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Wavelet Estimation of a Density From Observations of Almost Periodically Correlated Process Under Positive Quadrant Dependence

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Abstract

In this paper, we construct a new wavelet estimator of density for the component of a finite mixture under positive quadrant dependence. Our sample is extracted from almost periodically correlated processes. To evaluate our estimator we will determine a convergence speed from an upper bound for the mean integrated squared error (MISE). Our result is compared to the independent case which provides an optimal convergence rate.

Keywords: Wavelet estimation of density, mixture model, almost periodically correlated processes, wavelet basis, positive quadrant dependence, Besov balls

1. Motivations

We observe n random variables $X_1, ..., X_n$ extracted from the continuous time process $X = (X_t)_{t\in\mathbb{R}}$ uniformly almost periodically correlated, of square integrable and zero mean, such that, for any $i \in \{1, ..., n\}$, the density of X_i is the following finite mixture:

$$\Upsilon_i(x) = \sum_{d=1}^{m} \Pi_d(i) f_d(x), \quad x \in \mathbb{R},$$

where $m \in \mathbb{N}^*$ (nonzero positive integer).

* $(\Pi_d(i))_{(i,d)\in\{1,...,n\}\times\{1,...,m\}}$ are taken as known weights of the mixture model such that, for any $i \in \{1, ..., n\}$, $0 \leq \Pi_d(i) \leq 1$ and $\sum_{d=1}^{m} \Pi_d(i) = 1$,

* $f_1, ..., f_m$ are densities (unknown) that we are going to estimate.

For any $\nu \in \{1, ..., m\}$, we are going to estimate f_ν from our sample $X_1, ..., X_n$ which consists of random variables which are pairwise positive quadrant dependent (PPQD).

The estimation of f_ν for finite mixture model has been extensively studied by Cai and Roussas (1997), Dewan and Prakasa Rao (1999), Masry (2001), Prakasa Rao (2003), Pokhyl'ko (2005), Chaubey and al (2006) and Christophe Chesneau (2010). Several estimators have been provided.

However, we propose a new estimator which proposes to considerably reduce the mean integrated square error (MISE) from an assumption on the appropriate wavelet.

Our study will be based on wavelet methodology inspired by Pokhyl'ko (2005) and Prakasa Rao (2010). We build a wavelet estimator, assuming that f_ν belongs to a Besov ball, and we evaluate its performance by determining the convergence speed from an upper bound of the mean integrated square error (MISE).

This paper is structured as follows. In section 2 a quick description of wavelet bases on $[0, 1]$, Besov balls, and almost periodically correlated processes. In section 3, we introduce additional assumptions and some notations on the model. The wavelet estimator and its parameters are detailed in section 4. The theorem on the convergence speed from the upper bound of the (MISE) is defined in section 5. Section 6 is dedicated to the proof of the result provided.

2. Wavelet basis, Besov Balls and Almost Periodically Correlated Processes

Wavelet basis. Let N be a nonzero positive integer, ϕ be a father wavelet of a multi-resolution analysis on $\mathbb{R}$ and ψ be the associated mother wavelet. Assume that

- there is an odd integer $\omega > 0$ such that $\phi^\omega = \phi$.

- $\phi \in C^1([0,1[)$

(For instance, Haar wavelet. $\phi(t) = 1$ if $t \in [0,1[$ and 0 otherwise).

Set

$$\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k), \;\; \psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k).$$

Then according to Cohen (1993), we can find an integer τ satisfying $2^\tau \geq 2N$ such that the set

$$\mathcal{B} = \{\phi_{\tau,k}(.); \psi_{j,k_1}(.)\}_{k\in\{0,\dots,2^\tau-1\}, j\in\{\tau,\tau+1,\dots\}, k_1\in\{0,\dots,2^j-1\}}$$

(with adequate treatment at the boundaries) is an orthonormal basis of $\mathbb{L}^2([0,1])$, the set of square-integrable functions on the interval $[0,1]$.

Hence for any integer $l \geq \tau$, any $g \in \mathbb{L}^2([0,1])$ can be expanded on the orthonormal basis $\mathcal{B}$ as

$$g(x) = \sum_{k=0}^{2^l-1} \alpha_{l,k}\phi_{l,k}(x) + \sum_{j=l}^{\infty}\sum_{k=0}^{2^j-1} \beta_{j,k}\psi_{j,k}(x),$$

where

$$\alpha_{l,k} = \int_0^1 g(x)\phi_{l,k}(x)dx, \;\; \beta_{j,k} = \int_0^1 g(x)\psi_{j,k}(x)dx.$$

$\alpha_{l,k}$ and $\beta_{j,k}$ are the wavelet coefficients of g.

Besov balls. Let $R > 0$, $s > 0$, $p \geq 1$ and $r \geq 1$. A measurable function g on $[0,1]$ and $\epsilon \geq 0$

$$\triangle_\epsilon (g)(x) = g(x+\epsilon) - g(x)$$

$\triangle_\epsilon^2 (g)(x) = \triangle_\epsilon (\triangle_\epsilon (g))(x)$ and identically, $\triangle_\epsilon^N (g)(x) = \triangle_\epsilon^{N-1} (\triangle_\epsilon (g))(x)$ for any nonzero positive integer N. Let

$$\rho^N(t,g,p) = \sup_{\epsilon\in[-t,t]} \|\triangle_\epsilon^N (g)\|_p$$

then, for $s \in [0,N[$, we define the Besov ball $B^s_{p,r}(R)$ of radius $R > 0$ by

$$B^s_{p,r}(R) = \Big\{g \in \mathbb{L}^p([0,1]); (\int_0^1 (\frac{\rho^N(t,g,p)}{t^s})^r \frac{dt}{t})^{1/r} \leq R\Big\}$$

if $p = \infty$ or $r = \infty$ we apply the usual modifications of norm.

We have the equivalence below to wavelet coefficients. (Hardle and al.(1998) corollary 9.1). A measurable function g belongs to $B^s_{p,r}(R)$ if and only if there exists a constant $R^* > 0$ (depending on R) such that the associated wavelet coefficients satisfy

$$2^{\tau(\frac{1}{2}-\frac{1}{p})}\Big(\sum_{k=0}^{2^\tau-1} |\alpha_{\tau,k}|^p\Big)^{1/p} + \Big(\sum_{j=\tau}^{\infty}\Big(2^{j(s+\frac{1}{2}-\frac{1}{p})}\Big(\sum_{k=0}^{2^j-1}|\beta_{j,k}|^p\Big)^{1/p}\Big)^r\Big)^{1/r} \leq R^*.$$

If a function g belongs to $B^s_{2,\infty}(R)$

$$\sup_{j\geq\tau} 2^{2js}\sum_{k\in\Lambda_j}\beta^2_{j,k} \leq R^*.$$

We set $\beta_{\tau-1,k} = \alpha_{\tau,k}$. p and r are norm parameters and s is a smoothness parameter. According to Meyer (1990), for a specific choice of s, p, and r, $B^s_{p,r}(R)$ contain the Holder and Sobolev balls.

Almost Periodically Correlated Processes

Before defining an almost periodically correlated process, let us introduce the notion of almost periodic functions. Other definitions equivalent to the following can be found in Corduneanu (1968).

Definition 2.1. *(Dehay.D , Monsan.V (2007)).*

Consider a complex-valued function $C : \mathbb{S}\times\mathbb{T} \to \mathbb{C}$ with $\mathbb{S} = \mathbb{Z}$ or $\mathbb{R}$, and $\mathbb{T} = \mathbb{Z}$ or $\mathbb{R}$. The function $C(s,\tau)$ is said to be almost periodic in s uniformly with respect to τ varying in $\mathbb{T}$, whenever for any $\epsilon > 0$ there exists $L_\epsilon > 0$ such that for any interval I with length greater than L_ϵ, there is $p_\epsilon \in I \cap \mathbb{S}$ with

$$\sup_{\tau\in\mathbb{T}}\sup_{s\in\mathbb{S}} |C(s+p_\epsilon,\tau) - C(s,\tau)| < \epsilon.$$

Moreover, whenever $\mathbb{S} = \mathbb{R}$ for simplicity, we assume that the function $(s, \tau) \mapsto C(s, \tau)$ is continuous on $\mathbb{R} \times \mathbb{T}$, any function $C : \mathbb{R} \times \mathbb{T} \to \mathbb{C}$ which is almost periodic uniformly on $\mathbb{T}$, satisfies the following uniform continuity property (Corduneanu, 1968)

$$\lim_{h\to 0} \sup_{\tau\in\mathbb{T}} \sup_{|s-t|<h} |C(t, \tau) - C(s, \tau)| = 0.$$

Next, we define the almost periodically correlated processes (Gladyshev (1963); Hurd (1991)).

Definition 2.2. *A real-valued process with zero mean $X = \{X_t : t \in \mathbb{S}\}$ is uniformly almost periodically correlated when $\mathbb{E}(X_t^2) < \infty$, for any $s \in \mathbb{S}$, the (shifted) covariance function $C(s, \tau) = Cov(X_s, X_{s+\tau}) = \mathbb{E}(X_s X_{s+t})$ is almost periodic in s uniformly in τ varying in $\mathbb{T} = \mathbb{S}$.*

3. Assumptions

In this section we start by introducing additional assumptions to the assumptions of section 1 then we provide notations.

(A_1) **ASSUMPTIONS ON** $X_1, ..., X_n$.

$X_1, ..., X_n$ are pairwise positive quadrant dependence (PPQD) i.e. for any $i, l \in \{1, ..., n\}$ with $i \neq l$ and any $(y, z) \in [0, 1]^2$

$$\mathbb{P}(X_i > y, X_l > z) \geq \mathbb{P}(X_i > y)\mathbb{P}(X_l > z)$$

This form of dependence was introduced by Lehmann (1966) and it generalizes other forms of dependence.

We recall $X_1, ..., X_n$ is a discrete-time process extracted from the continuous time process $X = (X_t)_{t\in\mathbb{R}}$ uniformly almost periodically correlated, of square-integrable and zero means.
- For any $i, l \in \{1, ..., n\}$

$$Cov(X_i, X_l) = \mathbb{E}(X_i X_l) \leq \sup_{t\in\mathbb{R}} \mathbb{E}(X_t^2) = b_0 < \infty,$$

(A_2) **ASSUMPTIONS ON** $f_1, ..., f_m$.

Without losing all generality, for any $d \in \{1, ..., m\}$, we assume that the support of f_d is $[0, 1]$.

(A_3) **ASSUMPTIONS ON THE WEIGHTS OF THE MIXTURE**.

We assume that the matrix Γ_n

$$\Gamma_n = \Big(\frac{1}{n}\sum_{i=1}^{n} \Pi_k(i)\Pi_l(i)\Big)_{(k,l)\in\{1,...,m\}^2}$$

is non-singular i.e. $det(\Gamma_n) > 0$. For fixed ν (the one that refers to our estimate of f_ν) and any $i \in \{1, ..., n\}$ we set

$$a_\nu(i) = \frac{1}{det(\Gamma_n)} \sum_{k=1}^{m} (-1)^{k+\nu} \gamma^n_{\nu,k} \Pi_k(i),$$

where $\gamma^n_{\nu,k}$ represents the determinant of the minor (ν, k) of the matrix Γ_n. $(a_\nu(1), ..., a_\nu(n))$ is unique solution of the following quadratic objective with linear constraints (quadratic optimization)

$$\underset{(b_1,...,b_n)\in\mathbb{R}^n}{\operatorname{argmin}} \frac{1}{n}\sum_{i=1}^{n} b_i^2,$$

such that for any $d \in \{1, ..., m\}$

$$\frac{1}{n}\sum_{i=1}^{n} b_i \Pi_d(i) = \delta_{\nu,d},$$

where $\delta_{\nu,d}$ represents the Kronecker delta.

For technical details see Maiboroda (1996) and Pokhyl'ko (2005).

We set

$$z_n = \frac{1}{n}\sum_{i=1}^{n} a_\nu^2(i).$$

We assume that $z_n < n$ for technical reasons.

4. Linear Estimator

Suppose that $f_\nu \in B^s_{p,r}(R)$ with $R > 0$, $s > 0$, $r \geq 1$ and $p \geq 2$, we define the estimator $\hat{f}$ by

$$\hat{f}(x) = \sum_{k=0}^{2^{j_0}-1} \hat{\alpha}_{j_0,k}\phi_{j_0,k}(x), \quad x \in [0, 1]$$

where

$$\hat{\alpha}_{j_0,k} = \frac{1}{n}\sum_{i=1}^{n} a_\nu(i)\phi^\omega_{j_0,k}(X_i), \ \ with \ \ \phi^\omega_{j,k}(x) = 2^{\frac{j}{2}}\phi^\omega(2^j x - k),$$

with odd integer $\omega > 0$. j_0 is the integer satisfying

$$\frac{1}{2}(\frac{n}{z_n})^{1/(2s+\omega+3)} < 2^{j_0} \leq (\frac{n}{z_n})^{1/(2s+\omega+3)}.$$

$\hat{\alpha}_{j_0,k}$, is defined taking into account the pairwise positive quadrant dependence (PPQD) case, j_0 is chosen to minimize the mean integrated squared error (MISE) of $\hat{f}$.

5. Result

We formulate the main result of the article, the upper bound of the estimation error of $\hat{f}$ is given in Theorem 5.1 belongs.

Theorem 5.1. *Let $X_1, ..., X_n$ be n random variables as described in Section 1 under the assumptions (A_1), (A_2) and (A_3). Suppose that $f_\nu \in B^s_{p,r}(R)$ with $p \geq 2$, $s > 0$, $r \geq 1$ and $\omega > 0$ a odd integer. Then there exists a constant $C > 0$ such that*

$$\mathbb{E}\Big(\int_0^1 (\hat{f}(x) - f_\nu(x))^2 dx\Big) \leq C(\frac{z_n}{n})^{2s/(2s+\omega+3)}.$$

In the case where $m = 1$, $\Pi_1(1) = ... = \Pi_1(n) = 1$, $z_n = 1$ and $f_\nu = f_1 = f$, the rate of convergence attained by $\hat{f}$ becomes $(\frac{1}{n})^{2s/(2s+\omega+3)}$.

The case of independent random variables has been studied by Pokhyl'ko who provides an optimal convergence rate i.e. $(\frac{z_n}{n})^{2s/(2s+1)}$.

6. Proofs

In the sequel, C denotes a constant that may differ from one term to another. Its value can depend on ω, ϕ or ψ but does not depend on j, k and n.

To establish the proof of theorem 5.1 we use an inequality of moments on a particular decomposition of the mean integrated squared error (MISE).

Proposition 6.1. *Let $X_1, ..., X_n$ be n random variables as described in Section 1 under the assumptions (A_1), (A_2) and (A_3). For any $k \in \{0, ..., 2^{j_0} - 1\}$. Then there exists a constant $C > 0$ such that*

$$\mathbb{E}\Big((\hat{\alpha}_{j_0,k} - \alpha_{j_0,k})^2\Big) \leq C2^{j_0(\omega+2)}\frac{z_n}{n}$$

Proof. Proof of Proposition 6.1.

$$\begin{aligned}
\mathbb{E}(\hat{\alpha}_{j_0,k}) &= \frac{1}{n}\sum_{i=1}^{n} a_\nu(i)\mathbb{E}(\phi^\omega_{j_0,k}(X_i)) \\
&= \frac{1}{n}\sum_{i=1}^{n} a_\nu(i)(\sum_{d=1}^{n}\Pi_d(i)\int_0^1 f_d(x)\phi^\omega_{j_0,k}(x)dx) \\
&= \sum_{d=1}^{n}\int_0^1 f_d(x)\phi^\omega_{j_0,k}(x)dx(\frac{1}{n}\sum_{i=1}^{n} a_\nu(i)\Pi_d(i)) \\
&= \int_0^1 f_\nu(x)\phi^\omega_{j_0,k}(x)dx = \int_0^1 f_\nu(x)\phi_{j_0,k}(x)dx, \quad (\phi^\omega = \phi) \\
&= \alpha_{j_0,k}.
\end{aligned}$$

Therefore

$$\mathbb{E}\Big((\hat{\alpha}_{j_0,k}-\alpha_{j_0,k})^2\Big) = Var(\hat{\alpha}_{j_0,k}) \\ \leq \frac{1}{n^2}\sum_{i=1}^{n}\sum_{l=1}^{n}|a_\nu(i)|.|a_\nu(l)|.|Cov(\phi^{\omega}_{j_0,k}(X_i),\phi^{\omega}_{j_0,k}(X_l))|.$$

From the assumption (A_1), $X_1,...,X_n$ are pairwise positive quadrant dependence (PPQD), It arises from (Newman 1980, Lemma 3) that, for any $i,l\in\{1,...,n\}$ with $i\neq l$,

$$|Cov(\phi^{\omega}_{j_0,k}(X_i),\phi^{\omega}_{j_0,k}(X_l))| \leq (\sup_{x\in[0,1]}|(\phi^{\omega}_{j_0,k}(x))^{'}|)^2 Cov(X_i,X_l).$$

Therefore

$$\frac{1}{n^2}\sum_{i=1}^{n}\sum_{l=1}^{n}|a_\nu(i)|.|a_\nu(l)|.|Cov(\phi^{\omega}_{j_0,k}(X_i),\phi^{\omega}_{j_0,k}(X_l))| \leq \frac{1}{n^2}A.B$$

where

$$A = (\sup_{x\in[0,1]}|(\phi^{\omega}_{j_0,k}(x))^{'}|)^2,$$

$$B = \sum_{i=1}^{n}\sum_{l=1}^{n}|a_\nu(i)|.|a_\nu(l)|.\sup_{t\in\mathbb{R}}\mathbb{E}(X_t^2),$$

with $\sup_{t\in[0,1]}\mathbb{E}(X_t^2) = b_0 < \infty$.

Now let's find a bound for A and B in turn.

Upper bound for A. Since $\phi\in C^1([0,1])$, we have $(\phi^{\omega}_{j_0,k}(x))^{'} = 2^{j_0(3/2)}.\omega.\phi^{\omega-1}(2^{j_0}x-k).\phi^{'}(2^{j_0}x-k)$, so

$$\sup_{x\in[0,1]}|(\phi^{\omega}_{j_0,k}(x))^{'}| \leq 2^{j_0((\omega+2)/2)}\sup_{x\in[0,1]}|\omega\phi^{'}(x)\phi^{\omega-1}(x)| = C2^{j_0((\omega+2)/2)}.$$

$\omega > 0$ a odd integer. Hence

$$A \leq C2^{j_0(\omega+2)}.$$

Determine an upper bound for B. We have

$$B \leq b_0 nz_n + 2\sum_{i=2}^{n}\sum_{l=1}^{i-1}|a_\nu(i)|.|a_\nu(l)|.b_0 \\ \leq b_0[nz_n + \sum_{i=2}^{n}\sum_{l=1}^{i-1}(a_\nu^2(i)+a_\nu^2(l))] \\ \leq b_0[nz_n + \sum_{i=2}^{n}a_\nu^2(i) + \sum_{l=1}^{i-1}a_\nu^2(l)] \\ \leq b_0[nz_n+nz_n+nz_n] = 3nz_nb_0.$$

Hence

$$B \leq C.n.z_n.$$

We obtain

$$\mathbb{E}\Big((\hat{\alpha}_{j_0,k}-\alpha_{j_0,k})^2\Big) \leq C2^{j_0(\omega+2)}\frac{z_n}{n}.$$

This completes the proof of Proposition 6.1. □

Proof. Proof of Theorem 5.1.

We expand the function f_ν on $\mathcal{B}$ as

$$f_\nu(x) = \sum_{k=0}^{2^{j_0}-1}\alpha_{j_0,k}\phi_{j_0,k}(x) + \sum_{j=j_0}^{\infty}\sum_{k=0}^{2^j-1}\beta_{j,k}\psi_{j,k}(x),$$

where $\alpha_{j_0,k}$ and $\beta_{j,k}$ are the wavelet coefficients of f_ν defined by

$$\alpha_{j_0,k} = \int_0^1 f_\nu(x)\phi_{j_0,k}(x)dx, \;\; \beta_{j,k} = \int_0^1 f_\nu(x)\psi_{j,k}(x)dx.$$

We have

$$\hat{f}(x) - f_\nu(x) = \sum_{k=0}^{2^{j_0}-1}(\hat{\alpha}_{j_0,k} - \alpha_{j_0,k})\phi_{j_0,k}(x) - \sum_{j=j_0}^{\infty}\sum_{k=0}^{2^j-1}\beta_{j,k}\psi_{j,k}(x).$$

Since $\mathcal{B}$ is an orthonormal basis of $\mathbb{L}^2([0,1])$, we obtain

$$\mathbb{E}\Big(\int_0^1 (\hat{f}(x) - f_\nu(x))^2 dx\Big) \leq F + G$$

where

$$F = \sum_{k=0}^{2^{j_0}-1}\mathbb{E}\Big((\hat{\alpha}_{j_0,k} - \alpha_{j_0,k})^2\Big)$$

and

$$G = \sum_{j=j_0}^{\infty}\sum_{k=0}^{2^j-1}\beta_{j,k}^2.$$

Using the definition of j_0 and Proposition 6.1, we have

$$F \leq C2^{j_0}2^{j_0(\omega+2)}\frac{z_n}{n} \leq C.\Big(\frac{z_n}{n}\Big)^{2s/(2s+\omega+3)}.$$

A deterministic calculation makes it possible to obtain G. Since $p \geq 2$, we have $B^s_{p,r}(R) \subset B^s_{2,\infty}(R)$. Hence

$$G \leq C2^{-2j_0s} \leq C.\Big(\frac{z_n}{n}\Big)^{2s/(2s+\omega+3)}.$$

Therefore

$$\mathbb{E}\Big(\int_0^1 (\hat{f}(x) - f_\nu(x))^2 dx\Big) \leq C.\Big(\frac{z_n}{n}\Big)^{2s/(2s+\omega+3)}$$

This completes the proof of Theorem 5.1.

□

References

Cai, Z. W., & Roussas, G. G. (1997). Effcient estimation of a distribution function under quadrant dependence. *Scand. J. Statist., 24,* 1-14.

Chaubey, Y. P., Doosti, H., & Prakasa Rao, B. L. S. (2006). Wavelet based estimation of the derivatives of a density with associated variables. *International Journal of Pure and Applied Mathematics, 27*(1), 97-106.

Christophe, C. (2010). Wavelet linear estimation of a density from observations of mixtures under quadrant dependence. 2010. hal-00510690v1.

Cohen, A., Daubechies, I., Jawerth, B., & Vial, P. (1993). Wavelets on the interval and fast wavelet transforms. *Applied and Computational Harmonic Analysis, 24*(1), 54-81. https://doi.org/10.1006/acha.1993.1005

Corduneanu, C. (1968). *Almost periodic functions*. Wiley, New-York.

Dewan, I., & Prakasa Rao, B. L. S. (1999). A general method of density estimation for associated random variables. *J. Nonparametric Statistics, 10*, 405-420. https://doi.org/10.1080/10485259908832769

Dehay, D., & Monsan, V. (2007). Discrete Periodic Sampling with Jitter and Almost Periodically Correlated Processes. *Stat Infer Stoch Process, 10*, 223-253. https://doi.org/10.1007/s11203-006-0004-3

Gladyshev, E. (1963). Periodically and almost periodically correlated random processes with continuous time parameter. *Th. Probability App., l8,* 173-177. https://doi.org/10.1137/1108016

Hardle, W., Kerkyacharian, G., Picard, D., & TSYBAKOV, A. (1998). Wavelet, Approximation and Statistical Applications. Lectures Notes in Statistics, 129, Springer Verlag, New York. https://doi.org/10.1007/978-1-4612-2222-4

Hurd, H. L. (1991) Correlation theory for the almost periodically correlated processes with continuous time parameter. *J Multivariate Anal, 37*(1), 24-45. https://doi.org/10.1016/0047-259X(91)90109-F

Lehmann, E. L. (1966). Some concepts of dependence. *Ann. Math. Statist., 37*, 1137-1153. https://doi.org/10.1214/aoms/1177699260

Maiboroda, R. E. (1996). Estimators of components of a mixture with varying concentrations. *Ukrain. Mat. Zh., 48*(4), 562-566. https://doi.org/10.1007/BF02390622

Masry, E. (2001). Multivariate probability density estimation for associated processes: Strong consistency and rates. *Statist. Probab. Lett., 58*, 205-219. https://doi.org/10.1016/S0167-7152(02)00105-0

Meyer, Y. (1990). *Ondelettes et Operateurs*. Hermann, Paris.

Newman, C. M. (1980). Normal fuctuations and the FKG inequalities. *Commun. Math.Phys., 74*, 119-128. https://doi.org/10.1007/BF01197754

Pokhyl'ko, D. (2005). Wavelet estimators of a density constructed from observations of a mixture. *Theor. Prob. and Math. Statist., 70*, 135-145. https://doi.org/10.1090/S0094-9000-05-00637-X

Prakasa Rao, B. L. S. (2003). Wavelet linear density estimation for associated sequences. *Journal of the Indian Statistical Association, 41,* 369-379.

Prakasa Rao, B. L. S. (2010). Wavelet linear estimation for derivatives of a density from observations of mixtures with varying mixing proportions. *Indian Journal of Pure and Applied Mathematics, 41*(1), 275-291.