x)P(V > y)P(W > \max(x,y))$$

$$= e^{-\lambda_1 x} e^{-\lambda_2 y} e^{-\lambda \max(x,y)}$$

$$= e^{-\lambda_1 x - \lambda_2 y - \lambda \max(x,y)}$$

Note that, $H(x,y) - \tilde{H}(x,y) + 1 - F_X(x) + 1 F_Y(y) =$

$$\Rightarrow$$

$H(x,y) = 1 - e^{-(\lambda_1+\lambda)x} - e^{-(\lambda_2+\lambda)y} + e^{-\lambda_1 x - \lambda_2 y - \lambda \max(x,y)}$

**Archimedean Copula Models for Reliability**

Bivariate Exponential Distribution

Let $X$ be the stress and $Y$ be the strength of a material (say window glass panel). Let us further assume that the joint distribution of $X$ and $Y$ is bivariate exponential. As shown above, the cumulative joint distribution function for the bivariate exponential is of the form

$$H(x,y) = 1 - e^{-(\lambda_1+\lambda)x} - e^{-(\lambda_2+\lambda)y} + e^{-\lambda_1 x - \lambda_2 y - \lambda \max(x,y)} \tag{1}$$

From this equation we can infer by setting $y$ to $\infty$, the marginal distribution function of $X$ is

$$F(x) = 1 - e^{-(\lambda_1+\lambda)x} \tag{2}$$

From $(2)$ it is clear that the marginal distribution of $X$ is exponential.

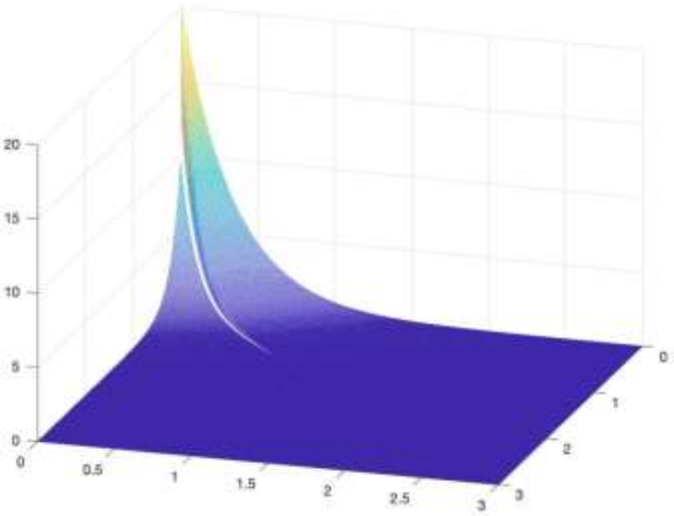
Similarly we can infer by setting $x$ to $\infty$, the marginal distribution function of $Y$ is

$$G(y) = 1 - e^{-(\lambda_2+\lambda)y} \tag{3}$$

From $(3)$ it is clear that the marginal distribution of $Y$ is exponential.

Moreover, the bivariate exponential density function is given by

$$f(x,y)=\begin{cases}(\lambda_1+\lambda)\lambda_2 e^{-(\lambda_1+\lambda)x-\lambda_2 y}, & x>y\\ \lambda_1(\lambda_2+\lambda)e^{-\lambda_1 x-(\lambda_2+\lambda)y}, & x<y\\ \lambda e^{-(\lambda_1+\lambda_2+\lambda)y}, & x=y\end{cases} \tag{4}$$




One can easily show that

$$E(XY)=\frac{(\lambda_1+\lambda_2+2\lambda)}{(\lambda_1+\lambda)(\lambda_2+\lambda)(\lambda_1+\lambda_2+\lambda)} \tag{5}$$

$$Cov(X,Y)=\frac{\lambda}{(\lambda_1+\lambda)(\lambda_2+\lambda)(\lambda_1+\lambda_2+\lambda)} \tag{6}$$

The theoretical correlation coefficient is given by

$$\rho=\frac{\lambda}{(\lambda_1+\lambda_2+\lambda)} \tag{7}$$

The theoretical reliability can be shown to be

$$\begin{aligned} R &= P(X\le Y)\\ &= P(X<Y)+P(X=Y)\\ &= \int_0^\infty\int_0^y f(x,y)\,dx\,dy + P(X=Y)\\ &= \int_0^\infty\int_0^y (\lambda_2+\lambda)\lambda_1 e^{-\lambda_1 x-(\lambda_2+\lambda)y}\,dx\,dy+\int_0^\infty \lambda e^{-(\lambda_1+\lambda_2+\lambda)y}\,dy\\ &= \frac{\lambda_1}{(\lambda_1+\lambda_2+\lambda)}+\frac{\lambda}{(\lambda_1+\lambda_2+\lambda)}\\ &= \frac{\lambda_1+\lambda}{(\lambda_1+\lambda_2+\lambda)} \end{aligned} \tag{8}$$

Next, we will use some Copula models for computing the reliability.

**Note:** As previously indicated $X$ is the stress endured by a glass panel and $Y$ is the strength of the glass material.

**Clayton Copula**

The Clayton Copula is defined as

$$C(x,y)=\left((F(x))^{-\alpha}+(G(y))^{-\alpha}-1\right)^{\frac{-1}{\alpha}} \tag{9}$$

and the Copula density is

$$\mathrm{c}(x,y) = \frac{\partial^2 C(x,y)}{\partial x\,\partial y} \tag{10}$$

$$c(x,y)=(1+\alpha)\left[\left(1-e^{-(\lambda_1+\lambda)x}\right)^{-\alpha}+\left(1-e^{-(\lambda_2+\lambda)y}\right)^{-\alpha}-1\right]^{-\frac{1}{\alpha}-2} .\left(1-e^{-(\lambda_1+\lambda)x}\right)^{-\alpha-1}\left(1-e^{-(\lambda_2+\lambda)y}\right)^{-\alpha-1}(\lambda_1+\lambda)(\lambda_2+\lambda)e^{-(\lambda_1+\lambda)x}e^{-(\lambda_2+\lambda)y} \tag{11}$$

For the Clayton Copula, we can estimate the dependence parameter $\alpha$ by using the equation

$$\frac{\alpha}{\alpha+2}\approx\rho=\frac{\lambda}{\lambda_1+\lambda_2+\lambda} \tag{12}$$

So, the theoretical reliability $R=P(X\le Y)$ can be approximated by the integral

$$\int_0^{\infty}\int_0^{y} c(x,y)\,dx\,dy$$

$$\approx\sum_{j=0}^{n^2}\sum_{i=0}^{[n\,y]}\left\{\left(1-e^{-\frac{i(\lambda_1+\lambda)}{n}}\right)^{-\alpha}+\left(1-e^{-\frac{j(\lambda_2+\lambda)}{n}}\right)^{-\alpha}-1\right\}^{-\frac{1}{\alpha}-2}\left(1-e^{-\frac{i(\lambda_1+\lambda)}{n}}\right)^{-\alpha-1}\left(1-e^{-\frac{j(\lambda_2+\lambda)}{n}}\right)^{-\alpha-1}e^{-\frac{i(\lambda_1+\lambda)}{n}}e^{-\frac{j(\lambda_2+\lambda)}{n}}\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)$$

Where $n$ means the number of very small partitions of the interval $(0,y)$. (13)

**Gumbel Copula**

The Gumbel Copula is defined as

$$C(x,y)=e^{-\left\{\left[-\ln F(x)\right]^{\alpha}+\left[-\ln G(y)\right]^{\alpha}\right\}^{\frac{1}{\alpha}}} \tag{14}$$

and the Copula density is

$$\mathrm{c}(x,y) = \frac{\partial^2 C(x,y)}{\partial x\,\partial y}$$

$$= e^{-\{[-\ln F(x)]^{\alpha}+[-\ln G(y)]^{\alpha}\}^{\frac{1}{\alpha}}}\left(-\ln F(x)\right)^{\alpha-1}\left(-\ln G(y)\right)^{\alpha-1}\frac{1}{F(x)}\frac{1}{G(y)}f(x)\,g(y)\left\{\left(-\ln F(x)\right)^{\alpha}+\left(-\ln G(y)\right)^{\alpha}\right\}^{\frac{1}{\alpha}-2} .\left\{(\alpha-1)+\left[\left(-\ln F(x)\right)^{\alpha}+\left(-\ln G(y)\right)^{\alpha}\right]^{\frac{1}{\alpha}}\right\} \tag{15}$$

For the Gumbel Copula, we can estimate the dependence parameter $\alpha$ by using the equation

$$1-\alpha^{-1}\approx\rho=\frac{\lambda}{\lambda_1+\lambda_2+\lambda} \tag{16}$$

So, again, the theoretical reliability $R=P(X\le Y)$ can be approximated by the integral

$$\int_0^{\infty}\int_0^{y} c(x,y)\;dx\,dy \approx\sum_{j=0}^{n^2}\sum_{i=0}^{[n\,y]}\left\{c\left(\frac{i}{n},\frac{j}{n}\right)\right\}\left(\frac{1}{n}\right)\left(\frac{1}{n}\right) \tag{17}$$

Where $n$ means the number of very small partitions of the interval $(0, y)$.

**Farlie-Gumbel Morgenstern (FGM) Copula**

The Farlie-Gumbel Morgenstern Copula is defined as

$$C(x, y) = F(x)G(y)\left(1+\alpha\left(1-F(x)\right)\left(1-G(y)\right)\right) \tag{18}$$

$$c(x,y) = \frac{\partial^2 C(x,y)}{\partial x \partial y}$$

$$= f(x)g(y)\left\{1+\alpha\left(1-2F(x)\right)\left(1-2G(y)\right)\right\} \tag{19}$$

For the Farlie-Gumbel Morgenstern Copula, we can estimate the dependence parameter $\alpha$ by using the equation,

$$\frac{1}{3^{\alpha}} \approx \rho = \frac{\lambda}{\lambda_1+\lambda_2+\lambda} \tag{20}$$

The reliability $R = P(X \leq Y)$ is approximated by the integral

$$\int_0^{\infty}\int_0^{y} c(x,y)\ dx\,dy$$

$$\approx \sum_{j=0}^{n^2}\sum_{i=0}^{[n\,y]} c\left(\frac{i}{n},\frac{j}{n}\right)\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)$$

Where $n$ means the number of very small partitions of the interval $(0, y)$ (21)

**Frank Copula**

The Frank Copula is defined as

$$C(x,y) = \frac{1}{\alpha}\ln\left\{1+\frac{\left(e^{\alpha F(x)}-1\right)\left(e^{\alpha G(y)}-1\right)}{\left(e^{\alpha}-1\right)}\right\} \tag{22}$$

$$c(x,y) = \frac{\partial^2 C(x,y)}{\partial x \partial y}$$

$$= \frac{\alpha}{\left(e^{\alpha}-1\right)} e^{\alpha F(x)}\, e^{\alpha G(y)} \left\{1+\frac{\left(e^{\alpha F(x)}-1\right)\left(e^{\alpha G(y)}-1\right)}{\left(e^{\alpha}-1\right)}\right\}^{-2} f(x)g(y) \tag{23}$$

For the Frank Copula, we can estimate the dependence parameter $\alpha$ by using the equation,

$$1-\frac{4}{\alpha}\left\{D_1(-\alpha)-1\right\} \approx \rho = \frac{\lambda}{\lambda_1+\lambda_2+\lambda} \tag{24}$$

where $D_1(x) = \frac{1}{x}\int_0^x \frac{t}{(e^t - 1)} dt$

and $D_1(-x) = D_1(x) + \frac{x}{2}$ for the negative values of $x$. (25)

Table 1. Archimedean Copula based reliability for bivariate exponential distribution

**Numerical Results for Reliability:** ($n = 200$ and 1000 simulation runs)

| Index | $\lambda_1$ | $\lambda_2$ | $\lambda$ | Theoretical | Clayton | Gumbel | FGM | Frank |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.5 | 0.3 | 0.2 | 0.700 | 0.751 | 0.751 | 0.751 | 0.751 |
| 2 | 0.3 | 0.2 | 0.3 | 0.750 | 0.697 | 0.697 | 0.697 | 0.697 |
| 3 | 0.4 | 0.2 | 0.1 | 0.714 | 0.809 | 0.809 | 0.809 | 0.809 |
| 4 | 0.5 | 2.5 | 1.0 | 0.375 | 0.340 | 0.341 | 0.341 | 0.344 |
| 5 | 1.5 | 1.0 | 0.5 | 0.667 | 0.729 | 0.729 | 0.728 | 0.729 |
| 6 | 3.0 | 2.0 | 2.0 | 0.714 | 0.700 | 0.696 | 0.696 | 0.698 |

**Bi-Variate Normal Distribution**

Next, we investigate the Bivariate Normal distribution. Let $X$ and $Y$ be random variables following a bivariate normal distribution.

$$f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{\frac{-0.5}{(1-\rho^2)}\left\{\left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \left(\frac{y-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right)\right\}} \tag{26}$$

The reliability

$$\begin{aligned} R &= P(X \le Y) \\ &= P(X < Y) + P(X = Y) \\ &= \int_{-\infty}^{\infty}\int_{x}^{\infty} f(x,y)\,dy\,dx + 0 \\ &= \int_{-\infty}^{\infty}\int_{x}^{\infty} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{\frac{-0.5}{(1-\rho^2)}\left\{\left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \left(\frac{y-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right)\right\}}\,dy\,dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_1} e^{-0.5\left(\frac{x-\mu_1}{\sigma_1}\right)^2} \left\{\int_{x}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_2\sqrt{1-\rho^2}} e^{\frac{-0.5}{\sigma_2^2(1-\rho^2)}\left(y-\mu_2-\rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)\right)^2}\right\} dy\,dx \end{aligned} \tag{27}$$

$$= E\left(1 - \Phi\left(\frac{x-\mu_2-\rho\left(\frac{\sigma_2}{\sigma_1}\right)(x-\mu_1)}{\sigma_2\sqrt{1-\rho^2}}\right)\right) \tag{28}$$

**Note:** In order to use the copula based approximations, we used the following results.

$$\mu_1 = E(X) = \frac{1}{\lambda_1 + \lambda} \quad , \quad \sigma_1 = \frac{1}{\lambda_1 + \lambda}$$

$$\mu_2 = E(Y) = \frac{1}{\lambda_2 + \lambda} \quad , \quad \sigma_2 = \frac{1}{\lambda_2 + \lambda}$$

$$\text{and} \quad \rho = \frac{\lambda}{\lambda_1 + \lambda_2 + \lambda}$$

**Clayton Copula**

For the bivariate normal distribution, the copula density based on the Clayton Copula is

$$c(x,y) = \frac{\partial^2 C(x,y)}{\partial x \partial y}$$

$$= (1+\alpha)\left\{\left[\Phi\left(\frac{x-\mu_1}{\sigma_1}\right)\right]^{-\alpha} + \left[\Phi\left(\frac{y-\mu_2}{\sigma_2}\right)\right]^{-\alpha} - 1\right\}^{\frac{-1}{\alpha}-2} \frac{1}{\sigma_1}\phi\left(\frac{x-\mu_1}{\sigma_1}\right)\frac{1}{\sigma_2}\phi\left(\frac{y-\mu_2}{\sigma_2}\right) \tag{29}$$

Where $\Phi(x)$ and $\phi(x)$ are the standard normal distribution and the standard normal density function respectively. So, the theoretical reliability $R = P(X \le Y)$ can be approximated by the integral

$$\int_0^\infty \int_0^y c(x,y)\, dx\, dy$$

$$\approx \sum_{j=0}^{n^2} \sum_{i=0}^{[ny]} \left\{\left(\Phi\left(\frac{\left(i/n\right)-\mu_1}{\sigma_1}\right)\right)^{-\alpha} + \left(\Phi\left(\frac{\left(j/n\right)-\mu_2}{\sigma_2}\right)\right)^{-\alpha} - 1\right\}^{-\frac{1}{\alpha}-2} \left(\Phi\left(\frac{\left(i/n\right)-\mu_1}{\sigma_1}\right)\right)^{-\alpha-1} \left(\Phi\left(\frac{\left(j/n\right)-\mu_2}{\sigma_2}\right)\right)^{-\alpha-1}$$

$$\phi\left(\frac{\left(i/n\right)-\mu_1}{\sigma_1}\right)\phi\left(\frac{\left(j/n\right)-\mu_2}{\sigma_2}\right)\left(\frac{1}{\sigma_1}\right)\left(\frac{1}{\sigma_2}\right)\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)(1+\alpha)$$

Where $n$ means the number of very small partitions of the interval $(0, y)$ (30)

**Frank Copula:**

The Frank Copula is defined as

$$C(x,y) = \frac{1}{\alpha}\ln\left\{1 + \frac{\left(e^{\alpha\Phi\left(\frac{x-\mu_1}{\sigma_1}\right)} - 1\right)\left(e^{\alpha\Phi\left(\frac{y-\mu_2}{\sigma_2}\right)} - 1\right)}{\left(e^{\alpha} - 1\right)}\right\} \tag{31}$$

$$c(x,y) = \frac{\partial^2 C(x,y)}{\partial x \partial y}$$

$$= \frac{\alpha}{\left(e^{\alpha}-1\right)} e^{\alpha\Phi\left(\frac{x-\mu_1}{\sigma_1}\right)} e^{\alpha\Phi\left(\frac{y-\mu_2}{\sigma_2}\right)} \left\{1 + \frac{\left(e^{\alpha\Phi\left(\frac{x-\mu_1}{\sigma_1}\right)} - 1\right)\left(e^{\alpha\Phi\left(\frac{y-\mu_2}{\sigma_2}\right)} - 1\right)}{\left(e^{\alpha}-1\right)}\right\}^{-2} \phi\left(\frac{x-\mu_1}{\sigma_1}\right)\phi\left(\frac{y-\mu_2}{\sigma_2}\right)\left(\frac{1}{\sigma_1}\right)\left(\frac{1}{\sigma_2}\right) \tag{32}$$

For the Frank Copula, we can estimate the dependence parameter $\alpha$ by using the equation,

$$1-\frac{4}{\alpha}\left\{D_1(-\alpha)-1\right\}\approx\rho=\frac{\lambda}{\lambda_1+\lambda_2+\lambda} \tag{33}$$

where $D_1(x)=\frac{1}{x}\int_0^x\frac{t}{\left(e^t-1\right)}dt$

and

$$D_1(-x)=D_1(x)+\frac{x}{2} \quad \text{for the negative values of } x. \tag{34}$$

**Gumbel Copula**

The Gumbel Copula density is

$$c(x,y)=\frac{\partial^2C(x,y)}{\partial x\,\partial y}$$

$$=e^{-\left\{\left[-\ln\Phi\left(\frac{x-\mu_1}{\sigma_1}\right)\right]^{\alpha}+\left[-\ln\Phi\left(\frac{y-\mu_2}{\sigma_2}\right)\right]^{\alpha}\right\}^{\frac{1}{\alpha}}}\left(-\ln\Phi\left(\frac{x-\mu_1}{\sigma_1}\right)\right)^{\alpha-1}\left(-\ln\Phi\left(\frac{y-\mu_2}{\sigma_2}\right)\right)^{\alpha-1}\frac{1}{\Phi\left(\frac{x-\mu_1}{\sigma_1}\right)}\frac{1}{\Phi\left(\frac{y-\mu_2}{\sigma_2}\right)}$$

$$.\left(\frac{1}{\sigma_1}\right)\left(\frac{1}{\sigma_2}\right)\varphi\left(\frac{x-\mu_1}{\sigma_1}\right)\varphi\left(\frac{y-\mu_2}{\sigma_2}\right)\left\{\left(-\ln\Phi\left(\frac{x-\mu_1}{\sigma_1}\right)\right)^{\alpha}+\left(-\ln\Phi\left(\frac{y-\mu_2}{\sigma_2}\right)\right)^{\alpha}\right\}^{\frac{1}{\alpha}-2} \tag{35}$$

$$.\left\{(\alpha-1)+\left[\left(-\ln\Phi\left(\frac{x-\mu_1}{\sigma_1}\right)\right)^{\alpha}+\left(-\ln\Phi\left(\frac{y-\mu_2}{\sigma_2}\right)\right)^{\alpha}\right]^{\frac{1}{\alpha}}\right\}$$

Gumbel Copula, we can estimate the dependence parameter $\alpha$ by using the equation

$$1-\alpha^{-1}\approx\rho=\frac{\lambda}{\lambda_1+\lambda_2+\lambda} \tag{36}$$

So, again, the theoretical reliability $R=P(X\le Y)$ can be approximated by the integral

$$\int_0^{\infty}\int_0^{y}c(x,y)\;dx\,dy$$

$$\approx\sum_{j=0}^{n^2}\sum_{i=0}^{[n\,y]}\left\{c\left(\frac{i}{n},\frac{j}{n}\right)\right\}\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)$$

Where $n$ means the number of very small partitions of the interval $(0,y)$ (37)

**Farlie-Gumbel Morgenstern (FGM) Copula**

The Farli-Gumbel Morgenstern Copula is defined as

$$C(x,y)=F(x)G(y)+\alpha\left(1-F(x)\right)\left(1-G(y)\right) \tag{38}$$

$$c(x,y) = \frac{\partial^2 C(x,y)}{\partial x \partial y}$$

$$= f(x)g(y)\{1+\alpha(1-2F(x))(1-2G(y))\} \tag{39}$$

For the Farlie-Gumbel Morgenstern Copula, we can estimate the dependence parameter $\alpha$ by using the equation,

$$\frac{1}{3^{\alpha}} \approx \rho = \frac{\lambda}{\lambda_1+\lambda_2+\lambda} \tag{40}$$

The reliability $R = P(X \le Y)$ is approximated by the numerical integral as in the other copulas.

Again note that in the context of bivariate normal distribution we are using the parameters,

$$\mu_1 = \sigma_1 = 1/(\lambda_1+\lambda)$$

$$\mu_2 = \sigma_2 = 1/(\lambda_2+\lambda)$$

$$\rho = \lambda/(\lambda_1+\lambda_2+\lambda)$$

Table 2. Archimedean Copula based reliability for bivariate normal distribution

**Numerical Results for Reliability:** **(** $n = 200$ and 1000 simulation runs**)**

| Index | $\lambda_1$ | $\lambda_2$ | $\lambda$ | Theoretical | Clayton | Gumbel | FGM | Frank |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.5 | 0.3 | 0.2 | 0.614 | 0.426 | 0.411 | 0.400 | 0.215 |
| 2 | 0.3 | 0.2 | 0.3 | 0.571 | 0.368 | 0.344 | 0.341 | 0.197 |
| 3 | 0.4 | 0.2 | 0.1 | 0.657 | 0.504 | 0.492 | 0.483 | 0.232 |
| 4 | 0.5 | 2.5 | 1.0 | 0.296 | 0.099 | 0.097 | 0.093 | 0.085 |
| 5 | 1.5 | 1.0 | 0.5 | 0.601 | 0.401 | 0.390 | 0.375 | 0.209 |
| 6 | 3.0 | 2.0 | 2.0 | 0.583 | 0.381 | 0.362 | 0.355 | 0.201 |

As you can see from the numerical results, the Archimedean Copulas such as Clayton, Gumbel, Frank, and Farlie-Gumbel Morgenstern (FGM) are not suitable for computing the reliability based on a bivariate normal distribution.

Next, we will investigate the tri-variate exponential distribution.

**Tri-variate Exponential Distribution**

In this section, we deal with the tri-variate exponential distribution.

Let us define the following **independen**t variables as follows

$$U \sim \exp(\lambda_1)$$

$$V \sim \exp(\lambda_2)$$

$$W \sim \exp(\lambda_3)$$

$$Z_1 \sim \exp(\lambda_{123})$$

$$Z_2 \sim \exp(\lambda_{12})$$

$$Z_3 \sim \exp(\lambda_{23})$$

$$Z_4 \sim \exp(\lambda_{13})$$

$$X = \min(U, Z_1, Z_2, Z_4)$$

$$Y = \min(V, Z_1, Z_2, Z_3)$$

$$Z = \min(W, Z_1, Z_3, Z_4) \tag{41}$$

We can verify the joint distribution as follows.

$$\begin{aligned} &P(X \geq x, Y \geq y, Z \geq z) \\ &= P(U \geq x, Z_1 \geq x, Z_2 \geq x, Z_4 \geq x, V \geq y, Z_1 \geq y, Z_2 \geq y, Z_3 \geq y, W \geq z, Z_1 \geq z, Z_3 \geq z, Z_4 \geq z) \\ &= P(U \geq x, V \geq y, W \geq z, Z_1 \geq \max(x,y,z), Z_2 \geq \max(x,y), Z_3 \geq \max(y,z), Z_4 \geq \max(x,z)) \\ &= P(U > x)P(V > y)P(W > z)P(Z_1 > \max(x,y,z))P(Z_2 > \max(x,y)) \\ &\quad . \ P(Z_3 > \max(y,z))P(Z_4 > \max(x,z)) \\ &= e^{-\lambda_1 x} e^{-\lambda_2 y} e^{-\lambda_3 z} e^{-\lambda_{123}\max(x,y,z)} e^{-\lambda_{12}\max(x,y)} e^{-\lambda_{23}\max(y,z)} e^{-\lambda_{13}\max(x,z)} \\ &= e^{-\lambda_1 x - \lambda_2 y - \lambda_3 z - \lambda_{12}\max(x,y) - \lambda_{13}\max(x,z) - \lambda_{23}\max(y,z) - \lambda_{123}\max(x,y,z)} \end{aligned} \tag{42}$$

$\Rightarrow$

$$\begin{aligned} F_{xyz}(x,y,z) = 1 &- e^{-(\lambda_1+\lambda_{12}+\lambda_{13}+\lambda_{123})x} - e^{-(\lambda_2+\lambda_{12}+\lambda_{23}+\lambda_{123})y} - e^{-(\lambda_3+\lambda_{13}+\lambda_{23}+\lambda_{123})z} \\ &+ e^{-(\lambda_1+\lambda_{13})x-(\lambda_2+\lambda_{23})y-(\lambda_{12}+\lambda_{123})\max(x,y)} \\ &+ e^{-(\lambda_2+\lambda_{12})y-(\lambda_3+\lambda_{13})z-(\lambda_{23}+\lambda_{123})\max(y,z)} + e^{-(\lambda_1+\lambda_{12})x-(\lambda_3+\lambda_{23})z-(\lambda_{13}+\lambda_{123})\max(x,z)} \\ &- e^{-\lambda_1 x-\lambda_2 y-\lambda_3 z-\lambda_{12}\max(x,y)-\lambda_{13}\max(x,z)-\lambda_{23}\max(y,z)-\lambda_{123}\max(x,y,z)} \end{aligned} \tag{43}$$

$\Rightarrow$

$$F_1(x) = 1 - e^{-(\lambda_1+\lambda_{12}+\lambda_{13}+\lambda_{123})x}$$

$$F_2(y) = 1 - e^{-(\lambda_2+\lambda_{12}+\lambda_{23}+\lambda_{123})y}$$

$$F_3(z) = 1 - e^{-(\lambda_3+\lambda_{13}+\lambda_{23}+\lambda_{123})z}$$

Case 1: $x < y < z$

$$F_{123}\left(x,y,z\right)=1-e^{-\left(\lambda_1+\lambda_{12}+\lambda_{13}+\lambda_{123}\right)x}-e^{-\left(\lambda_2+\lambda_{12}+\lambda_{23}+\lambda_{123}\right)y}-e^{-\left(\lambda_3+\lambda_{13}+\lambda_{23}+\lambda_{123}\right)z}$$
$$+e^{-\left(\lambda_1+\lambda_{13}\right)x-\left(\lambda_2+\lambda_{23}\right)y-\left(\lambda_{12}+\lambda_{123}\right)y}$$
$$+e^{-\left(\lambda_2+\lambda_{12}\right)y-\left(\lambda_3+\lambda_{13}\right)z-\left(\lambda_{23}+\lambda_{123}\right)z}+e^{-\left(\lambda_1+\lambda_{12}\right)x-\left(\lambda_3+\lambda_{23}\right)z-\left(\lambda_{13}+\lambda_{123}\right)z}$$
$$-e^{-\lambda_1x-\lambda_2y-\lambda_3z-\lambda_{12}y-\lambda_{13}z-\lambda_{23}z-\lambda_{123}z} \tag{44}$$

The density function

$$f_{123}\left(x,y,z\right)=\lambda_1\left(\lambda_2+\lambda_{12}\right)\left(\lambda_3+\lambda_{13}+\lambda_{23}+\lambda_{123}\right)e^{-\lambda_1x-\left(\lambda_2+\lambda_{12}\right)y-\left(\lambda_3+\lambda_{13}+\lambda_{23}+\lambda_{123}\right)z}\quad,\quad x<y<z$$

Similarly for the other cases, the density function can be found and represented by the subscript order.

Suppose that we want to compute the probability that

$$R_1=P\left(X<Y<Z\right)$$

$$=\int_0^\infty\int_x^\infty\int_y^\infty f_{123}\left(x,y,z\right)dz\,dy\,dx$$

$$=\frac{\lambda_1\left(\lambda_2+\lambda_{12}\right)}{\left(\lambda_1+\lambda_2+\lambda_3+\lambda_{12}+\lambda_{13}+\lambda_{23}+\lambda_{123}\right)\left(\lambda_2+\lambda_{12}+\lambda_3+\lambda_{13}+\lambda_{23}+\lambda_{123}\right)} \tag{45}$$

Case 2: $y<x<z$

$$R_2=P\left(Y<X<Z\right)$$

$$=\int_0^\infty\int_y^\infty\int_x^\infty f_{213}\left(x,y,z\right)dz\,dx\,dy$$

$$=\frac{\lambda_2\left(\lambda_1+\lambda_{12}\right)}{\left(\lambda_1+\lambda_2+\lambda_3+\lambda_{12}+\lambda_{13}+\lambda_{23}+\lambda_{123}\right)\left(\lambda_1+\lambda_{12}+\lambda_3+\lambda_{13}+\lambda_{23}+\lambda_{123}\right)} \tag{46}$$

Case 3: $x<z<y$

$$R_3=P\left(X<Z<Y\right)$$

$$=\int_0^\infty\int_x^\infty\int_z^\infty f_{132}\left(x,y,z\right)dy\,dz\,dx$$

$$=\frac{\lambda_1\left(\lambda_3+\lambda_{13}\right)}{\left(\lambda_1+\lambda_2+\lambda_3+\lambda_{12}+\lambda_{13}+\lambda_{23}+\lambda_{123}\right)\left(\lambda_2+\lambda_{12}+\lambda_3+\lambda_{13}+\lambda_{23}+\lambda_{123}\right)} \tag{47}$$

Case 4: $z<x<y$

$$R_4=P\left(Z<X<Y\right)$$

$$=\int_0^\infty\int_z^\infty\int_x^\infty f_{312}\left(x,y,z\right)dy\,dx\,dz$$

$$= \frac{\lambda_3(\lambda_1+\lambda_{13})}{(\lambda_1+\lambda_2+\lambda_3+\lambda_{12}+\lambda_{13}+\lambda_{23}+\lambda_{123})(\lambda_1+\lambda_{12}+\lambda_2+\lambda_{13}+\lambda_{23}+\lambda_{123})} \tag{48}$$

Case 5: $y < z < x$

$$R_5 = P(Y < Z < X)$$

$$= \int_0^\infty \int_y^\infty \int_z^\infty f_{231}(x,y,z)\,dx\,dz\,dy$$

$$= \frac{\lambda_2(\lambda_3+\lambda_{23})}{(\lambda_1+\lambda_2+\lambda_3+\lambda_{12}+\lambda_{13}+\lambda_{23}+\lambda_{123})(\lambda_1+\lambda_{12}+\lambda_3+\lambda_{13}+\lambda_{23}+\lambda_{123})} \tag{49}$$

Case 6: $z < y < x$

$$R_6 = P(Z < Y < X)$$

$$= \int_0^\infty \int_z^\infty \int_y^\infty f_{321}(x,y,z)\,dx\,dy\,dz$$

$$= \frac{\lambda_3(\lambda_2+\lambda_{23})}{(\lambda_1+\lambda_2+\lambda_3+\lambda_{12}+\lambda_{13}+\lambda_{23}+\lambda_{123})(\lambda_1+\lambda_{12}+\lambda_2+\lambda_{13}+\lambda_{23}+\lambda_{123})} \tag{50}$$

**Note 1:**

$$P(X = Y = Z) = \frac{\lambda_{123}}{\lambda_1+\lambda_2+\lambda_3+\lambda_{12}+\lambda_{13}+\lambda_{23}+\lambda_{123}} \tag{51}$$

**Note 2:**

$$Cov(X,Y) = \frac{\lambda_{12}+\lambda_{123}}{(\lambda_1+\lambda_{12}+\lambda_{13}+\lambda_{123})(\lambda_2+\lambda_{12}+\lambda_{23}+\lambda_{123})(\lambda_1+\lambda_2+\lambda_{12}+\lambda_{13}+\lambda_{23}+\lambda_{123})} \tag{52}$$

$\Rightarrow$

$$\rho_{X,Y} = \frac{\lambda_{12}+\lambda_{123}}{\lambda_1+\lambda_2+\lambda_{12}+\lambda_{13}+\lambda_{23}+\lambda_{123}} \tag{53}$$

where $\rho_{X,Y}$ means the correlation coefficient of $X$ and $Y$.

**Note 3:** $$P(X = Y) = \frac{\lambda_{123}}{\lambda_1+\lambda_2+\lambda_{12}+\lambda_{13}+\lambda_{23}+\lambda_{123}} \tag{54}$$

**Copula Construction**

Here we will investigate the Copula models for approximating the tri-variate exponential distribution.

Let $u_1 = P(X \le x)$

$$u_2 = P(Y \le y)$$

$$u_3 = P(Z \le z)$$

**Clayton Copula:**

$$C(u_1, u_2, u_3) = \left(u_1^{-\alpha} + u_2^{-\alpha} + u_3^{-\alpha} - 2\right)^{\frac{-1}{\alpha}} \tag{55}$$

**Notations**

$\alpha_{ij} =$ The dependence parameter of the pairwise copula linking variables $X_i$ and $X_j$

$\rho_{ij} =$ The correlation coefficient of the variables $X_i$ and $X_j$

$\alpha =$ The overall dependence parameter

$\rho = P(X = Y = Z)$

First, we will discuss this in the context of Clayton Copula.

$(i)$. Simple Reliability (probability) based approach where $\alpha = \frac{2\rho}{1-\rho}$ and $\rho$ is as described above in equation (51).

For other approaches listed below, in the case of Clayton Copula $\alpha_{ij} \approx \frac{2\rho_{ij}}{1-\rho_{ij}}$

$(ii)$. $\alpha$ is the harmonic mean of $\alpha_{12}, \alpha_{23}, \alpha_{13}$.

$(iii)$. $\alpha$ is the geometric mean of $\alpha_{12}, \alpha_{23}, \alpha_{13}$.

$(iv)$. $\alpha$ is the arithmetic mean of $\alpha_{12}, \alpha_{23}, \alpha_{13}$.

**Gumbel Copula**

$$C(u_1, u_2, u_3) = e^{-\left\{(-\ln(u_1))^{\alpha} + (-\ln(u_2))^{\alpha} + (-\ln(u_3))^{\alpha}\right\}^{\frac{1}{\alpha}}} \tag{56}$$

Similarly, we propose methods to estimate the dependence parameter $\alpha$. We will discuss this in the context of Gumbel Copula.

$(i)$. Simple Reliability (probability) based approach where $\alpha = \frac{1}{1-\rho}$ and $\rho$ is as described in equation (51).

For other approaches, in the case of Gumbel Copula, $\alpha_{ij} \approx \frac{1}{1-\rho_{ij}}$ where $\rho_{ij}$ is the correlation coefficient.

$(ii)$. $\alpha$ is the harmonic mean of $\alpha_{12}, \alpha_{23}, \alpha_{13}$.

$(iii)$. $\alpha$ is the geometric mean of $\alpha_{12}, \alpha_{23}, \alpha_{13}$.

$(iv)$. $\alpha$ is the arithmetic mean of $\alpha_{12}, \alpha_{23}, \alpha_{13}$.

**Frank Copula**

$$C\left(u_1,u_2,u_3\right)=\frac{1}{\alpha}\ln\left[1+\frac{\left(e^{\alpha u_1}-1\right)\left(e^{\alpha u_2}-1\right)\left(e^{\alpha u_3}-1\right)}{\left(e^{\alpha}-1\right)^2}\right] \tag{57}$$

Again along the same lines, we propose methods to estimate the dependence parameter $\alpha$ . We will discuss this in the context of Frank Copula.

$(i)$. Simple Reliability (probability) based approach where $\rho\approx 1-\frac{4}{\alpha}\left[D_1\left(-\alpha\right)-1\right]$ and $D_1\left(x\right)=\frac{1}{x}\int_0^x\frac{t}{\left(e^t-1\right)}dt$ is the "Debye" function and $D_1\left(-x\right)=D_1\left(x\right)+\frac{x}{2}$

For other approaches given below, we will use the correlation coefficient $\rho_{ij}$ .

$(ii)$. $\alpha$ is the harmonic mean of $\alpha_{12},\alpha_{23},\alpha_{13}$.

$(iii)$. $\alpha$ is the geometric mean of $\alpha_{12},\alpha_{23},\alpha_{13}$.

$(iv)$. $\alpha$ is the arithmetic mean of $\alpha_{12},\alpha_{23},\alpha_{13}$.

**Three variate Pairwise Hierarchical Copula**

As seen from the accompanying hierarchical copula diagram, at the top level, the generator function is $\psi_1$ . At the next level, the generator is $\psi_{12}$ . There is a hierarchy in the level arrangement. Note that $\alpha_1$ and $\alpha_{12}$ are the dependence parameters at the first level and second level respectively. The variables that exhibit the higher order of correlation are placed at the higher level.

So, one can write the hierarchical copula as follows.

$$C\left(u_1,u_2,u_3\right)=\psi_1^{-1}\left(\psi_1\left(C_2\left(u_1,u_2\right)\right)+\psi_1\left(u_3\right)\right) \tag{58}$$

**Clayton Copula (Hierarchical)**

$$C\left(u_1,u_2,u_3\right)=\left[1+\psi_1\left(C_2\left(u_1,u_2\right)\right)+\psi_1\left(u_3\right)\right]^{-\frac{1}{\alpha_1}}$$

$$=\left[1+\psi_1\left(\psi_{12}^{-1}\left(\psi_{12}\left(u_1\right)+\psi_{12}\left(u_2\right)\right)\right)+\psi_1\left(u_3\right)\right]^{-\frac{1}{\varepsilon_1}}$$

$$=\left[1+\left[\psi_{12}^{-1}\left(\psi_{12}\left(u_1\right)+\psi_{12}\left(u_2\right)\right)\right]^{-\alpha_1}-1+\psi_1\left(u_3\right)\right]^{-\frac{1}{\alpha_1}\,1}$$

$$=\left[\left[\psi_{12}^{-1}\left(\psi_{12}\left(u_1\right)+\psi_{12}\left(u_2\right)\right)\right]^{-\alpha_1}+u_3^{-\alpha_1}-1\right]^{-\frac{1}{\alpha_1}}$$

$$=\left[\left(1+\psi_{12}\left(u_1\right)+\psi_{12}\left(u_2\right)\right)^{\frac{\alpha_1}{\alpha_{12}}}+u_3^{-\alpha_1}-1\right]^{-\frac{1}{\alpha_1}}$$

$$=\left[\left(1+u_1^{-\alpha_{12}}+u_2^{-\alpha_{12}}-2\right)^{\frac{\alpha_1}{\alpha_{12}}}+u_3^{-\alpha_1}-1\right]^{-\frac{1}{\alpha_1}}$$

$$=\left[\left(u_1^{-\alpha_{12}}+u_2^{-\alpha_{12}}-1\right)^{\frac{\alpha_1}{\alpha_{12}}}+u_3^{-\alpha_1}-1\right]^{-\frac{1}{\alpha_1}} \tag{59}$$

**Gumbel Copula (Hierarchical)**

$$C\left(u_1,u_2,u_3\right)=e^{-\left\{\psi_1\left(C_{12}\left(u_1,u_2\right)\right)+\psi_1\left(u_3\right)\right\}^{\frac{1}{\alpha}}}$$

$$= e^{-\left\{\left[\left(-\ln(u_1)\right)^{\alpha_{12}} + \left(-\ln(u_2)\right)^{\alpha_{12}}\right]^{\frac{\alpha}{\alpha_{12}}} + \left(-\ln(u_3)\right)^{\alpha}\right\}^{\frac{1}{\alpha}}} \quad (60)$$

**Frank Copula (Hierarchical)**

$$C(u_1, u_2, u_3) = \frac{1}{\alpha}\ln\left[1 + \frac{\left(e^{\alpha_{12} u_1} - 1\right)\left(e^{\alpha u_2} - 1\right)\left(e^{\alpha u_3} - 1\right)}{\left(e^{\alpha_{12}} - 1\right)\left(e^{\alpha} - 1\right)}\right] \quad (61)$$

**Numerical Simulation**

Here, we compare the performance of Clayton, Gumbel, and Frank Copulas in approximating the Tri-variate Exponential Distribution. The results are based on 10000 simulation runs.

In order to compare the performance, we will use absolute percentage error $\left(\frac{|C - F_{xyz}|}{F_{xyz}}\right) \times 100\%$

where $F_{xyz}$ represents the cumulative distribution of the Tri-variate Exponential and $C$ is the Copula.

**NOTATIONS (in Table 3):**

$s$ : Simple reliability based dependence parameter

$a$ : Arithmetic Mean based dependence parameter

$h$ : Harmonic Mean based dependence parameter

$g$ : Geometric Mean based dependence parameter

$hi$ : Hierarchical Copula based parameter

Table 3. Absolute Percentage Error in **Percentage**

| $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_{12}$ | $\lambda_{13}$ | $\lambda_{23}$ | $\lambda_{123}$ | Clayton $s$ | Clayton $a$ | Clayton $hi$ | Clayton $h$ | Clayton $g$ | Gumbel $s$ | Gumbel $a$ | Gumbel $hi$ | Gumbel $h$ | Gumbel $g$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 6 | 7 | 0.9 | 0.6 | 1.4 | 1.2 | 22 | 13 | 19 | 14 | 14 | 29 | 23 | 27 | 23 | 23 |
| 1 | 2 | 3 | 0.9 | 0.6 | 1.4 | 1.2 | 27 | 14 | 22 | 15 | 14 | 36 | 26 | 33 | 26 | 26 |
| 3 | 4 | 5 | 0.9 | 0.6 | 1.4 | 1.2 | 25 | 15 | 21 | 15 | 15 | 32 | 25 | 29 | 25 | 25 |
| 7 | 6 | 5 | 0.9 | 0.6 | 1.4 | 1.2 | 22 | 14 | 20 | 15 | 14 | 29 | 23 | 27 | 23 | 23 |
| 3 | 2 | 1 | 0.9 | 0.6 | 1.4 | 1.2 | 29 | 16 | 25 | 17 | 16 | 38 | 28 | 34 | 28 | 28 |
| 5 | 4 | 3 | 0.9 | 0.6 | 1.4 | 1.2 | 26 | 16 | 23 | 16 | 16 | 33 | 26 | 31 | 26 | 26 |
| 5 | 6 | 7 | 0.6 | 0.9 | 1.2 | 1.4 | 22 | 15 | 20 | 15 | 15 | 29 | 23 | 28 | 24 | 24 |
| 1 | 2 | 3 | 0.6 | 0.9 | 1.2 | 1.4 | 26 | 15 | 23 | 15 | 15 | 36 | 27 | 34 | 27 | 27 |
| 3 | 4 | 5 | 0.6 | 0.9 | 1.2 | 1.4 | 24 | 16 | 22 | 16 | 16 | 33 | 31 | 26 | 26 | 26 |
| 7 | 6 | 5 | 0.6 | 0.9 | 1.2 | 1.4 | 23 | 15 | 21 | 16 | 16 | 30 | 24 | 28 | 24 | 24 |
| 3 | 2 | 1 | 0.6 | 0.9 | 1.2 | 1.4 | 28 | 17 | 26 | 18 | 17 | 38 | 29 | 36 | 29 | 29 |
| 5 | 4 | 3 | 0.6 | 0.9 | 1.2 | 1.4 | 25 | 17 | 23 | 17 | 17 | 33 | 27 | 32 | 27 | 27 |

Table 3. (continuation) for Frank Copula Absolute Percentage Error in **Percentage**

| $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_{12}$ | $\lambda_{13}$ | $\lambda_{23}$ | $\lambda_{123}$ | $s$ | $a$ | $hi$ | $h$ | $g$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | Frank | | | | |
| 5 | 6 | 7 | 0.9 | 0.6 | 1.4 | 1.2 | 45 | 44 | 57 | 44 | 44 |
| 1 | 2 | 3 | 0.9 | 0.6 | 1.4 | 1.2 | 48 | 48 | 58 | 48 | 48 |
| 3 | 4 | 5 | 0.9 | 0.6 | 1.4 | 1.2 | 41 | 41 | 58 | 41 | 41 |
| 7 | 6 | 5 | 0.9 | 0.6 | 1.4 | 1.2 | 37 | 37 | 63 | 37 | 37 |
| 3 | 2 | 1 | 0.9 | 0.6 | 1.4 | 1.2 | 49 | 49 | 59 | 49 | 49 |
| 5 | 4 | 3 | 0.9 | 0.6 | 1.4 | 1.2 | 43 | 42 | 78 | 42 | 42 |
| 5 | 6 | 7 | 0.6 | 0.9 | 1.2 | 1.4 | 37 | 36 | 76 | 36 | 36 |
| 1 | 2 | 3 | 0.6 | 0.9 | 1.2 | 1.4 | 50 | 50 | 81 | 50 | 50 |
| 3 | 4 | 5 | 0.6 | 0.9 | 1.2 | 1.4 | 50 | 50 | 58 | 50 | 50 |
| 7 | 6 | 5 | 0.6 | 0.9 | 1.2 | 1.4 | 38 | 38 | 42 | 38 | 38 |
| 3 | 2 | 1 | 0.6 | 0.9 | 1.2 | 1.4 | 51 | 51 | 59 | 51 | 51 |
| 5 | 4 | 3 | 0.6 | 0.9 | 1.2 | 1.4 | 43 | 43 | 53 | 43 | 43 |

## Discussion and Conclusion

At first this paper studied the use of Bivariate Copulas in the context of approximating the Bivariate Exponential distribution and the Bivariate Normal distribution. The Archimedean Copulas did reasonably well in approximating the Bivariate Exponential distribution while not doing well with respect to the Bivariate Normal distribution. So, we decided to use the Archimedean Copulas to approximate the Tri-variate Exponential distribution. In Archimedean Copula constructions in the context of higher dimensions, the question arises as to how one should estimate the dependence parameter. This paper aims to seek an answer for this question based on some commonly used Archimedean Copula models such as Clayton, Gumbel, and Frank models in the case of a three dimensional problem. Overall, the Clayton Copula model is seen to perform better in approximating the Tri-variate Exponential distribution. As you can see from the numerical results for the absolute percentage error, Clayton Copula gives the smallest percentage error.

Moreover, there is not much of a difference between the absolute percentage error calculated based on the dependence parameter estimates using the arithmetic mean or geometric mean or harmonic mean. The dependence parameter estimate based on the probability $P(X = Y = Z)$ yields a high error rate and therefore should not be recommended. Also, the use of hierarchical copulas should be avoided as these also yield high error rates.

## References

Bemis, B. M. (1971). *Some statistical inferences for the bivariate exponential distribution.* Doctoral Dissertation, Missouri University of Science and Technology.

Clayton, D. G. (1978). A Model for Association in Bivariate Life Tables and its applications in Epidemiological Studies of Familial Tendency in Chronic Disease Incidence. *Biometrika, 65,* 141-151.

Frees, E. W., & Valdez, E. A. (1998). *Understanding Relationships Using Copulas*, *NAAJ 2*(1), 1-25.

Gumbel, E. (1960). Distributions des valeurs extremes en plusieurs dimensions. *Publications de Institut de statistique de Universite de Paris, 9,* 171-173.

Joe, H. (1993). Parametric Families of Multivariate Distributions with given Marginals. *Journal of Multivariate Analysis, 46*, 262-282.

Marshall, A. W., & Olkin, I. (1966). *A multivariate exponential distribution*. *J. Amer. Statist. Soc., 62,* 30-44.

Nelsen, R. B. (1999). *An Introduction to Copulas,* New York: Springer, ISBN 978-0-387-98623-4.

Ristic, M. M., & Kundu, D. (2015). *Marshall-Olkin generalized exponential distribution*. Metron, Springer.

Sklar, A. (1959). Fonctions de Répartition à n Dimensions et Leurs Marges. *Publications de l'Institut Statistique de l'Université de Paris, 8,* 229-231.