International Journal of Statistics and Probability; Vol. 12, No. 3; May 2023
ISSN 1927-7032 E-ISSN 1927-7040
Published by Canadian Center of Science and Education

Uniform Distribution as the Limiting Form of a Density Function

Hasan A. Al-Halees

Correspondence: Department of Mathematical Sciences, Saginaw Valley State University, 7400 Bay Road, University Center, Michigan 48710, USA

Received: February 6, 2023 Accepted: May 17, 2023 Online Published: May 29, 2023

doi:10.5539/ijsp.v12n3p18 URL: https://doi.org/10.5539/ijsp.v12n3p18

Abstract

The uniform distribution, denoted by $U(x; A, B) = 1/(B - A)$ if $0 < A < x < B < \infty$ and zero otherwise, is the simplest probability density functions of a continuous random variable X. For a continuous random variable X on the interval (0, 1), a three parameters density function, denoted by $h(x; A, B, n)$, is constructed so that its limiting form is the uniform density function $U(x; A, B)$ in which $n \to \infty$.

Keywords: distribution, uniform, density, standard, normal, mean, mode, variance, cumulative

1. Introduction

The intention is to generalize the well-known uniform distribution $U(x; A, B)$ where $0 < A < x < B < \infty$. The power functions x^n and $x^{1/n}$ for all $n \in (1, \infty)$, are used to construct a probability density function, denoted by $h(x; n)$, whose limiting form, in which $n \to \infty$, is the uniform distribution $U(x; 0, 1)$, with $0 < x < 1$. This density function will be called the ***standard h-distribution***. A general form of $h(x; n)$, denoted by $h(x; A, B, n)$, is also constructed so that its limiting form is the uniform distribution $U(x; A, B)$ in which $n \to \infty$. The density function $h(x; n)$ and its general form are not only important as generalization of an existing distribution, but also, they are applicable in many branches of science, such as the medical field, mineral industry, and technology. First, let us recall the following well-known facts that are needed for this work.

The set of all possible outcomes of a statistical experiment is called a sample space. A sample space is called continuous if it contains a noncountable number of possibilities. A random variable X is a real-valued function on the sample space. When the random variable assumes noncountable number of values, it is called a continuous random variable. To calculate the probabilities that a continuous random variable assumes values from a certain interval of real numbers, we must derive its probability density function. The probability density function for a continuous random variable is constructed so that the area under its curve bounded by the x-axis is equal to 1. A function $f(x)$ is called a probability density function of a continuous random variable X, if the following hold.

$$1.\quad f(x) \geq 0 \text{ for all real numbers x} \qquad 2.\ \int_{-\infty}^{\infty} f(x)dx = 1 \qquad 3.\quad P(a < x < b) = \int_{a}^{b} f(x)dx \tag{0}$$

For a continuous random variable, nonzero probabilities are associated only with interval of numbers. Because of condition 3 in (0), the probability of occurrence of a single value of this random variable is zero.

The cumulative distribution function, the rth moment of the distribution, and the variance of a continuous random variable X with probability density function $f(x)$, are denoted, respectively, by

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt\,; \quad E(X^r) = \int_{-\infty}^{\infty} x^r f(x)dx\,, r = 1, 2, \dots.; \quad V(X) = E(X^2) - [E(X)]^2 \tag{1}$$

Standard deviation $\sqrt{V(X)}$ is the amount of dispersion from the mean of values that random variable X can assume.

Intermediate Value Theorem If f is a continuous function on the closed interval $[a, b]$, and K is a number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$, then there exists a number c in the open interval (a, b) such that $f(c) = \text{K}$.

2. Method

2.1 The Standard h-Distribution

The functions x^n and $x^{1/n}$ for all $n \in (1, \infty)$, are inverse to each other and their graphs are crossing each other at the points $x = 0$ and $x = 1$. The area between the two curves is

$$A_n = \int_0^1 \left(x^{1/n} - x^n\right) dx = \frac{n-1}{n+1} \tag{2}$$

The function denoted by

$$h(x;n) = \begin{cases} \frac{n+1}{n-1}\left(x^{1/n} - x^n\right) & 0 < x < 1 \\ 0 & OW \end{cases} \tag{3}$$

is a probability density function for a continuous random variable X (defined over a set of real numbers**)** due to

1. $h(x;n) \geq 0$ for all real numbers *x*, since for $n > 1$, $0 < x < 1$, the ratio $\frac{n+1}{n-1} > 0$ and $x^{1/n} - x^n > 0$.

2. $\int_0^1 h(x;n)\, dx = 1,$ by (2)

3. $P(a < x < b) = \int_a^b h(x;n) \quad where\ 0 < a < b < 1$

The probability density function in (3) is the **standard *h*-distribution**, where $n \in (1,\ \infty)$.

In the medical field, it is reasonable to claim that the time it takes to complete a surgery on part of a body is proportional to the size of the wound that is needed to perform the surgery through. The diameter of the needle that is needed to sew the wound is also proportional to the size of the wound. Hence, the diameter of the needle that is needed is, consequently, proportional to the time that is needed to sew the wound and complete the surgery. So, the diameter X of a certain needle may be taken to be a random variable that is distributed uniformly between, say, 0 and 1 units of time. Furthermore, the efficiency of a certain medical component that is used during a certain surgery may be taken to be a uniform random variable X between A and B units of time. By the way, Japan manufactures the smallest needle that can be used to sew the blood veins.

If $h(y;n)$ is defined by

$$h(y;n) = \begin{cases} \frac{n+1}{n-1}\left(y^{1/n} - y^n\right) & A < y < B,\ \ A, B \in (0,1) \\ 0 & OW \end{cases},$$

then, applying the linear transformation $X = \frac{y-A}{B-A}$ to it, we get back the distribution (3) for the random variable X over the interval (0, 1). That is,

$$h(x;n) = \begin{cases} \frac{n+1}{n-1}\left(x^{\frac{1}{n}} - x^n\right) & 0 < x < 1 \\ 0 & OW \end{cases}$$

a. The cumulative distribution function $\boldsymbol{F(x;n)}$ of a random variable $\boldsymbol{X}$ with the density function $\boldsymbol{h(x;n)}$.

$$F(x;n) = P(X \leq x) = \int_0^x h(t;n)\, dt = \int_0^x \frac{n+1}{n-1}\left(t^{\frac{1}{n}} - t^n\right) dt = \frac{1}{n-1}\left(nx^{\frac{n+1}{n}} - x^{n+1}\right)$$

$$= \frac{x}{n-1}\left(nx^{\frac{1}{n}} - x^n\right), \ \ 0 < x < 1 \tag{4}$$

In piecewise defined form,

$$F(x;n) = \begin{cases} 0 & x \le 0, \\ \dfrac{x}{n-1}\left(n\, x^{\frac{1}{n}} - x^n\right) & 0 < x < 1, \\ 1 & x \ge 1. \end{cases}$$

b. The expected value and variance of continuous random variable X with the density function $h(x;n)$.

For $r = 1, 2, \ldots$, the expected value of the statistic X^r (or the rth moment of the distribution of X) is

$$E(X^r;\ n) = \int_{-\infty}^{\infty} x^r h(x;n)\, dx = \frac{n+1}{n-1}\int_0^1 \left(x^{\frac{rn+1}{n}} - x^{n+r}\right) dx = \frac{(n+1)^2}{[(r+1)n+1](n+r+1)} \tag{5}$$

This implies that

$$\mu_{X,n} = E(X;\ n) = \frac{(n+1)^2}{(2n+1)(n+2)}, \qquad and \qquad E(X^2;\ n) = \frac{(n+1)^2}{(3n+1)(n+3)}, \tag{6}$$

Hence, the variance by (1) is

$$\sigma^2_{X,n} = \mathrm{V}(X;\mathrm{n}) = \frac{(n+1)^2}{(3n+1)(n+3)} - \frac{(n+1)^4}{(2n+1)^2(n+2)^2} = (n+1)^2\left[\frac{n^4+4n^3+7n^2+4n+1}{(3n+1)(n+3)(2n+1)^2(n+2)^2}\right] \tag{7}$$

Theorem 1 For all x in $(0,1)$, the sequence of functions $h(x;n)$ converges pointwise to the uniform distribution $U(x;0,1)$.

Proof. For any fixed x in $(0,1)$, $\lim_{n\to\infty} x^{1/n} = x^0 = 1$ and $\lim_{n\to\infty} x^n = 0$. Therefore,

$h(x;\infty) = \lim_{n\to\infty} h(x;n) = \lim_{n\to\infty} \frac{n+1}{n-1}\left(x^{\frac{1}{n}} - x^n\right) = 1(1-0) = 1$, for all x in $(0,1)$.

In piecewise defined form,

$$h(x;\ \infty) = \begin{cases} 1 & 0 < x < 1 \\ 0 & OW \end{cases}$$

which is indeed the uniform distribution $U(x;0,1)$ of the random variable X over the interval [0, 1]. ■

Figure 1.1 below shows how the *standard h-distribution* $h(x;n)$ approaches the uniform distribution $U(x;0,1)$ as n approaches infinity. It shows that, as n is getting bigger, the graph of $h(x;n)$ is getting closer to the graph of the density function $U(x;0,1)$.

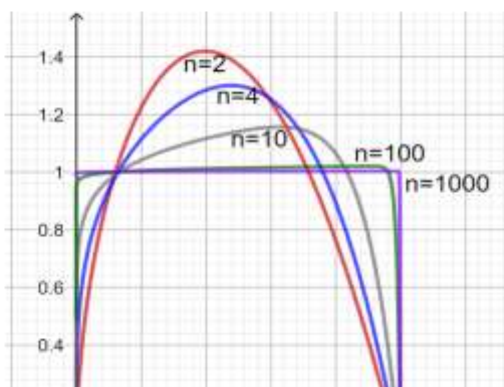

Figure 1.1. $h(x;n)$

For each $n > 1$, the density function $h(x;n)$ is skewed to the right and part of the graph is above the horizontal line $y = 1$. As n is getting bigger, the left part of the curve $h(x;n)$ is getting closer to the vertical line $x = 0$, the right part of the curve is getting closer to the vertical line $x = 1$, and the part of the curve that is above the line $y = 1$ is getting closer to this line. Only at infinity, the density function $h(x;n)$ equals the uniform distribution $U(x;0,1)$. The curve $h(x;n)$ has a bell shape. But due to a lack of symmetry and the heavy right skewness, its median that is very sensitive to extremely small or extremely large values, is greater than its mean for $n > 1$. However, the mean and mode of $h(x;n)$ must coincide at one value of the parameter n. See Proposition 6 below.

Corollary 2. The cumulative distribution function F(x), expected value E(X), and variance V(X) of the continuous random variable X that has the probability density function $h(x;\,\infty) = U(x;0,1)$ are the limiting form, in which $n \to \infty$, of the cumulative distribution function $F(X;n)$, expected value $E(X;n)$, and variance $V(X;n)$, respectively, of random value X that has the probability density function $h(x;\,n)$.

Proof. $\lim_{n\to\infty} x^{\frac{n+1}{n}} = x^{\lim_{n\to\infty}\frac{n+1}{n}} = x,\ \ 0 < x < 1$, Then we have

$$F(x) = \lim_{n\to\infty} F(x;n) = \lim_{n\to\infty} \frac{n+1}{n-1}\left(\frac{n}{n+1}x^{\frac{n+1}{n}} - \frac{1}{n+1}x^{n+1}\right) = 1(x-0) = x\,,\text{for all } 0 < x < 1.$$

$$E(X) = \lim_{n\to\infty} E(X;n) = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+1)(n+2)} = \frac{1}{2}$$

$$E(X^2) = \lim_{n\to\infty} E(X^2;n) = \lim_{n\to\infty} \frac{(n+1)^2}{(3n+1)(n+3)} = \frac{1}{3}\ \text{, and}$$

$$V(X) = \lim_{n\to\infty} V(X;n) = \lim_{n\to\infty} E(X^2;n) - \lim_{n\to\infty}[E(X;n)]^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12} \qquad \text{by (1)}$$

Indeed, $F(x)$, $E(X)$, and $V(X)$ are the cumulative distribution, expected value, and variance, respectively, of the random variable X with the probability density function $h(x;\,\infty) = U(x;0,1)$. ■

c. The mode (or modal value) of the standard h-distribution $\boldsymbol{h(x,n)}$.

The value of x, which maximizes the density function $h(x,n)$ over its domain, is called the mode or modal value.

Proposition 3 The mode of the density function $h(x,n)$ occurs at $x(n) = n^{2n/(1-n^2)}$ in the interval (0, 1), and the maximum height of the same function is $\frac{2n^2}{n^{1-n^2}}\,(n+1)^2$.

Proof. For simplicity, let us use, temporarily, the notation $h(x)$ for $h(x,n)$, which is clearly a continuous function on the closed interval $[0,1]$. Then

$$h'(x) = \frac{n+1}{n-1}\left(\frac{1}{n}x^{\frac{1-n}{n}} - nx^{n-1}\right) = 0, \text{ which implies that}$$

$$0 = x^{\frac{1-n}{n}} - n^2x^{n-1} = x^{\frac{1-n}{n}}(1 - n^2x^{(n^2-1)/n}).$$

Thus, 0 and $n^{2n/(1-n^2)}$ are the critical numbers of $h(x)$. Since $h(0) = h(1) = 0$, the optimal value of $h(x)$ occurs at $x(n) = n^{2n/(1-n^2)}$ in $(0,1)$. For $n > 1$, we have $1 - n^2 < 0$ and $\frac{2n}{1-n^2} < 0$. So, the exponential form $n^{\frac{2n}{1-n^2}} \neq 0$ must be between 0 and 1. Now,

$$h''(x) = \frac{n+1}{n-1}\left(\frac{1-n}{n^2}x^{\frac{1-2n}{n}} - n(n-1)\,x^{n-2}\right)$$

For $n > 1$, the numbers $\frac{1-n}{n^2}$ and $-n(n-1)$ are both negative, and the number $\frac{n+1}{n-1}$ is positive. Hence,

$h''(x) < 0$ for all x in $(0,1)$ and for sure $h''\left(n^{\frac{2n}{1-n^2}}\right) < 0$ since $n^{\frac{2n}{1-n^2}} \in (0,1)$. Thus, by the second derivative test, the density function $h(x;n)$ reaches its peak at $n^{2n/(1-n^2)}$. So, its mode is $x(n) = n^{2n/(1-n^2)}$. A routine calculation shows that the maximum height of the curve $h(x;n)$ is

$$h\left(n^{\frac{2n}{1-n^2}}\right) = \frac{2n^2}{n^{1-n^2}}\,(n+1)^2 \qquad \blacksquare$$

To show how this distribution can appear, we show that $x(n)$ is increasing function of n. Now, $x(n) = n^{2n/(1-n^2)}$ implies that $\ln x(n) = \frac{2n\ln n}{1-n^2}$. To show that $x(n)$ is increasing function, it is enough to show that $\ln x(n)$ is an increasing function because $x(n)$ is increasing if and only if $\ln x(n)$ is increasing. Note that

$$\frac{\mathrm{d}}{\mathrm{dn}}(\ln x(n)) = \frac{2(n^2\ln n + \ln n + 1 - n^2)}{(1-n^2)^2}$$

Let $f(n) = n^2\ln n + \ln n + 1 - n^2$. Then $f'(n) = 2n\ln n - n + \frac{1}{n}$ and $f''(n) = 2\ln n + 1 - \frac{1}{n^2}$. So $f''(n) > 0$ for all $n > 1$. Thus $f'(n)$ is increasing for all $n \geq 1$. Since $f'(1) = 0$, the function $f'(n) > 0$ for all $n > 1$. So, $f(n)$ is increasing for all $n \geq 1$. Since $f(1) = 0$, the function $f(n) > 0$ for all $n > 1$. Thus, $\frac{\mathrm{d}}{\mathrm{dn}}(\ln x(n)) > 0$. So, $\ln x(n)$ and $x(n)$ are increasing functions. Let y be the inverse function of $x(n)$. Therefore, if the mode $x(n)$ is given, then the power n can be recovered by $n = y(x)$.

Lemma 4 The mode $x(n) = n^{2n/(1-n^2)}$ and mean $\mu_{X,n} = \frac{(n+1)^2}{(2n+1)(n+2)}$ of $h(x;n)$, $n > 1$, are increasing sequences that approach 1 and $\frac{1}{2}$ as $n \to \infty$, respectively. The maximum height $\frac{2n^2}{n^{1-n^2}}\,(n+1)^2$ of the curve $h(x,n)$ is a decreasing sequence that approaches 1 as $n \to \infty$.

Proof. $\lim_{n\to\infty} n^{\frac{2n}{1-n^2}} = e^{\lim\limits_{n\to\infty}((2n\ln n)/(1-n^2))} = e^0 = 1$ by L'Hospital's Rule. It is clear that $\lim_{n\to\infty}\frac{(n+1)^2}{(2n+1)(n+2)} = \frac{1}{2}$.

The chart below shows that the maximum height $\frac{2n^2}{n^{1-n^2}}\,(n+1)^2$ of the curve $h(x,n)$ approaches 1 as $n \to \infty$, which is the maximum height of the density function $h(x;\infty) = U(x;0,1)$.

n	2	3	4	5	6	15
$\frac{2n^2}{n^{1-n^2}}(n+1)^2$	1.417411	1.350819	1.298809	1.259259	1.228649	1.110597

Indeed 1 is the mode and maximum height of the limiting form $U(x;0,1)$ of the *standard h-distribution* $h(x;n)$ in which $n \to \infty$.

$\blacksquare$

Remark 5 Since the mean of the density function $h(x,n)$ is $\mu_{X,n} = \frac{(n+1)^2}{(2n+1)(n+2)}$, the centroid of the region between the curves $x^{\frac{1}{n}}$ and x^n over the interval [0, 1] is located at the unique point on the line $y = x$, denoted by

$$(\bar{x},\bar{y}) = \left(\frac{(n+1)^2}{(2n+1)(n+2)}\,,\frac{(n+1)^2}{(2n+1)(n+2)}\right),$$

which can, also, be calculated using the following integrals

$$\bar{x} = \frac{1}{A}\int_0^1 x\left(x^{\frac{1}{n}} - x^n\right)dx, \qquad \bar{y} = \frac{1}{A}\int_0^1 \frac{1}{2}\left[\left(x^{\frac{1}{n}}\right)^2 - (x^n)^2\right]\ dx \ \ where\ A = \int_0^1 \left(x^{\frac{1}{n}} - x^n\right)dx$$

The following chart shows how the values of the mode $x(n)$ and mean $\mu_{X,n}$ behave as n is getting bigger.

n	2	3	4	5	6	20	100
$x(n) = n^{2n/(1-n^2)}$	0.396850	0.438691	0.477421	0.511402	0.54101	0.740578	0.912002
$\mu_{X,n} = \dfrac{(n+1)^2}{(2n+1)(n+2)}$	0 .45	0.457143	0.462963	0.467532	0.471154	0.490022	0.497561

The chart also shows that the mode of $h(x,n)$ is smaller than its mean if $n = 3$ and greater than its mean if $n = 4$.

Proposition 6 The mode and mean of the density $h(x,n)$ are the same at one value of n inside the interval $(3,\ 4)$.

Proof. The mode and the mean of the curve $h(x,n)$ are characterized by n. For a real number n greater than 1, the function $g(n) = n^{2n/(1-n^2)} - \frac{(n+1)^2}{(2n+1)(n+2)}$ is clearly continuous. Since $g(3) < 0$ and $g(4) > 0$, the Intermediate Value Theorem guarantees the existence of a number n_0 in the interval (3, 4) such that $g(n_0) = 0$. That is, n_0 is the number between $n = 3$ and $n = 4$ at which the mode and the mean of the curve $h(x,n)$ are equal. This means that when $n = n_0$, the density function $h(x,n)$ is approximately a normal distribution. So, the density function $h(x,n)$ is approximately a normal distribution when $n \in [3, 4]$

■

After a sequence of iterations, it was found that

$$g(3.54837) = -0.000000351 < 0 \qquad \text{and} \qquad g(3.54839) = 0.000000305 > 0$$

and the number $c \approx 3.54838$ is in the interval $(3.54837,\ 3.54839)$ such that $g(3.54838) = -0.000000023 \approx 0$

That is, at $c \approx 3.54838$, the mode and the mean of the curve $h(x,n)$ are equal.

The mode $x_n = n^{2n/(1-n^2)}$ and mean $\mu_{X,n} = \frac{(n+1)^2}{(2n+1)(n+2)}$ of the density function $h(x,n)$ depend on $n > 1$. So, n shifts the mode and the mean on the horizontal axis. Since the *standard h-distribution* $h(x,n)$ is characterized by $n > 1$, for each n, there is a corresponding *h-distribution*.

2.2 A General Form of the Standard h-Distribution $h(x;n)$.

A probability density function with three parameters A, B and n can be defined for a continuous random variable X, by

$$h(x;A,B,n) = \begin{cases} \frac{n+1}{n-1}(B-A)^{-1}\left[\left(\frac{x-A}{B-A}\right)^{1/n} - \left(\frac{x-A}{B-A}\right)^n\right] & 0 < A \le x \le B < 1 \\ 0 & OW \end{cases} \tag{8}$$

Note that when $A = 0$, and $B = 1$, the function in (8) in nothing but the *standard h-distribution* $h(x;n)$. The function in (8) can also be written (x replaced by y) as

$$h(y;A,B,n) = \frac{1}{B-A}h\left(\frac{y-A}{B-A};\ n\right), \quad 0 < A \le y \le B < 1 \tag{9}$$

The function (9) is a probability density function on $(A,\ B)$ due to the following:

1. Since $h(x;n) \ge 0$, by (8)

$$\frac{1}{B-A}\ h\left(\frac{y-A}{B-A};\ n\right) = h(y;A,B,n) \ge 0$$

$$2.\ \int_A^B h(y;A,B,n)\,dy = \int_A^B \frac{1}{B-A}\, h\left(\frac{y-A}{B-A};\ n\right)dy = \int_0^1 \frac{1}{B-A}\, h(x;\ n)\,(B-A)\,dx = \int_0^1 h(x;n)\,dx = 1,$$

using (9) and the substitution $x = \frac{y-A}{B-A}$

$$3.\ P(a < x < b) = \int_a^b h(x;A,B,n)dx.$$

Back to a surgery performed on somebody. The initial wound will be on the surface of the body followed by a smaller one on an interior organ inside the body at which the main operation is taking place. That is how the numbers A and B came to the picture.

a. The cumulative distribution of the random variable X with the probability density function $\boldsymbol{h(x;A,B,n)}$.

Using the substitution $x = (y-A)/(B-A)$ and equation (9), we have

$$F(y;A,B,n) = P(Y \le y) = \int_{-\infty}^{y} h(y;A,B,n)\,dy = \int_A^y \frac{1}{B-A}\, h\left(\frac{t-A}{B-A};\ n\right)dt = \int_0^{\frac{y-A}{B-A}} \frac{1}{B-A}\, h(x;\ n)\,(B-A)dx$$

$$= F\left(\frac{Y-A}{B-A},\ n\right)$$

$$= \frac{1}{n-1}\left(\frac{y-A}{B-A}\right)\left[n\left(\frac{y-A}{B-A}\right)^{\frac{1}{n}} - \left(\frac{y-A}{B-A}\right)^{n}\right], 0 < A < x < B < 1 \quad by\ (4) \qquad (10)$$

b. The expected value $\boldsymbol{E(X;A,B,n)}$ and variance $\boldsymbol{V(X;A,B,n)}$ of the random variable X with the probability density function $\boldsymbol{h(x;A,B,n)}$.

Again, using the equation (9) and the substitution $x = (y-A)/(B-A)$, we have

$$E(K(Y);A,B,n) = \int_{-\infty}^{\infty} K(y)\, h(y;A,B,n)\,dy = \int_A^B K(y)\frac{1}{B-A}\, h\left(\frac{y-A}{B-A};\ n\right)dy$$

$$= \int_0^1 K\big((B-A)x + A\big)\cdot\frac{1}{B-A} h(x;\ n)\,(B-A)\,dx$$

$$= \int_0^1 K\big((B-A)x + A\big)\, h(x;\ n)\ dx \qquad (i)$$

If $K(Y) = Y$, then by (i)

$$E(Y;A,B,n) = \int_0^1 \big((B-A)x + A\big)\, h(x;\ n)\,dx = (B-A)\int_0^1 x\, h(x;\ n)\ dx + A\int_0^1 h(x;\ n)\,dx$$

$$= (B-A)\,E(X;\ n) + A = \frac{(B-A)(n+1)^2}{(2n+1)(n+2)} + A \quad (11)$$

If $K(Y) = Y^2$, then by (i)

$$E(Y^2;A,B,n) = \int_A^B y^2\, h(x;A,B,n)\,dx = \int_0^1 [(B-A)x + A]^2\, h(x;\ n)\,dx$$

$$= \int_0^1 [(B-A)^2x^2 + 2(B-A)\,A\,x + A^2]\, h(x;\ n)\,dx$$

$$= (B-A)^2 \int_0^1 x^2 h(x;\ n)\, dx \ + \ 2(B-A)\, A \int_0^1 x\, h(x;\ n)\, dx + A^2 \int_0^1 h(x;\ n)\, dx$$

$$= \ (B-A)^2 E(X^2;\ n) + 2A(B-A)\, E(X;\ n) + A^2 \qquad \text{(ii)}$$

So, the variance of X with the probability density function $h(x; A, B, n)$ is

$$V(Y; A, B, n) \ = E(Y^2; A, B, n) - [E(Y; A, B, n)]^2$$

$$= (B-A)^2\, E(X^2;\ n) + 2A(B-A)\, E(X;\ n) + A^2 - [(B-A)\, E(X;\ n) \ + A\,]^2 \ \ by\ (ii)\ \ and\ (10)$$

$$= (B-A)^2 E(X^2;\ n) + 2A(B-A)\, E(X;\ n) + A^2 - \ (B-A)^2 [\,E(X;\ n)]^2 - \ 2A(B-A)\, E(X;\ n) - A^2$$

$$= (B-A)^2\, [E(X^2;\ n) - [\,E(X;\ n)]^2] = (B-A)^2\, V(X; n)$$

$$= (B-A)^2 \left[\frac{(n+1)^2(\mathrm{n}^4 + \ 4\mathrm{n}^3 \ + \ 7\mathrm{n}^2 \ + \ 4\mathrm{n} \ + \ 1)}{(3\mathrm{n}+1)(\mathrm{n}+3)(2\mathrm{n}+1)^2(\mathrm{n}+2)^2}\right] \qquad by\ (7) \qquad (12)$$

Theorem 7 The sequence of curves $h(x; A, B, n)$, for all x in $(0, 1)$, converges pointwise to the uniform distribution $U(x; A, B, n)$ on the interval [A, B], $0 < A < x < B < 1$.

Proof. $\lim_{n\to\infty} \frac{n+1}{n-1} = 1$, $e^{\lim_{n\to\infty} \frac{1}{n} \ln\left(\frac{x-A}{B-A}\right)} = e^0 = 1$, and $\lim_{n\to\infty} \left(\frac{x-A}{B-A}\right)^n = 0$ because $0 < (B-A) < 1$. Then

$$\lim_{n\to\infty} h(x; A, B, n) = \lim_{n\to\infty} \frac{n+1}{n-1} (B-A)^{-1} \left[\left(\frac{x-A}{B-A}\right)^{1/n} - \left(\frac{x-A}{B-A}\right)^n\right] = (B-A)^{-1}(1-0) = 1/(B-A)$$

which is indeed the uniform distribution $U(x; A, B\ )$ of the random variable X on the interval [A, B].

■

Figure 2.1 below shows how the sequence of curves $h(x; A, B, n)$ with $A = 0.1$ and $B = .09$ approaches the curve of the uniform distribution $U(x; 0.1,\ 0.9)$ as n approaches infinity.

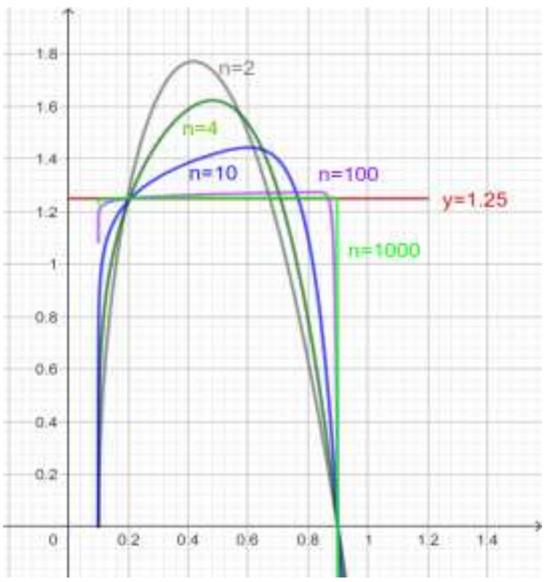

Figure 2.1. $h(x;\ 0.1, 0.9, n)$

We have $B - A = 0.8$ and $(B-A)^{-1} = \frac{1}{0.8} = 1.25$. For each n, the curve $h(x; 0.1, 0.9, n)$ is positively skewed, and part of it is located above the line $y = 1.25$. As $n \to \infty$, the right part of the curve coincides with the line $x = 0.9$, the left part of the curve coincides with the line $x = 0.1$, and the part above the line $y = 1.25$ coincides with this line. So, the uniform density $\mathrm{U}(x; \mathrm{A}, \mathrm{B})$ is the limiting form of the density function $h(x; A, B, n)$ in which $n \to \infty$.

Corollary 8 The cumulative distribution function F(x), expected value E(X), and variance V(X) of random variable X with uniform distribution $U(x; A, B)$, are the limiting form, in which $n \to \infty$, of the cumulative distribution function $\mathrm{F}(\mathrm{X}; \mathrm{A}, \mathrm{B}, \mathrm{n})$, expected value $\mathrm{E}(\mathrm{X}; \mathrm{A}, \mathrm{B}, \mathrm{n})$, and variance $\mathrm{V}(\mathrm{X}; \mathrm{A}, \mathrm{B}, \mathrm{n})$ of the random variable X with the probability density function $h(x; A, B, n)$, respectively.

Proof. Since $\lim_{n\to\infty} \frac{n}{n-1} = 1$, $\lim_{n\to\infty} \frac{n+1}{n} = 1$, and $\lim_{n\to\infty} \frac{1}{n-1} = 0$,

$$F(x) = \lim_{n\to\infty} F(x; A, B, n) = \lim_{n\to\infty} [\frac{n}{n-1} \left(\frac{x-A}{B-A}\right)^{(n+1)/n} - \frac{1}{n-1} \left(\frac{x-A}{B-A}\right)^{n+1}] = \frac{x-A}{B-A} \ ;$$

$$E(X) = \lim_{n\to\infty} E(X; A, B, n) = \lim_{n\to\infty} \left[\frac{(n+1)^2(B-A)}{(2n+1)(n+1)} + A\right] = \frac{1}{2}(B-A) + A = \frac{1}{2}(A+B);$$

$$V(X) = \lim_{n\to\infty} V(X; A, B, n) = \lim_{n\to\infty} \left[(B-A)^2 \left[\frac{(n+1)^2(n^4+4n^3-n^2-4n-1)}{(3n+1)(n+3)(2n+1)^2(n+2)^2}\right]\right] = (B-A)^2/12.$$

Indeed, $F(x)$, $E(X)$, and $V(X)$ are the cumulative distribution function, expected value, and variance, of the random variable X with the uniform distribution $U(x; A, B)$, respectively. ■

3. Conclusion

The density function $h(x; A, B, n)$ that is constructed in this paper, is a generalization of the well-known uniform distributions $U(x; A, B), where\ 0 < A < x < B < 1$. It has applications in several physical phenomena including the medical field, minerals industry, and technology. When the parameter n, on its way up, fall in the closed interval [3 ,4], the density $h(x; A, B, n)$ is approximated by the normal distribution. So, the density $h(x; A, B, n)$ is a good addition to those densities that may be approximated by the normal distribution under certain conditions such as the *binomia*l, the *hypergeometric*, the *Poisson*, and the *gamma*. But only when $n \to \infty$ will the distribution $h(x; A, B, n)$ equals the uniform distribution $U(x; A, B)$, where $0 < A < x < B < 1$. Thus, the uniform curve $U(x; A, B)$ is the curve $h(x; A, B, n)$ with n equals ∞. If the mode $x(n) = n^{2n/(1-n^2)}$ of $h(x; n)$ is given, then the power n can be recovered and the distribution $h(x; n)$ can appear. The discussion of properties of the standard *h-distribution* $h(x; n)$ is as follows.

1. For each $n \in (1, \infty)$, the curve $h(x; n)$ is of bell shape over the interval (0, 1) but not always symmetric.
2. There are infinitely many *standard h-distributions* $h(x; n)$, one for every value of the parameter n.
3. For each $n > 1$, the curve $h(x; n)$ has two parts: one above the line $y = 1$, and the other below the same line.
4. As $n \to \infty$, the sequence of curves $h(x; n)$ approaches the uniform distribution curve $U(x; 0, 1)$.
5. The density function $h(x, n)$ and its mode, mean, and maximum height, are characterized by the parameter n.
6. Its mode $x(n) = n^{2n/(1-n^2)}$is an increasing sequence that converges to 1, which is the mode of $U(x; 0, 1)$.
7. Its mean $\mu_{X,n} = \frac{(n+1)^2}{(2n+1)(n+2)}$ is an increasing sequence that converges to $\frac{1}{2}$, which is the mean of $U(x; 0, 1)$.
8. The curve $h(x; n)$ peaks at only one value n_0 of the parameter n that is inside the open interval (3, 4).
9. If $n < n_0$, the mode of $h(x; n)$ is smaller than its mean. If $n > n_0$, its mode is greater than its mean.
10. For $n \in [3, 4]$, its mode $x(n)$ and mean $\mu_{X,n}$ are very close to each other. That is, $h(x; n)$ is very close to the normal distribution.

Acknowledgment

Thanks goes to Saginaw Valley State University for the Sabbatical Leave award that was approved at the December 17, 2021, Board of Control meeting. The Sabbatical was for winter semester of 2023 to put this work together.

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