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An Approximate Confidence Interval for the Variance of Random Effects of One-Way Analysis of Variance in the Completely Randomized Design

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Abstract

We proposed three methods to find an approximate confidence interval for the variance of the random effects for a one-way analysis of the variance model in completely randomized design. We compared the proposed methods with some other methods reported in the literature. Several criteria are used for the empirical comparisons: the mean width of the confidence interval, the variance of the width, and the coverage probability. We use Simulation and Monte-Carlo techniques to perform the comparison study. We use R language to facilitate the simulation procedures. We found that one of the proposed methods was in general superior to the others.

Keywords: random effects model, one-way analysis of variance, variance of the random effects, variance components

Introduction

Experimentation is the essential component of any research activity in all disciplines. It is therefore important for the researcher to have a general knowledge of the design and analysis of experiments. A model is an appropriate way to describe the relationship between variables in the experiment. Therefore, it is important for the researcher to distinguish between different types of experimental design models. There are three main models, which are the fixed effects model, the random effects model, and the mixed effects model. In this paper, we are interested in finding the variance of the random effects in a one-way analysis of variance in a completely randomized design.

The random effects analysis of the variance component model can be traced as far back as the works of the astronomers Airy (1861) and Chauvenet (1863). Many years later, statisticians re-invented the model beginning with Fisher (1925) who introduced the concept of analysis of variance, and Tippet (1931) who clarified the analysis of variance method of variance component estimation (as cited in Khuri and Sahai, 1985).

The one-way random effects analysis of variance (ANOVA) model can be written as:

$$y_{ij} = \mu + \tau_i + e_{ij} \qquad \begin{cases} i = 1,2,\dots,k \\ j = 1,2,\dots,n \end{cases}$$

where y_{ij} is the j-th observation made at the i-th level of the factor (treatment), μ is the overall mean, τ_i is the unknown effect due to the i-th level, e_{ij} is a random error, k is the number of treatments, and n is the replication of each treatment. It is assumed that the errors e_{ij} are independently and identically distributed normal random variables with mean 0 and variance σ^2, and that the treatment effects τ_i are independently and identically distributed normal random variables with mean 0 and variance σ_τ^2, and that e_{ij} and τ_i are independent.

There are several different quantities that may be of interest in the variance component analysis. One quantity that may be of interest is the variance of random effects σ_τ^2, which is a measure of the variability between the population group means. Another quantity is the total variance $\sigma_\tau^2 + \sigma^2$.

Confidence intervals are one of the most important and informative summary results in statistical applications. There is no exact confidence interval for σ_τ^2 (Searle S.R, Casella G, McCulloch C.E (2006) and Montgomery, D. C, (2017)). Therefore, scientists sought to find approximate confidence intervals for σ_τ^2, and we mention here some of them.

Montgomery, D.C, (2017) mentioned that Satterthwaite (1941, 1946) proposed an approximate confidence interval for σ_τ^2 based on a linear combination of mean squares. In addition, he mentioned that Graybill and Wang (1980) proposed a procedure called the modified large-sample method, which can be a very useful alternative to Satterthwaite's method. Also, Welch (1956) proposed an approximate confidence interval for σ_τ^2 based on a normal approximation instead of the Satterthwaite procedure.

Bross (1950) derived a fiducial interval for σ_τ^2. Tukey (1951) and others have commented adversely on this procedure because the resultant limits fail to satisfy certain boundary properties. Anderson and Bancroft (1952), while discussing some of the available procedures for confidence intervals on σ_τ^2, they proposed a modified version of the Bross procedure which satisfies the boundary conditions as stated in (Sahai and Ojeda ,2004).

A method for establishing a confidence interval for σ_τ^2 has been independently proposed by Tukey (1951) and Williams (1962). The Tukey-Williams method is based on two quadratic forms in normal variables, which are exactly distributed as multiples of χ^2 −distributed random variables. A good description of deriving this Williams interval is given in Graybill (1976).

Moriguti (1954) and Bulmer (1957) independently developed the same confidence limits for σ_τ^2. Boardman (1974) showed that these two methods were identical, and Bulmer (1957) showed that the method gives very accurate approximations. Scheffee (1959) gave general formulas for obtaining approximate confidence limits for variance components based on Bulmer's method.

Sahai and Ojeda (2004) stated that Howe (1974) proposed a general procedure for constructing confidence intervals on a difference between two expected mean squares. An approximate procedure that seems to provide a shorter interval and has better coverage probability is given by Ting et al. (1990), and Burdick and Graybill (1992). Also, approximate procedures are given in Graybill (1961) and Searle (1971).

Thomas and Hultquist (1978) obtained an approximate confidence interval in the case of unbalanced data. Bottai and Orsini (2004) obtained confidence intervals for the variance component of random-effects linear models. Taoufik et al. (2007) presented sequential confidence intervals for variance components in one-way random models.

Researchers use statistical software packages to analyze their data. Two common software packages used in the field are SAS and SPSS. According to the SAS/STATR 9.1 User's Guide, PROCMIXED provides chi-squared based confidence intervals for the variance components using a Satterthwaite approximation (SAS Institute 2004). Another procedure is given by Wald approximate Confidence Intervals (see Scheffé (1959)). Finally, there are several other estimators of σ_τ^2 have been proposed in the literature (see Searle S.R, Casella G, McCulloch, C.E, (2006) for detailed reviews and derivations) as mentioned by Yandell BS (1997, pp307).

Since the variance component model was first formally introduced in the 1930's, it has been used to model experiments in many disciplines, including Astronomy, Agriculture, Animal Breeding, Medicine, Engineering, Education, and other fields. In all these disciplines, researchers need both point and interval estimates of the variance components to take decisions or test hypotheses.

Background

The error mean of squares (MS_E) and the treatment mean of squares (MST_r) can be converted into chi-square variables by multiplying each one of them by corresponding degree of freedom and then dividing by the corresponding expected mean square, that is: $\frac{(N-k)MS_E}{\sigma^2}\sim\chi^2_{(N-k)}$, where ($N = nk$), and $\frac{(k-1)MST_r}{\sigma^2+n\sigma_\tau^2}\sim\chi^2_{(k-1)}$. Therefore, it can be easily finding a confidence interval for the variance component σ^2, where the exact 100(1- α) % confidence interval for σ^2 is:

$$\frac{(N-k)MS_E}{\chi^2_{\left(\frac{\alpha}{2},N-k\right)}} \le \sigma^2 \le \frac{(N-k)MS_E}{\chi^2_{\left(1-\left(\frac{\alpha}{2}\right),N-k\right)}}$$

where $\chi^2_{\left(\frac{\alpha}{2},N-k\right)}$ and $\chi^2_{\left(1-\left(\frac{\alpha}{2}\right),N-k\right)}$ are the $\frac{\alpha}{2}-$ th and $\left(1-\frac{\alpha}{2}\right)-$ th percentiles of $\chi^2_{(N-k)}$, respectively.

Regarding the variance component σ_τ^2, it is known that an unbiased point estimator of it is:

$$\hat{\sigma}_\tau^2 = \frac{MST_r - MS_E}{n}$$

The distribution of $\hat{\sigma}_\tau^2$ is a linear combination of two chi-square random variables, say:

$$\frac{\sigma^2+n\sigma_\tau^2}{n(k-1)}\chi^2_{(k-1)} - \frac{\sigma^2}{n(N-k)}\chi^2_{(N-k)}$$

Unfortunately, there is no exact formula (closed form) for the distribution of this linear combination (Sahai, H. & Ojeda,

M. M. (2004); Searle, S. R., Casella G, McCulloch, C. E. (2006) and Montgomery, D.C, (2017). Therefore, an exact confidence interval for σ_τ^2 cannot be found. Different approximate confidence intervals have been suggested in the literature. In this paper, we suggested new approximate confidence intervals for σ_τ^2.

Methodology

We compared the proposed three methods for finding approximate confidence intervals for σ_τ^2 with some other methods reported in the literature. Several criteria are used for the empirical comparisons, which are the mean and the variance of the width of the confidence interval and the coverage probability. We use Simulation and Monte-Carlo techniques to perform the comparison study. We use R language to facilitate the simulation procedures.

The Proposed Methods

It is known that an unbiased estimator of the variance component σ^2 is MS_E, and an unbiased estimator of the variance component σ_τ^2 is $\hat{\sigma}_\tau^2 = (MST_r - MS_E)/n$. Also, it is known that $\frac{(N-k)MS_E}{\sigma^2} \sim \chi^2_{(N-k)}$, and an exact $100(1-\alpha)\%$ confidence interval (C.I.) for σ^2 is $L < \sigma^2 < U$, where:

$$L = \frac{(N-k)MS_E}{\chi^2_{\frac{\alpha}{2},(N-k)}} \text{ and } U = \frac{(N-k)MS_E}{\chi^2_{1-\frac{\alpha}{2},(N-k)}}$$

Also, it is known that $\frac{(k-1)MST_r}{\sigma^2+n\sigma_\tau^2} \sim \chi^2_{(k-1)}$, $F = \frac{MS_E}{MST_r}\frac{\sigma^2+n\sigma_\tau^2}{\sigma^2} \sim F_{(N-k,k-1)}$, and an exact $100(1-\alpha)\%$ Confidence interval for σ_τ^2 is:

$$\frac{\sigma^2}{n}\left[\frac{MST_r}{MS_E}\, F_{1-\frac{\alpha}{2},(N-k,k-1)} - 1\right] < \sigma_\tau^2 < \frac{\sigma^2}{n}\left[\frac{MST_r}{MS_E}\, F_{\frac{\alpha}{2},(N-k,k-1)} - 1\right]$$

Since σ^2 is unknown, we suggested the following methods for finding confidence intervals:

$$W_1: \frac{MST_r\, F_{1-\frac{\alpha}{2},(N-k,k-1)} - MS_E}{n} < \sigma_\tau^2 < \frac{MST_r\, F_{\frac{\alpha}{2},(N-k,k-1)} - MS_E}{n}$$

$$W_2: \frac{U}{n}\left[\frac{MST_r}{MS_E}\, F_{1-\frac{\alpha}{2},(N-k,k-1)} - 1\right] < \sigma_\tau^2 < \frac{L}{n}\left[\frac{MST_r}{MS_E}\, F_{\frac{\alpha}{2},(N-k,k-1)} - 1\right]$$

$$W_3: \frac{L}{n}\left[\frac{MST_r}{MS_E}\, F_{1-\frac{\alpha}{2},(N-k,k-1)} - 1\right] < \sigma_\tau^2 < \frac{U}{n}\left[\frac{MST_r}{MS_E}\, F_{\frac{\alpha}{2},(N-k,k-1)} - 1\right]$$

We will investigate these three approximate confidence intervals in a coming section.

Some Approximate Intervals Found in the Literature

1. Williams' confidence interval:

The following approximate interval is found in (Searle S.R., Casella G, McCulloch C.E. (2006), and Sahai, H. & Ojeda, M. M. (2004)):

$$\text{Wil: } \frac{(k-1)(MST_r - MS_E * F_{\left(k-1,N-k\,;\,\frac{\alpha}{2}\right)}}{n\chi^2_{\left(k-1\,;\,\frac{\alpha}{2}\right)}} \le \sigma_\tau^2 \le \frac{(k-1)(MST_r - MS_E * F_{\left(k-1,N-k\,;\,1-\frac{\alpha}{2}\right)}}{n\chi^2_{\left(k-1\,;\,1-\frac{\alpha}{2}\right)}}$$

2. Wald's Confidence Interval:

The following approximate interval is found in (Kraemer, Kari. (2012)):

$$\text{WW: } \hat{\sigma}_\tau^2 \pm Z_{1-\frac{\alpha}{2}} \sqrt{\frac{2\left(\hat{\sigma}_\tau^2 + \frac{\hat{\sigma}^2}{n}\right)^2}{k-1} + \frac{2\left(\frac{\hat{\sigma}^2}{n}\right)^2}{N-k}}$$

3. SAS confidence interval:

The following approximate interval is found in (Kraemer, Kari. (2012)) and used by SAS:

$$\text{SAS: } \frac{u\,\hat{\sigma}_\tau^2}{\chi^2_{\left(u\,;\frac{\alpha}{2}\right)}} \le \sigma_\tau^2 \le \frac{u\,\hat{\sigma}_\tau^2}{\chi^2_{\left(u\,;1-\frac{\alpha}{2}\right)}} \qquad \text{where: } u = \frac{(MST_r - MS_E)^2}{\frac{MST_r{}^2}{(k-1)} + \frac{MS_E{}^2}{(N-k)}}.$$

Simulations and Empirical Comparisons:

We use Monte Carlo simulation method to calibrate coverage probabilities and construct the approximate confidence intervals and facilitating the comparisons.

We choose six values of the number of treatments (k=5, 10, 15, 20, 25 and 30) and four values of replications (n=5, 10, 20, and 30) for a total of 24 different cases. We make simulations for the random effects and errors assuming that they are independently and normally distributed. We choose three different values of the variance of the random effects (σ_τ^2) and the variance of the errors (σ^2) which are (1,1), (4,1) and (1,4) respectively and the constant μ was set to zero (without loss of generality). We use these data to calculate the various approximate confidence intervals. For each of the 24 cases, we generate 10,000 Monte Carlo trials, we calculate (95 and 99) % approximate confidence intervals for σ_τ^2 for each method, and we obtain the coverage probabilities by classifying each interval as contains or does not contain the true value of σ_τ^2.

In Simulation process, we calibrated the empirical confidence coefficients to be almost the same for all the confidence intervals of σ_τ^2 . We make this simulation by using a program R. We follow the following steps to find the empirical coverage probability:

1. Generate a random a sample of size k, $(\tau_1, \tau_2, \dots, \tau_k)$, from N (0, σ_τ^2).
2. For each τ_i, generate a random sample of size n, $(\epsilon_{i1}, \epsilon_{i2}, \dots, \epsilon_{in})$, from N(0,$\sigma^2$).
3. Calculate $Y_{ij} = \tau_i + \epsilon_{ij}$.
4. Calculate L, U, MS_E, and MST_r.
5. Calculate $(1-\alpha)100\%$ approximate confidence intervals for σ_τ^2 using the six methods.
6. For each method (i.e., for each confidence interval), calculate:

$$M_r = \begin{cases} 1; & \sigma_\tau^2 \in C.I. \\ 0; & \sigma_\tau^2 \notin C.I. \end{cases}$$

7. Repeat steps (from 1 to 6) 10,000 times $(r = 1, 2, \dots, 10000)$.
8. Calculate the following proportion (empirical coverage probability) for each method:

$$p = \frac{\text{number of times } \sigma_\tau^2 \text{ belongs to the C.I.}}{10000} = \frac{\sum_{r=1}^{10000} M_r}{10000}$$

Results and Findings

We demonstrate the comparisons' results using graphs. We compare the average width of the six approximate confidence intervals as well as the variance of the width. The best confidence interval would be that one with the smallest average width and the smallest variance of the width.

The First Case

For this case, the standard deviation of the random effects is σ_τ= 1, and the standard deviation of the errors is σ = 1. The graphs for this case are Figure 1 to Figure 6 which follow:

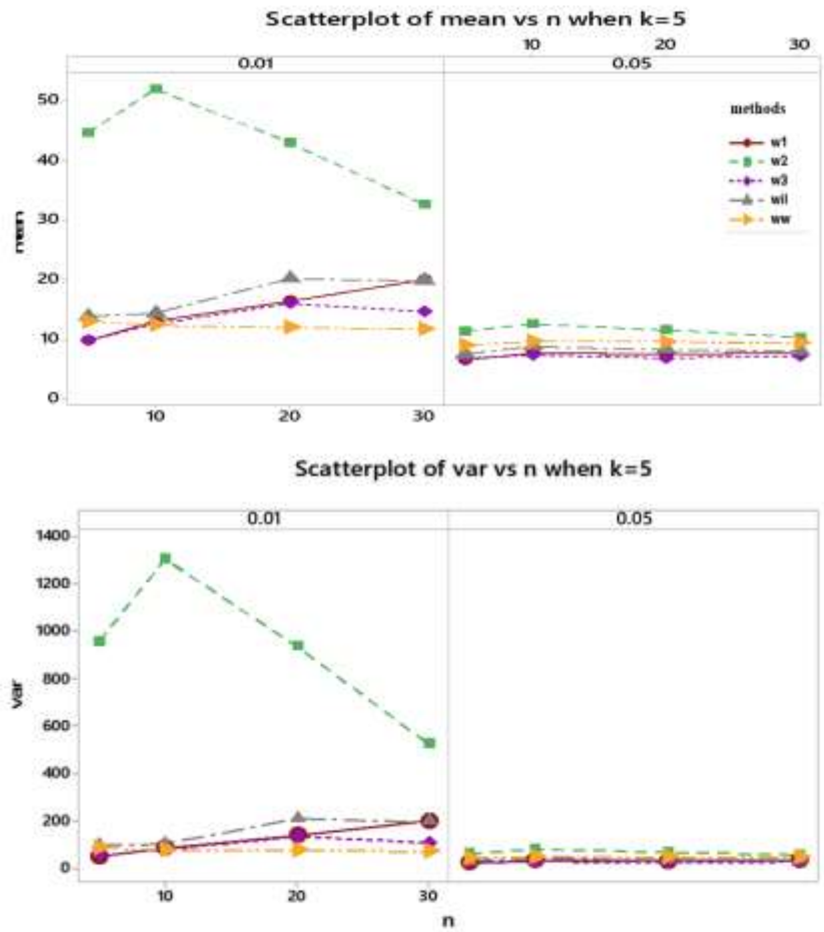


Figure 1. Mean and Variance for the Six Intervals of the First Case When K=5 and α=0.01 (left), 0.05 (right)

It is known that the lower of the α, the wider of the interval, and this is what we notice in this Figure. As for the average width of the intervals, we notice that at α = 0.01, W_3 and W_1 have the smallest width average for small values of n. For large values of n, Wald interval has the smallest width average (but we recall that the Wald interval in this case is at α between 0.03-0.04, so the comparison here is unfair with other methods. So, it can be said in general that W_3 is the best for this case. For α =0.05, W_3 and W_1 have the smallest width average for all values of n. The same remarks are said for the variance.

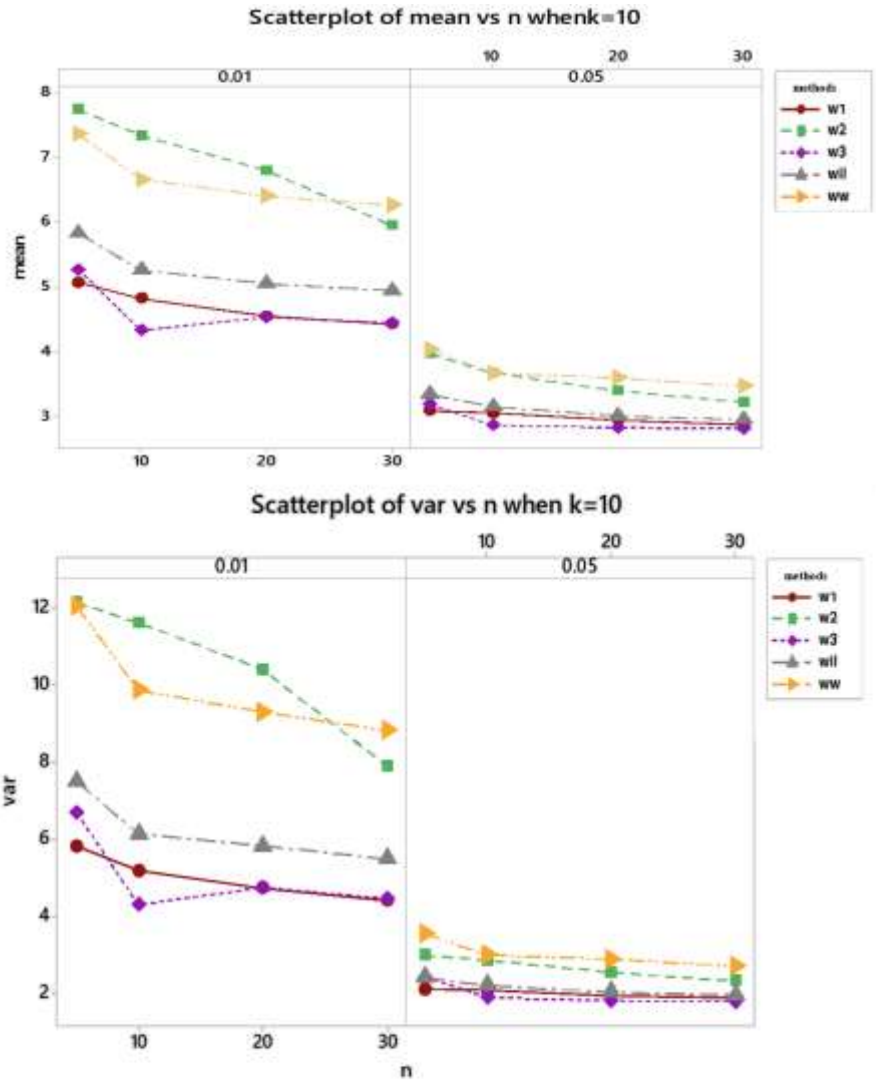


Figure 2. Mean and Variance for the Six Intervals of the First Case When K=10 and α=0.01 (left), 0.05 (right)

Here we notice that for all α values, W_1 is the best for small values of n and then W_3 competes it strongly with the remaining n values . So, we notice that W_3 is the best in general. The same remarks are said for the variance.

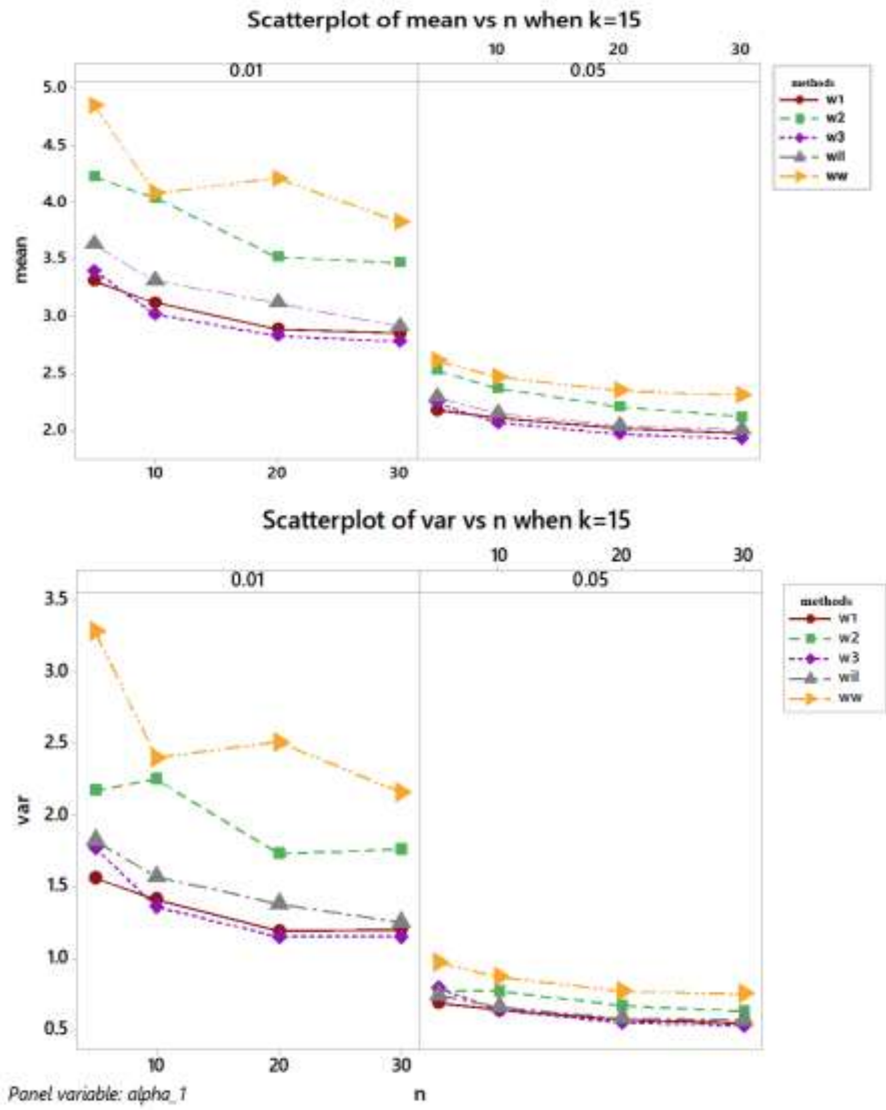


Figure 3. Mean and Variance for the Six Intervals of the First Case When K=15 and α=0.01 (left), 0.05 (right)

Here, we notice that the best interval is W_3 at α = 0.01 and at α = 0.05, then W_1 comes next and then the Williams interval and Wald come last in the average width. The same applies to the variance of the width of the intervals.

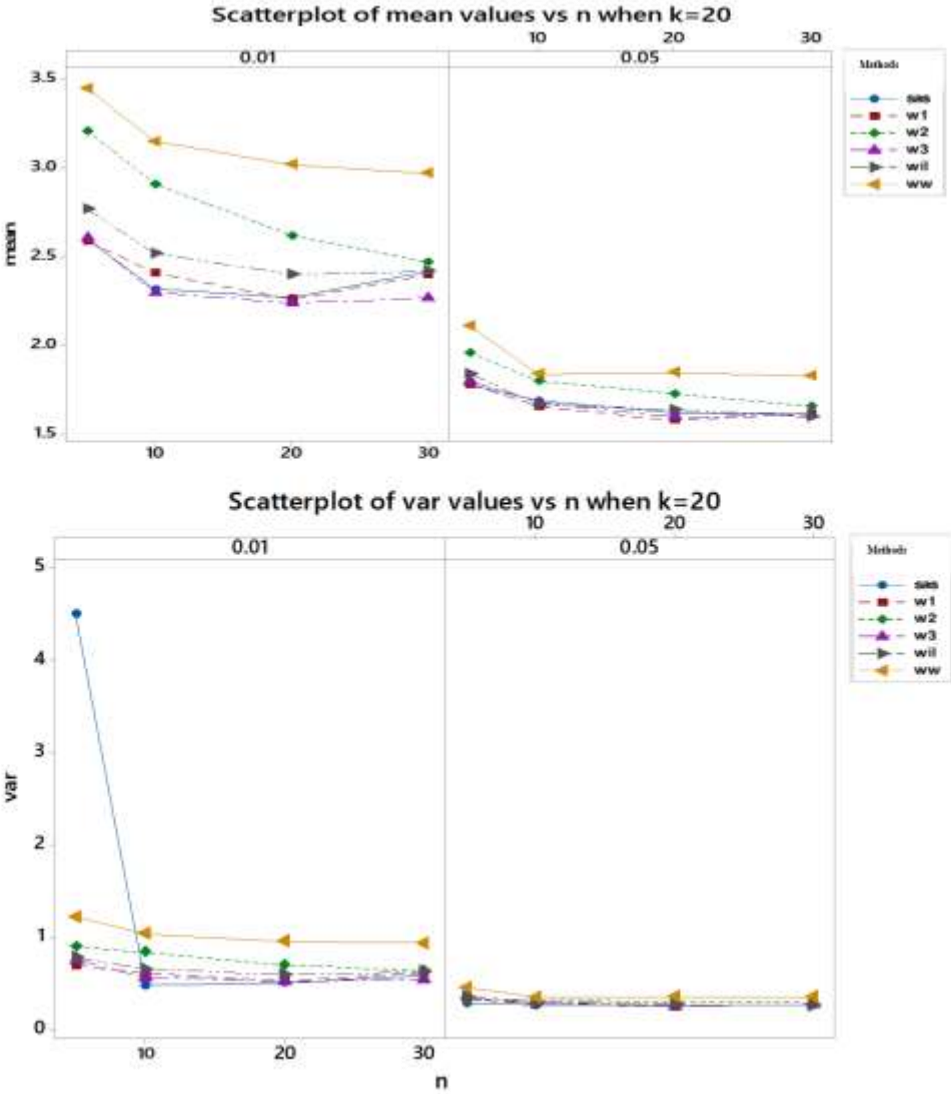


Figure 4. Mean and Variance for the Six Intervals of the First Case When K=20 and α=0.01 (left), 0.05 (right)

Here, it is good that the SAS interval enter in the comparisons. But W_3 is still the best, then comes the SAS interval and then W_1 at α = 0.01. The case is different at α = 0.05, where we notice that W_1 is the best, then W_3 competes with the SAS interval, then comes the Williams interval, then W_2 and finally the Wald interval. Also, we notice that W_2 approaches good intervals at large values of n. As for the variance, at α = 0.01, we notice that SAS is the best except for small n values. For α = 0.05, The variances for all intervals are very similar except for Wald interval.

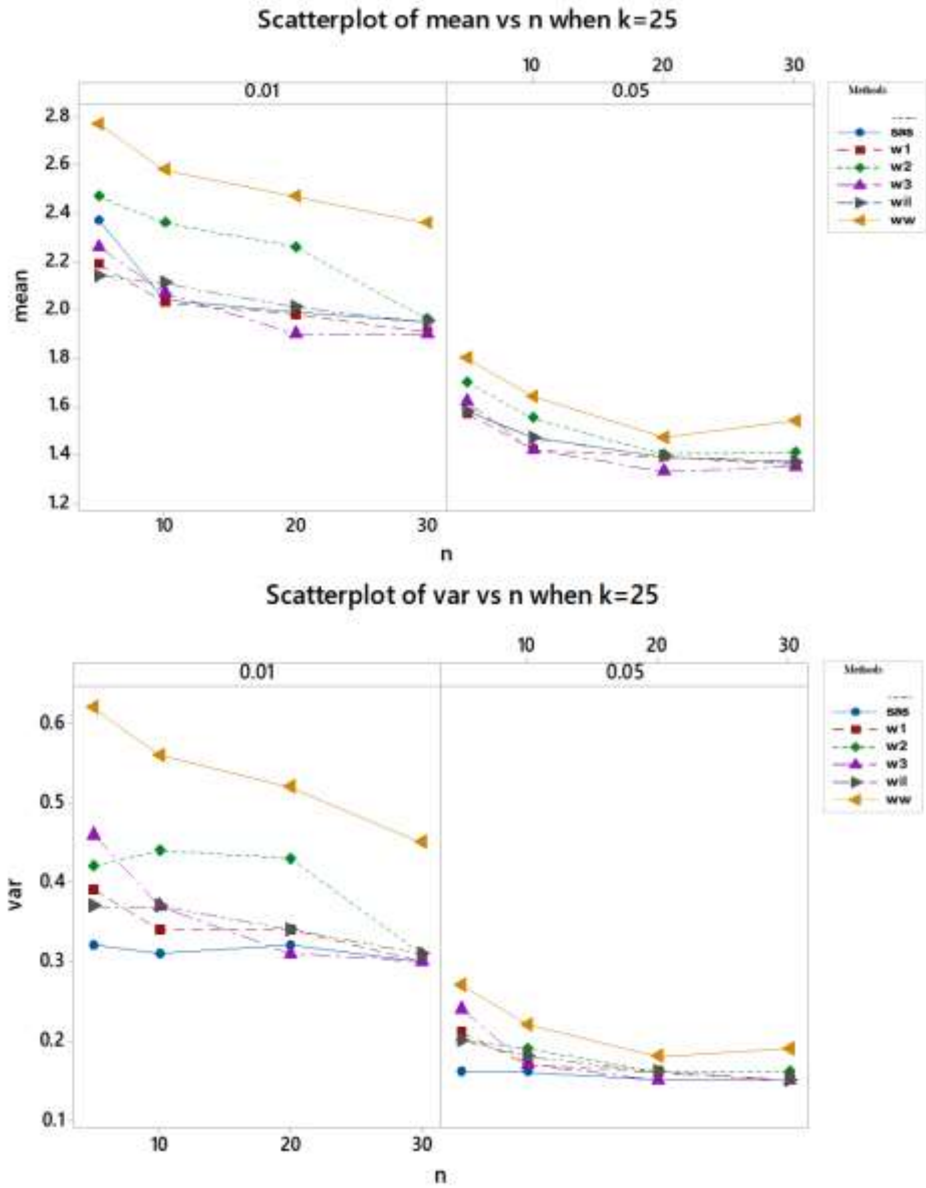


Figure 5. Mean and Variance for the Six Intervals of the First Case When K=25 and α=0.01 (left), 0.05 (right)

Here, we notice that as the values of k increase, the average width of the intervals approaches each other very significantly, and W_3 is still the best in general. For the variance, SAS interval is almost the best then W_3 rivals it.

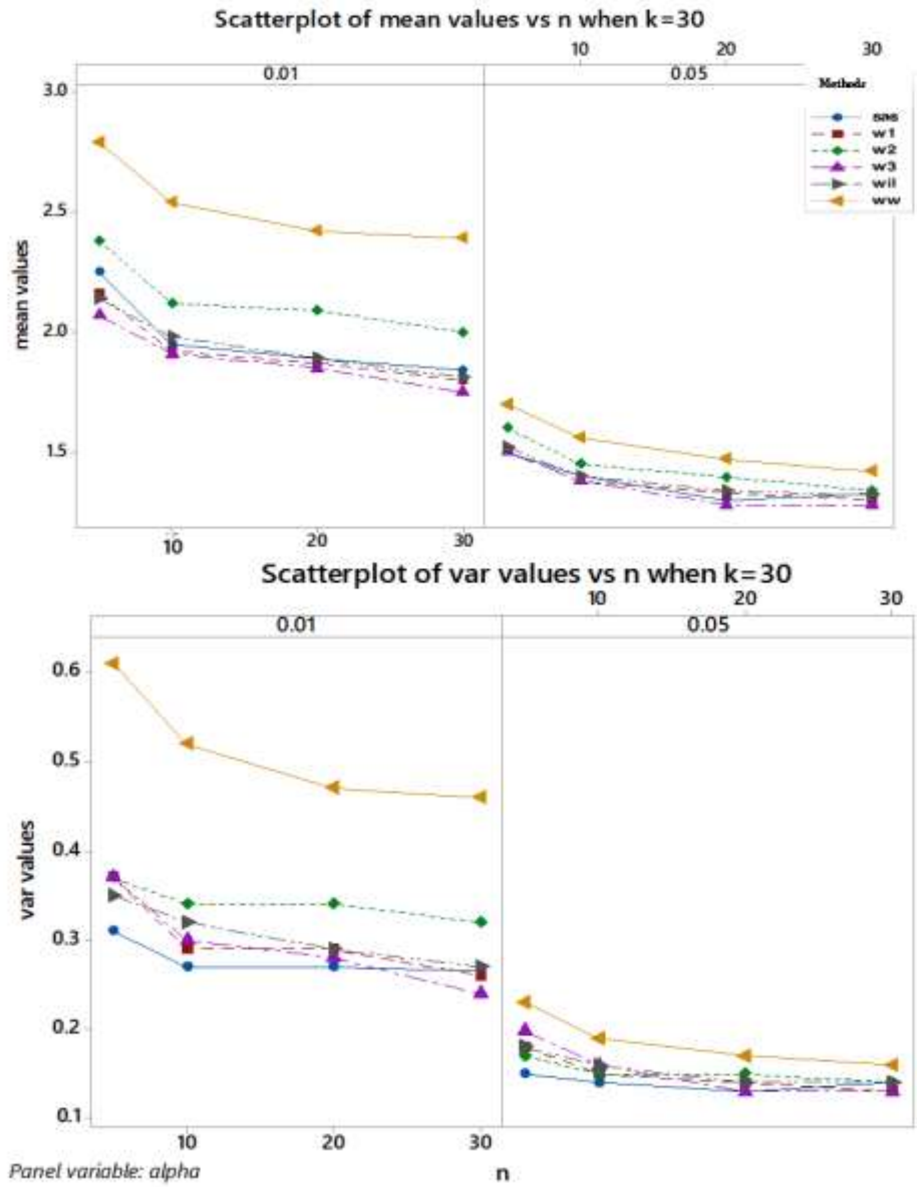


Figure 6. Mean and Variance for the Six Intervals of the First Case When K=30 and α=0.01 (left), 0.05 (right)

Here, we notice that W_3 is the best then W_1, then SAS and Williams intervals are compete. As for variance, SAS is best for small values of n and W_3 is best for large values of n.

Conclusion of the first case:

Regarding the average width of the interval, and in general, W_3 is the best, then W_1 and then the Williams interval. If the SAS interval is presented in the comparison, it competes strongly with W_1 but it is faulted for its excessive sensitivity when n and k are small. In addition, W_2 and Wald interval approach the other intervals if k and n are increased. As for the variance, it takes the same behavior as the average, but when SAS interval entering in comparison, it is the best for small and moderate n and W_3 is the best for large n.

There is a problem that many researchers have had in finding SAS interval, such as Kraemer, Kari, (2012) which is the sensitivity of the SAS interval to small values of n and k. By examining the program, we found that some values of q1sas and q2sas are equal to zero, which are values in the denominator of sasll and sasul, which results in "NAN". So, if there are some values of "NAN" then the variance of the length of the intervals will of course be a missing value (NA).

The Second Case

In this case we would like to see what happen if we change the variance for random effects. For this case, the standard deviation of the random effects is $\sigma_\tau = 2$, and the standard deviation of the errors is $\sigma = 1$. The graphs for this case are Figure 7 to Figure 12 which follow:

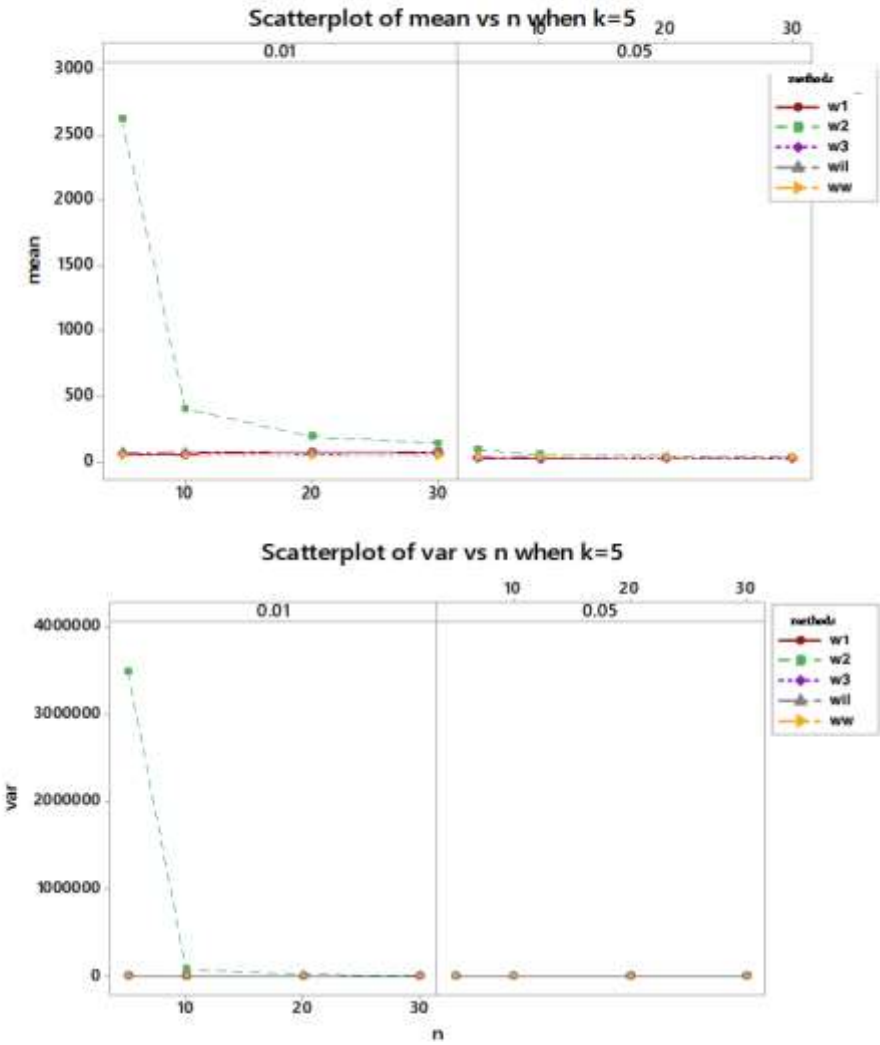


Figure 7. Mean and Variance for the Six Intervals of the Second Case When K=5 and α=0.01 (left), 0.05 (right)

In this case, we notice that W_3 is the best followed by W_1 then William's interval for all α values. In addition, we exclude the Wald interval from the comparison because α in the left part of the graph is between 0.03-0.04 not at 0.01, but at α =0.05, we notice that it approaches the other intervals strongly when the value of n increases. As for the variance of the width of the intervals, we find that W_3 is the least and therefore it is the best, followed by W_1.

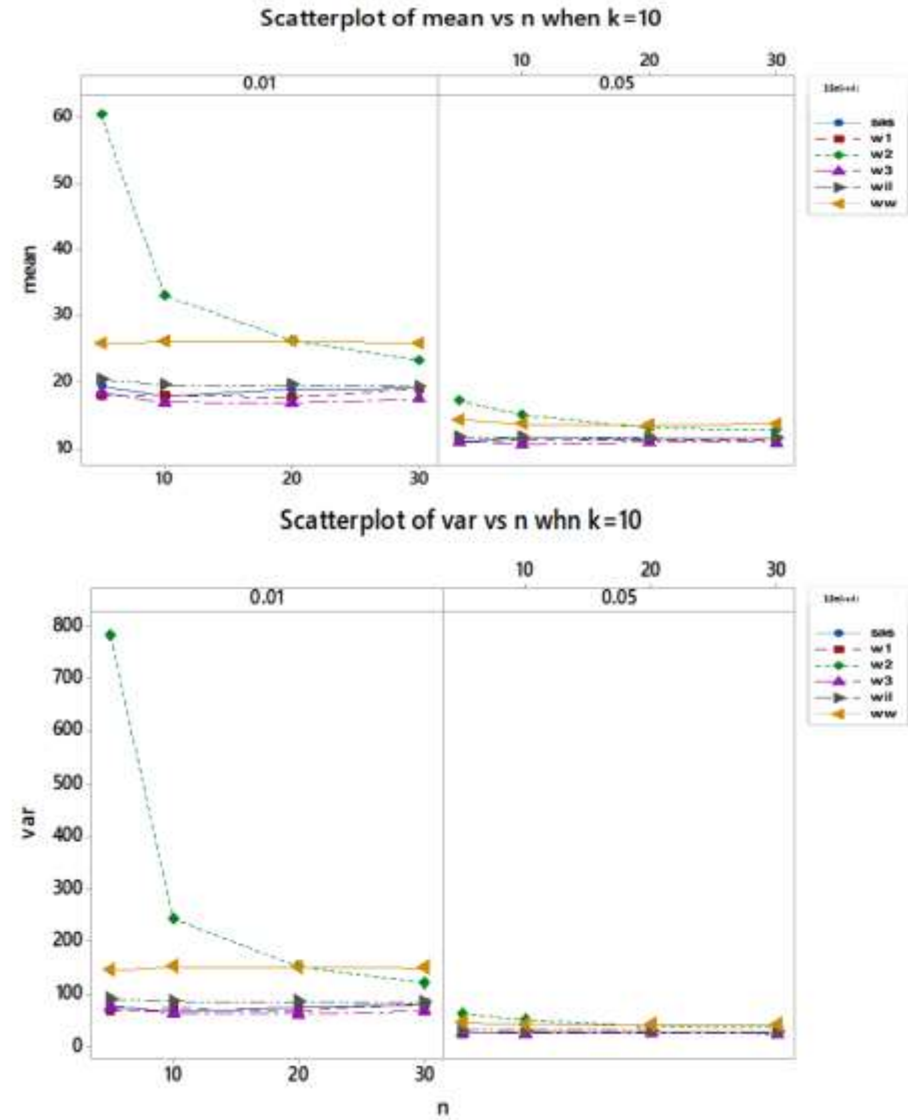


Figure 8. Mean and Variance for the Six Intervals of the Second Case When K=10 and α=0.01 (left), 0.05 (right)

In general, we notice that W_3 is the best for all α values, then the comparison is difficult between W_1 and SAS interval, then William's interval. The same applies to variance as well.

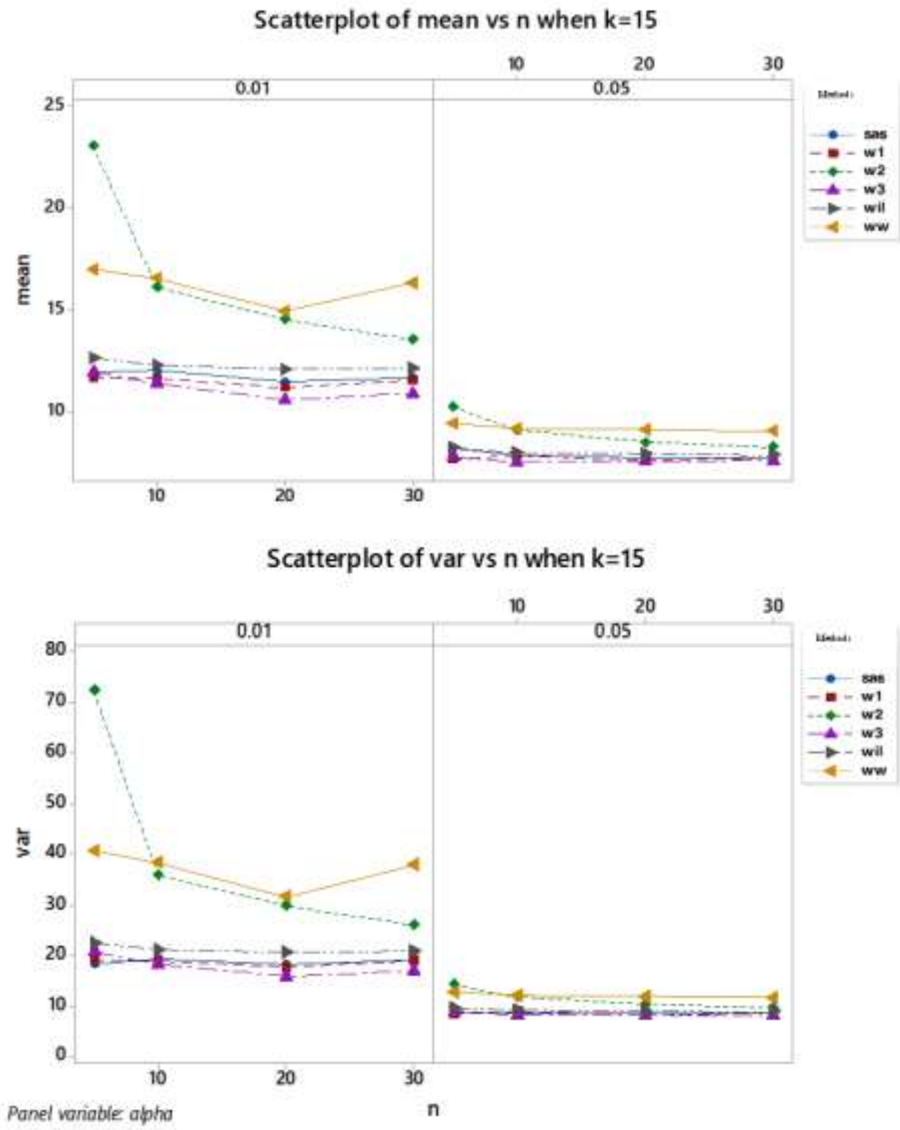


Figure 9. Mean and Variance for the Six Intervals of the Second Case When K=15 and α=0.01 (left), 0.05 (right)

In general, W_3 is still the best of all α values for the mean and the variance.

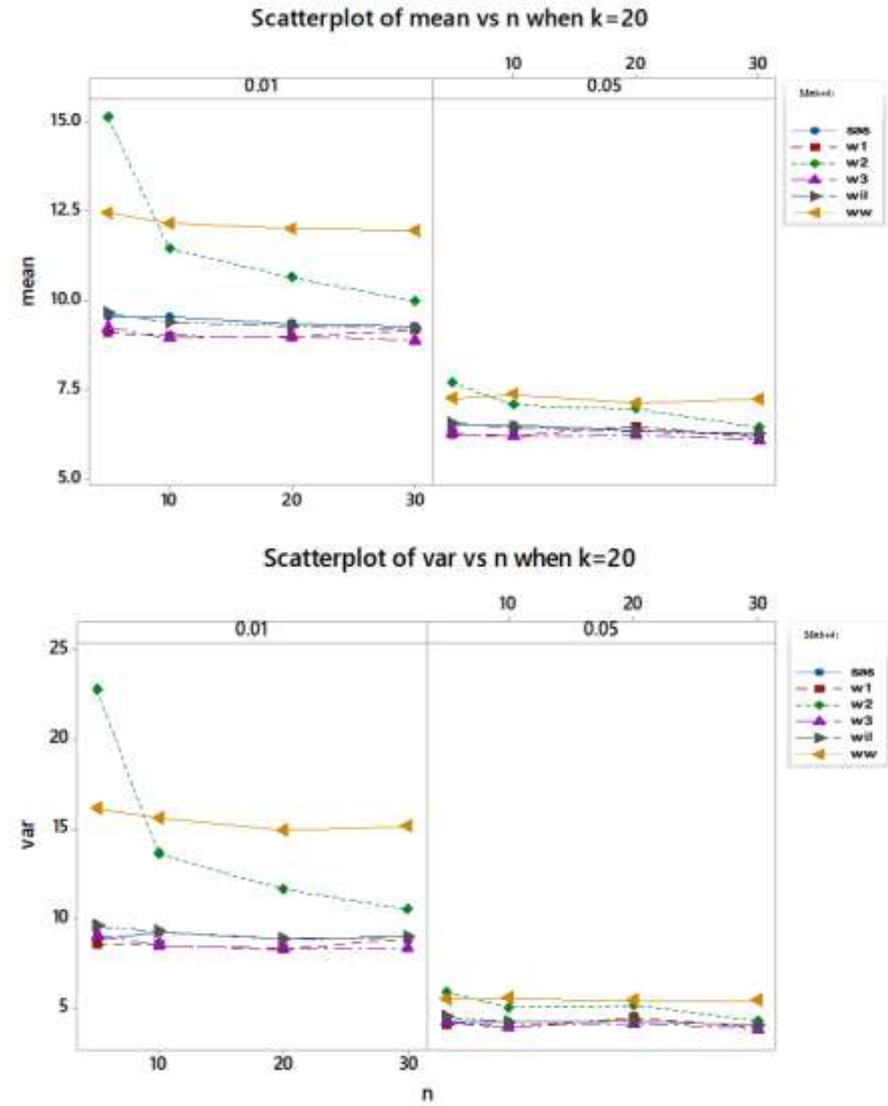


Figure 10. Mean and Variance for the Six Intervals of the Second Case When K=20 and α=0.01 (left), 0.05 (right)

For all α values, comparison between W_3 and W_1 in terms of preference is difficult, followed by the Williams and SAS intervals. For the variance, we notice the same between W_1 and W_3, then comes the Williams and SAS intervals.

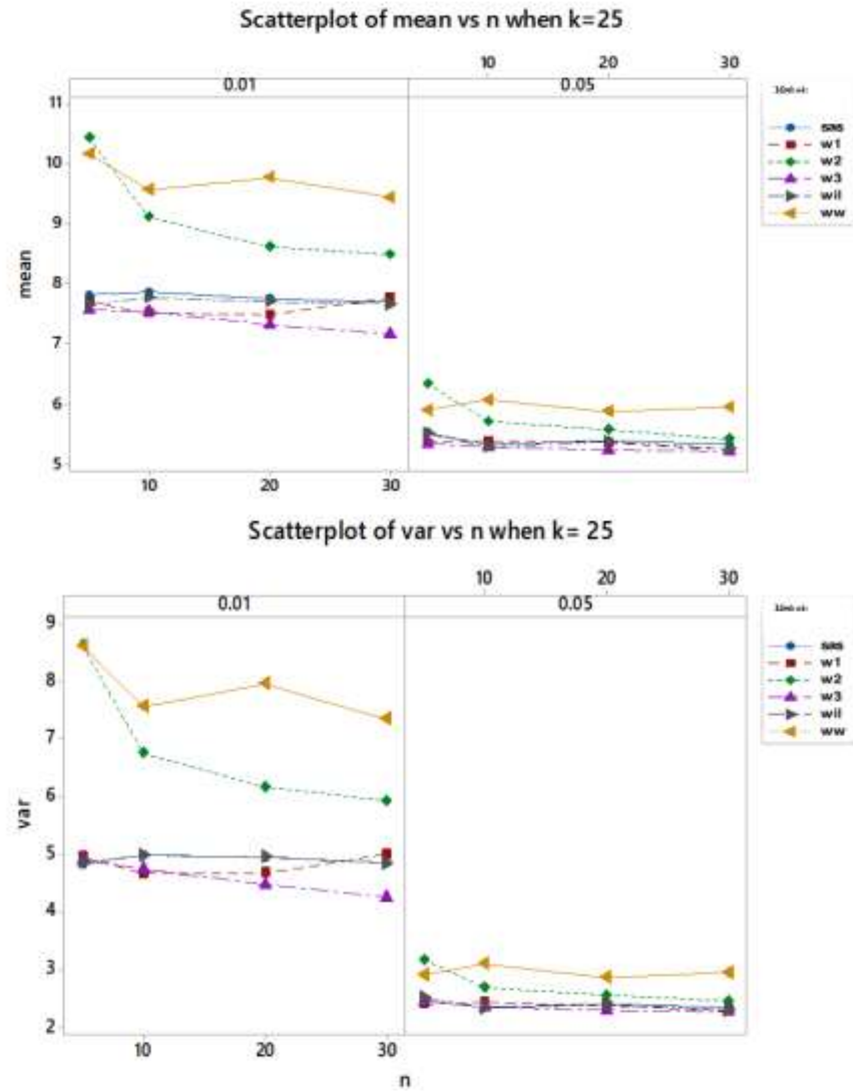


Figure 11. Mean and Variance for the Six Intervals of the Second Case When K=25 and α=0.01 (left), 0.05 (right)

In general, W_3 is the best then W_1 for all α values and for both mean and variance.

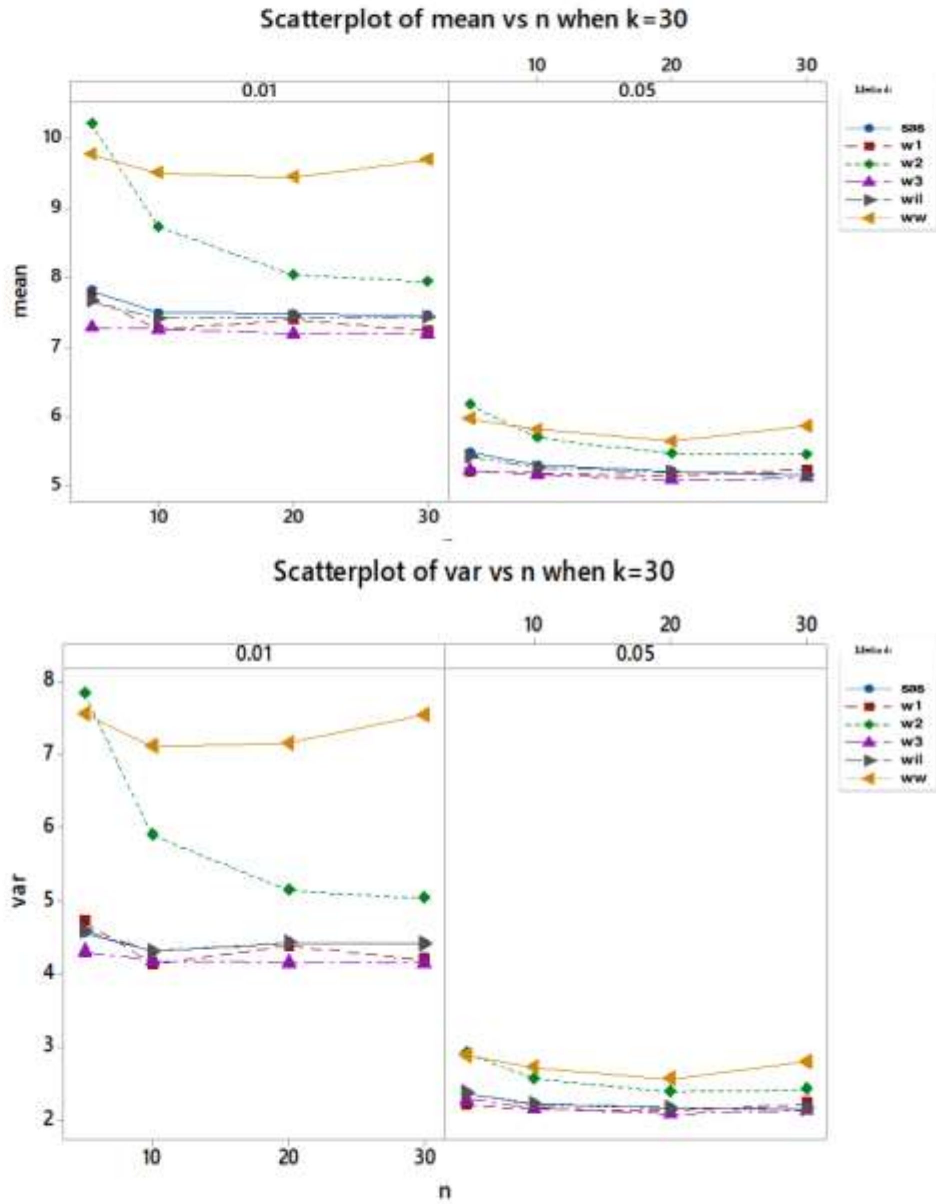


Figure 12. Mean and Variance for the Six Intervals of the Second Case When K=30 and α=0.01 (left), 0.05 (right)

In general, W_3 is the best for all α values for both mean and variance.

Conclusion of the second case:

For this case (σ_τ = 2, σ =1), we notice that W_3 is still the best at all α values, then W_1, and then SAS interval which competes with the Williams interval. As for the variance, it takes the same behavior as the mean. The variance of W_3 is the least, and W_1 competes with it, and then SAS interval which competes with the Williams interval.

The Third Case

In this case we would like to see what happen if we change the variance for random errors. For this case, the standard deviation of the random effects is σ_τ= 1, and the standard deviation of the errors is σ = 2. The graphs for this case are Figure 13 to Figure 18 which follow:

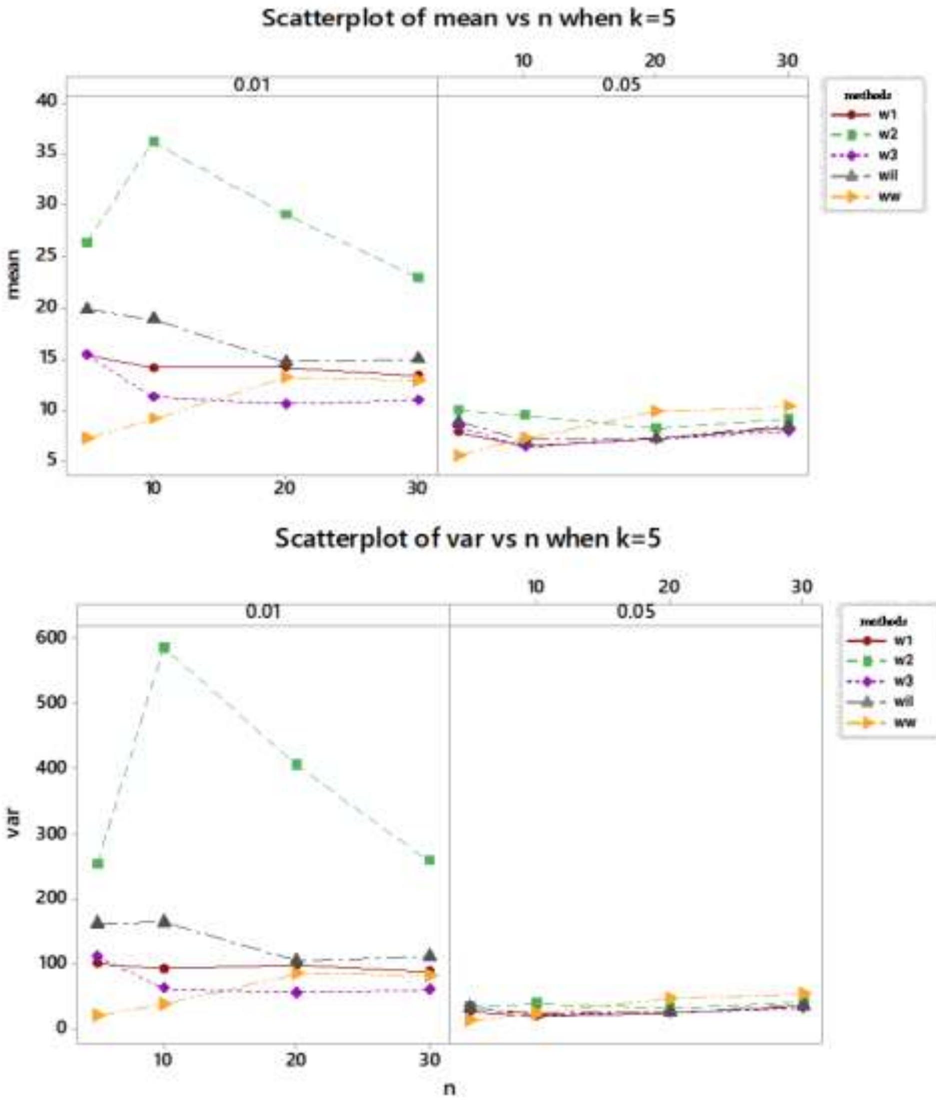


Figure 13. Mean and Variance for the Six Intervals of the Third Case When K=5 and α=0.01 (left), 0.05 (right)

Here, we notice that if we increase the variance of random errors, the situation has changed. For all α values, Wald interval is better for small values of n and W_3 is better for large values of n. As for the variance, it is generally that Wald interval is the best at small n values. At $\alpha = 0.01$, W_3 is better for large n values. At $\alpha = 0.05$, it becomes very difficult to compare between W_3 and W_1 for large n values.

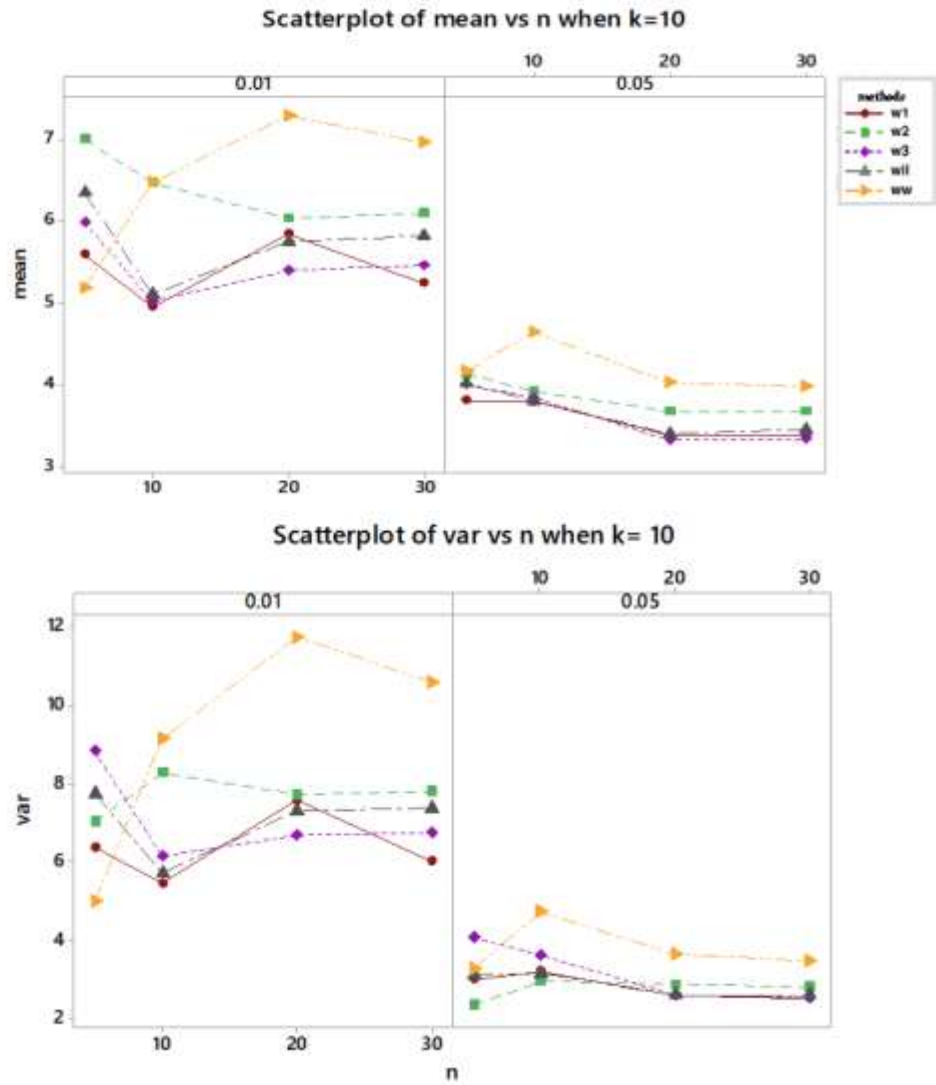
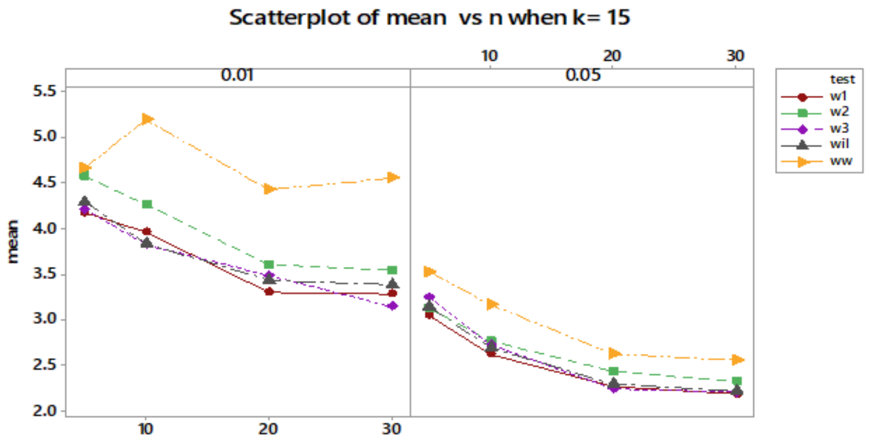


Figure 14. Mean and Variance for the Six Intervals of the Third Case When K=10 and α=0.01 (left), 0.05 (right)

For the mean, in general and at α =0.01, W_1 is the best then W_3. But at α =0.05, the competition between them is strong. While for the variance, at α =0.01, Wald interval then W_1 are better for small values of n, and W_1 competes with W_3 for large values of n. For α = 0.05, W_2 is the best for small n values, then W_1, W_3 and William's interval are competing at higher n values.



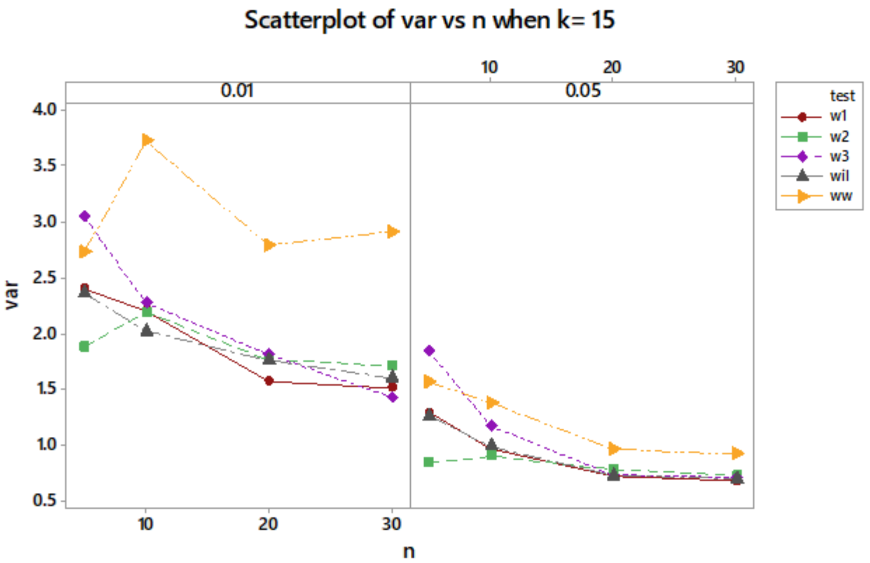


Figure 15. Mean and Variance for the Six Intervals of the Third Case When K=15 and α=0.01 (left), 0.05 (right)

For the average, at α = 0.01, W_1 and W_3 are better in general. Also, at α = 0.05, W_1 is the best in general, and then W_3 competes with the Williams interval. As for the variance, we notice that W_2 is the best for only small n values for all α values, then W_3, W_1 and the Williams interval compete strongly.

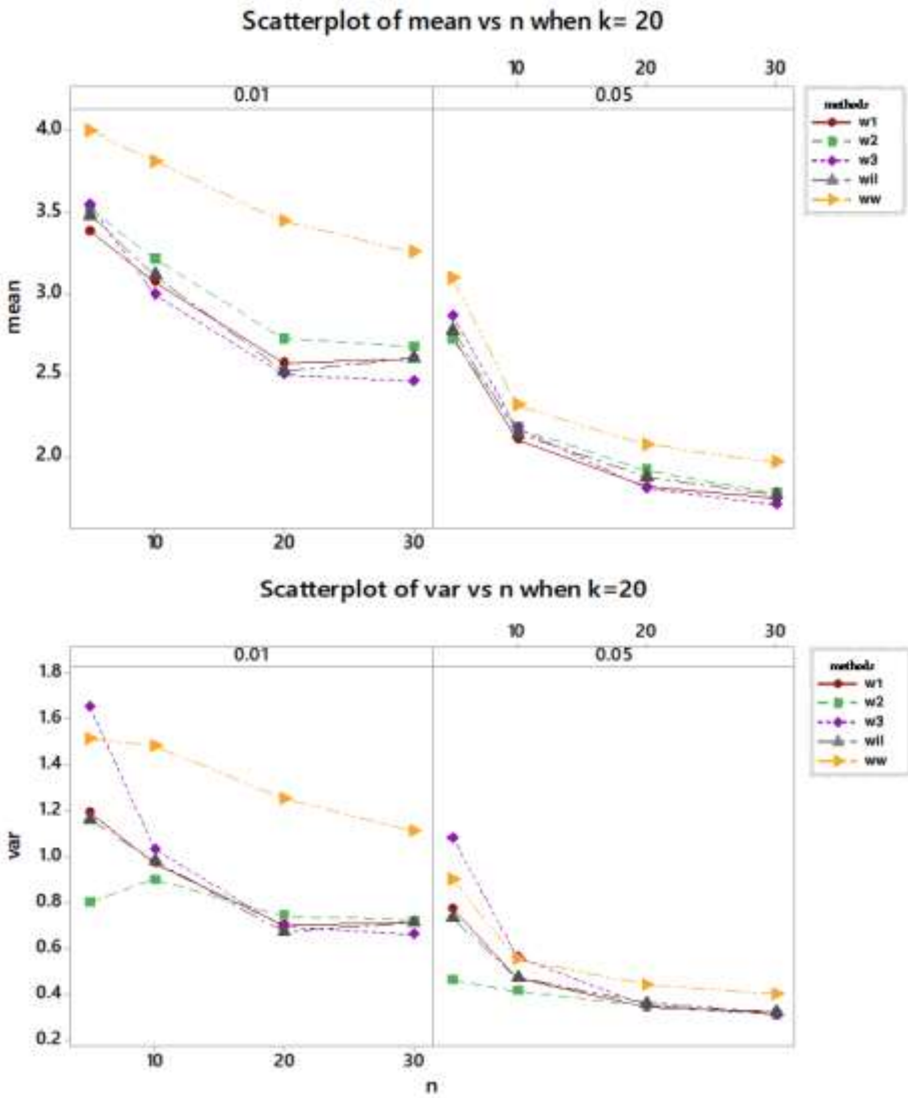


Figure 16. Mean and Variance for the Six Intervals of the Third Case When K=20 and α=0.01 (left), 0.05 (right)

Here, we find that at α = 0.01, W_1 is best for small values of n and W_3 is the best for larger values of n. At α = 0.05, W_2 is the best for small values of n, but for larger values of n, W_1 competes with W_3. The case is different for the variance, we find that W_2 is the best at small n values, followed by W_1, W_3 and the Williams interval in a different way for all α values.

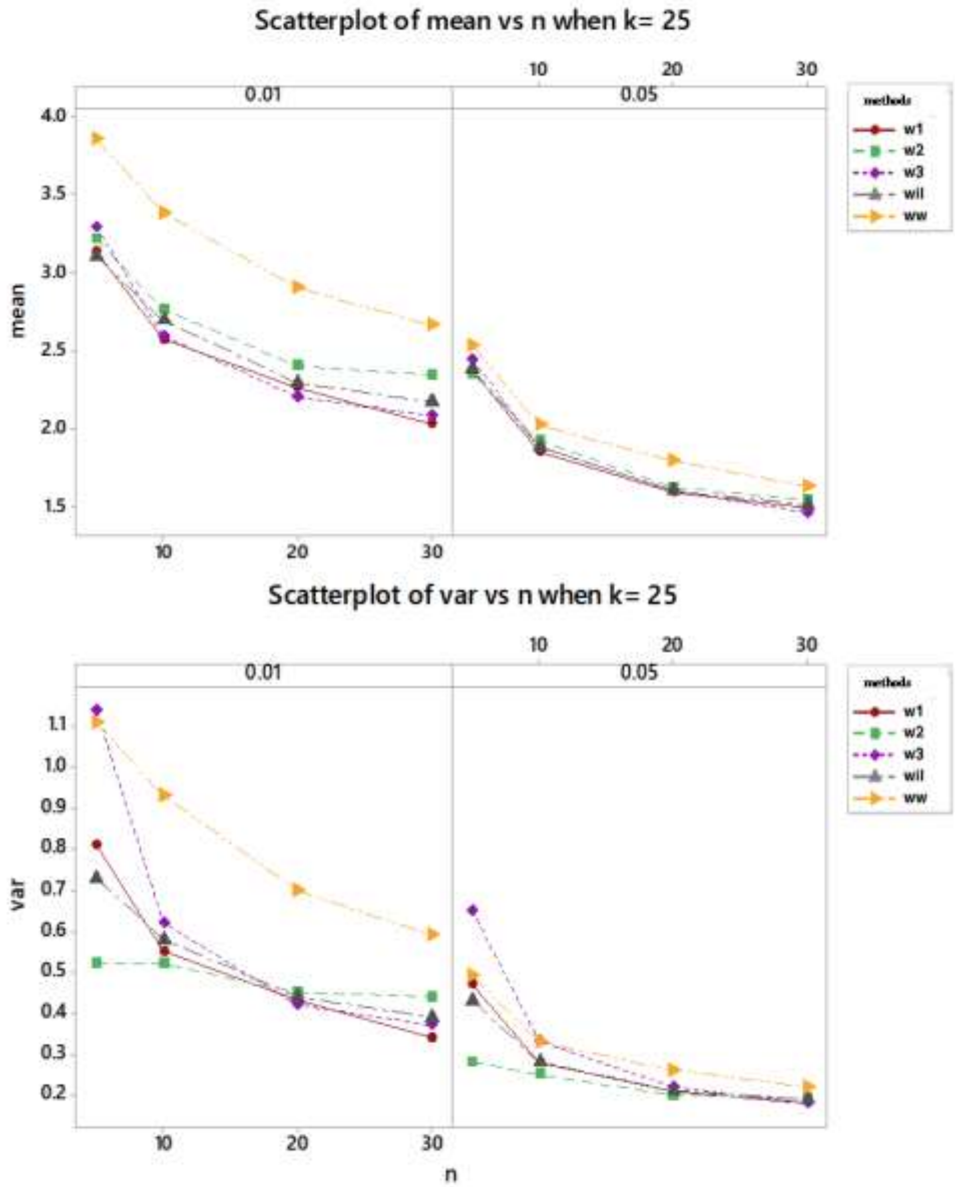


Figure 17. Mean and Variance for the Six Intervals of the Third Case When K=25 and α=0.01 (left), 0.05 (right)

In general, for the average, we notice that W_1 and W_3 are the best. As for the variance, W_2 is the best for small values of n, and for the largest value of n, we notice that W_1 and W_3 are the best.

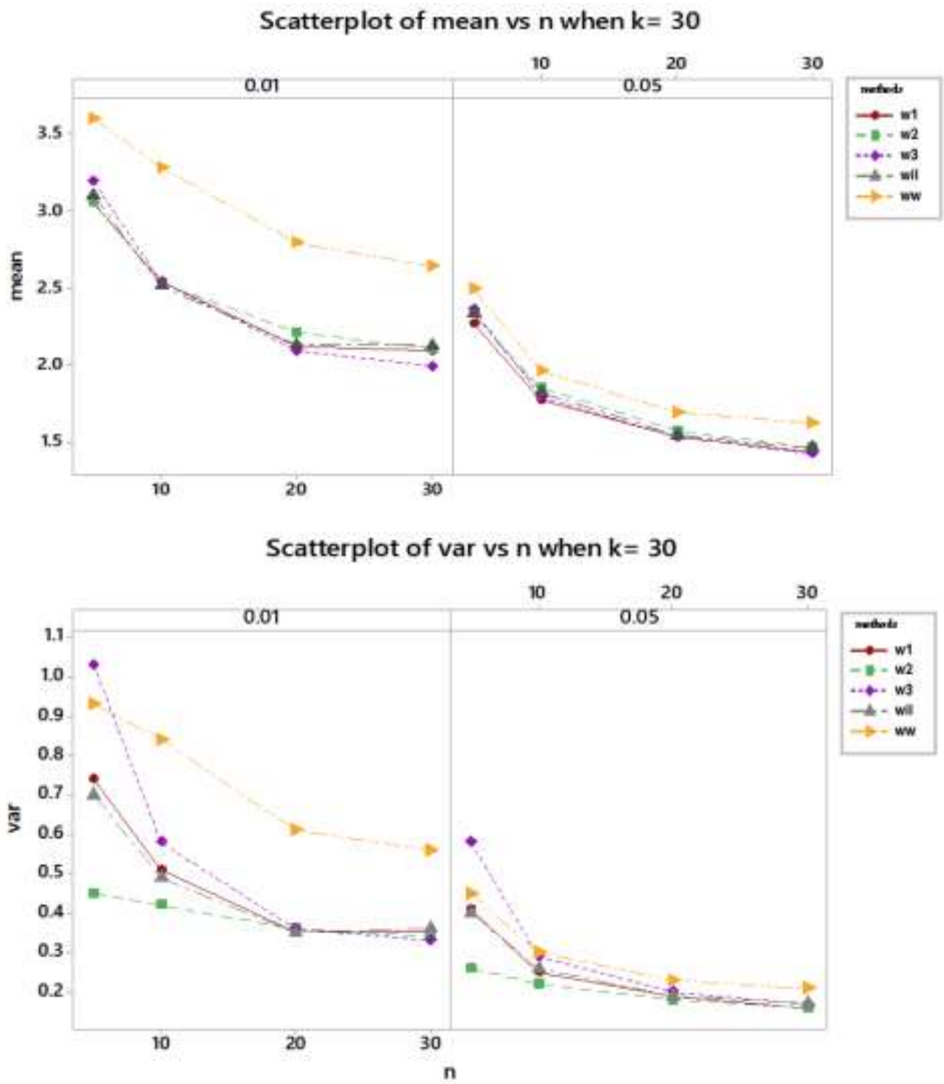


Figure 18. Mean and Variance for the Six Intervals of the Third Case When K=30 and α=0.01 (left), 0.05 (right)

Here, we notice that the higher the value of k, the closer the average width of the intervals to each other. But, in general, we notice that W_3 and W_1 are the best. While for variance, W_2 remains the best for most values of n.

Conclusion of the third case:

For this case (σ_τ = 1, σ =2), we notice that there is a difference in the behavior of the mean width of the intervals according to the value of k. In the case of too small k, we find that the average width of the Wald interval is better for small values of n and W_3 is better for large values of n. For large value of k, and in general, the comparison between W_1 and W_3 is difficult in terms of preference, and in some cases W_2 is better only for small n values. As for the variance, we notice that for too small values of k, Wald is better for small values of n and W_3 and W_1 are better for large values of n, while for high and moderate values of k, W_2 is better at small values of n, and $W_1$and W_3 and the Williams interval are the best at the higher value of n. Finally, for too large value of k, we notice that W_2 is better for most values of n.

General Conclusion

When comparing our proposed intervals ($\mathrm{W_1, W_2, and\ W_3}$) with some of the intervals in the literature (Wald, Williams, and SAS), we noticed the following:

1. The higher the value of k, the closer the six intervals are to each other in the mean and variance of the interval's width, and they almost become equal when we increase the number of treatments to 80.

2. In general, W_3 is the best among all intervals, although SAS interval competes with it when entering the

comparisons, but it is weakness for not showing its values at small values of n and k, especially since most designs do not take large values of k.

3. In most cases, we found a strong competition between W_3 and W_1, then Williams competes with SAS interval, but W_3 maintains the lead.

4. W_2's interval and Wald's interval approach other intervals at large values of n and k.

5. When we allow an increase in the variance of errors (to 4 for example) then W_3 is better for large values of n while Wald and W_2 intervals are better for small values of n.

In conclusion, we empirically found that W_3 is the best for most cases. This is because of calibration and adjusting the confidence level since they are all approximate methods, and they are somewhat conservative.

Reference

Anderson, R. L., & Bancroft, T. A. (1952). *Statistical Theory in Research, McGraw-Hill*. New York.

Boardman, T. J. (1974). Confidence intervals for variance components-a comparative Monte Carlo study. *Biometrics, 30*, 251-262. https://doi.org/10.2307/2529647

Bottai, M., & Orsini, N. (2004), Confidence intervals for the variance component of random-effects linear models. *The Stata Journal, 4*(4), 429–435. https://doi.org/10.1177/1536867X0400400405

Bross, I. (1950). Fiducial intervals for variance components. *Biometrics, 6*, 136-144. https://doi.org/10.2307/3001494

Bulmer, M. G. (1957). Approximate confidence limits for components of variance. *Biometrika, 44*, 159-167. https://doi.org/10.1093/biomet/44.1-2.159

Burdick, R. K., & Graybill, F. A. (1992). *Confidence intervals for variance components.* New York: Marcel Dekker. https://doi.org/10.1201/9781482277142

Graybill, F. A. (1961). *An Introduction to Linear Statistical Models*. Vol. 1. McGraw-Hill, New York.

Graybill, F. A. (1976). *Theory and Application of the Linear Model.* Duxbury Press, North Scituate.

Graybill, F. A., & Wang, C. M. (1980). Confidence Intervals on Nonnegative Linear Combinations of Variances. *Journal of the American Statistical Association, 75*, 869–873. https://doi.org/10.1080/01621459.1980.10477565

Howe, W. G. (1974). Approximate confidence limits on the mean of X +Y where X and Y are two tabled independent random variables. *J. Amer. Statist. Assoc., 69*, 789-794. https://doi.org/10.2307/2286019

Khuri, A. I., & Sahai, H. (1985). Variance components analysis: A selective literature survey. *International Statistical Review, 53*(3), 279–300. https://doi.org/10.2307/1402893

Kraemer, K. (2012). *Confidence intervals for variance components and functions of variance components in the random effects model under non-normality*. Iowa State University.

Montgomery, D. C. (2017). *Design and Analysis of Experiments.* (9th ed.). Wiley.

Moriguti, S. (1954). Confidence limits for a variance component. *Reports of Stat. Applications in Research, Japanese Union of Scientists and Engineers, 3*(2), 29–41.

Sahai, H., & Ojeda, M. M. (2004). Analysis of Variance for Random Models: Volume I: Balanced Data Theory, Methods, Applications and Data Analysis, Birkhauser Boston, New York. https://doi.org/10.1007/978-0-8176-8168-5

SAS Institute Inc. (2004). *SAS/ STATR 9.1 User's Guide.* Cary, NC: SAS Institute Inc.

Satterthwaite, F. E. (1946). *An Approximate Distribution of Estimates of Variance Components. Biometrics Bull., 2*, 110–112. https://doi.org/10.2307/3002019

Scheffé, H. (1959). *The Analysis of Variance.* Wiley, New York.

Searle, S. R. (1971). *Linear Models.* Wiley, New York.

Searle, S. R., Casella, G., & McCulloch, C. E. (2006). *Variance Components.* Wiley, New York.

Taoufik, Z., & Mohamed, Y., & El-Bassiouni, (2007). Sequential confidence intervals for variance components in one way random models. *Journal of Statistical Planning and Inference, 137*, 1185-1198. https://doi.org/10.1016/j.jspi.2006.04.004

Thomas, J. D., & Hultquist, R. A. (1978). Interval estimation for the unbalanced case of the one-way random effects model. *The Annals of Statistics*, *6*, 582-587. https://doi.org/10.1214/aos/1176344202

Ting, N., Burdick, R. K., Graybill, F. A., Jeyaratnam, S., & Lu, T. F. C. (1990). Confidence intervals on linear combinations of variance components that are unrestricted in sign. 1. *Stat. Compo Simul.*, *35*, 135-143. https://doi.org/10.1080/00949659008811240

Tukey, J. W. (1951). Components in regression. *Biometrics*, *7*, 33-69. https://doi.org/10.2307/3001602

Welch, B. L. (1956), On linear combinations of several variables. *J. Amer. Statist. Assoc.*, *51*, 132-148. https://doi.org/10.1080/01621459.1956.10501316

Williams, J. S. (1962). A confidence interval for variance components. *Biometrika*, *49*, 278-281. https://doi.org/10.1093/biomet/49.1-2.278

Yandell, B. S. (1997). *Practical Data Analysis for Designed Experiments*. Springer US. https://doi.org/10.1007/978-1-4899-3035-4