Wavelet Estimation of a Density From Observations of Almost Periodically Correlated Process Under Positive Quadrant Dependence

Moussa Koné¹ & Vincent Monsan¹

¹ UFR de Mathématiques et Informatique, Université Félix Houphouët Boigny, Abidjan, Côte d’Ivoire

Correspondence: UFR de Mathematiques et Informatique, Universit Flix Houphouët Boigny, Abidjan, Côte dIvoire. E-mail: papyrusdegypte@yahoo.fr

Received: November 30, 2022 Accepted: January 16, 2023 Online Published: February 21, 2023
doi:10.5539/ijsp.v12n2p1 URL: https://doi.org/10.5539/ijsp.v12n2p1

Abstract
In this paper, we construct a new wavelet estimator of density for the component of a finite mixture under positive quadrant dependence. Our sample is extracted from almost periodically correlated processes. To evaluate our estimator we will determine a convergence speed from an upper bound for the mean integrated squared error (MISE). Our result is compared to the independent case which provides an optimal convergence rate.

Keywords: Wavelet estimation of density, mixture model, almost periodically correlated processes, wavelet basis, positive quadrant dependence, Besov balls

1. Motivations
We observe \( n \) random variables \( X_1, \ldots, X_n \) extracted from the continuous time process \( X = (X_t)_{t \in \mathbb{R}} \) uniformly almost periodically correlated, of square integrable and zero mean, such that, for any \( i \in \{1, \ldots, n\} \), the density of \( X_i \) is the following finite mixture:

\[
\Upsilon_i(x) = \sum_{d=1}^{m} \Pi_d(i)f_d(x), \quad x \in \mathbb{R},
\]

where \( m \in \mathbb{N}^* \) (nonzero positive integer).

* \( \Pi_d(i)(\omega(i), n) \) are taken as known weights of the mixture model such that, for any \( i \in \{1, \ldots, n\} \), \( 0 \leq \Pi_d(i) \leq 1 \) and \( \sum_{d=1}^{m} \Pi_d(i) = 1 \),

* \( f_1, \ldots, f_m \) are densities (unknown) that we are going to estimate.

For any \( \nu \in \{1, \ldots, m\} \), we are going to estimate \( f_\nu \) from our sample \( X_1, \ldots, X_n \) which consists of random variables which are pairwise positive quadrant dependent (PPQD).

The estimation of \( f_\nu \) for finite mixture model has been extensively studied by Cai and Roussas (1997), Dewan and Prakasa Rao (1999), Masry (2001), Prakasa Rao (2003), Pokhyl’ko (2005), Chaubey and al (2006) and Christophe Chesneau (2010). Several estimators have been provided.

However, we propose a new estimator which proposes to considerably reduce the mean integrated square error (MISE) from an assumption on the appropriate wavelet.

Our study will be based on wavelet methodology inspired by Pokhyl’ko (2005) and Prakasa Rao (2010). We build a wavelet estimator, assuming that \( f_\nu \) belongs to a Besov ball, and we evaluate its performance by determining the convergence speed from an upper bound of the mean integrated square error (MISE).

This paper is structured as follows. In section 2 a quick description of wavelet bases on \([0, 1]\), Besov balls, and almost periodically correlated processes. In section 3, we introduce additional assumptions and some notations on the model. The wavelet estimator and its parameters are detailed in section 4. The theorem on the convergence speed from the upper bound of the (MISE) is defined in section 5. Section 6 is dedicated to the proof of the result provided.

2. Wavelet basis, Besov Balls and Almost Periodically Correlated Processes

Wavelet basis. Let \( N \) be a nonzero positive integer, \( \phi \) be a father wavelet of a multi-resolution analysis on \( \mathbb{R} \) and \( \psi \) be the associated mother wavelet. Assume that

- there is an odd integer \( \omega > 0 \) such that \( \phi^{\omega} = \phi \).
\[ \phi \in C^1([0,1]) \]

(For instance, Haar wavelet. \( \phi(t) = 1 \) if \( t \in [0, 1] \) and 0 otherwise).

Set
\[ \phi_{jk}(x) = 2^{j/2} \phi(2^j x - k), \quad \psi_{jk}(x) = 2^{j/2} \psi(2^j x - k). \]

Then according to Cohen (1993), we can find an integer \( \tau \) satisfying \( 2^\tau \geq 2N \) such that the set
\[ \mathcal{B} = \{ \phi_{jk}(\cdot); \psi_{jk}(\cdot) \}_{k\in[0,2^\tau-1], j\in[0,\ldots,2^\tau-1]} \]

(with adequate treatment at the boundaries) is an orthonormal basis of \( L^2([0,1]) \), the set of square-integrable functions on the interval \([0,1]\).

Hence for any integer \( l \geq \tau \), any \( g \in L^2([0,1]) \) can be expanded on the orthonormal basis \( \mathcal{B} \) as

\[ g(x) = \sum_{k=0}^{2^{l-1}} \alpha_{j,k} \phi_{j,k}(x) + \sum_{j=l}^{\infty} \sum_{k=0}^{2^{j-1}} \beta_{j,k} \psi_{j,k}(x), \]

where
\[ \alpha_{j,k} = \int_0^1 g(x) \phi_{j,k}(x) dx, \quad \beta_{j,k} = \int_0^1 g(x) \psi_{j,k}(x) dx. \]

\( \alpha_{j,k} \) and \( \beta_{j,k} \) are the wavelet coefficients of \( g \).

**Besov balls.** Let \( R > 0, s > 0, p \geq 1 \) and \( r \geq 1 \). A measurable function \( g \) on \([0,1]\) and \( \epsilon \geq 0 \)

\[ \Delta_\epsilon \phi(x) = g(x + \epsilon) - g(x) \]

\[ \Delta_\epsilon (g)(x) = \Delta_\epsilon (\Delta_\epsilon (g))(x) \]

and identically, \( \Delta_\epsilon^N (g)(x) = \Delta_\epsilon^N (\Delta_\epsilon (g))(x) \) for any nonzero positive integer \( N \). Let

\[ \rho^N(t, g, p) = \sup_{\epsilon \in [0,1]} \| \Delta_\epsilon^N (g) \|_p \]

then, for \( s \in [0,N[ \), we define the Besov ball \( B^s_p(R) \) of radius \( R > 0 \) by

\[ B^s_p(R) = \{ g \in L^p([0,1]): \left( \int_0^1 \left( \frac{\rho^N(t, g, p)}{t^{s/r}} \right)^p \frac{dt}{t} \right)^{1/p} \leq R \} \]

if \( p = \infty \) or \( r = \infty \) we apply the usual modifications of norm.

We have the equivalence below to wavelet coefficients. (Hardle and al.(1998) corollary 9.1). A measurable function \( g \) belongs to \( B^s_p(R) \) if and only if there exists a constant \( R^* > 0 \) (depending on \( R \)) such that the associated wavelet coefficients satisfy

\[ 2^{s/2} \left( \sum_{k=0}^{2^{l-1}} |\alpha_{j,k}|^p \right)^{1/p} + \left( \sum_{j=\tau}^{\infty} 2^{j/2} \left( \sum_{k=0}^{2^{j-1}} |\beta_{j,k}|^p \right)^{1/p} \right)^{1/r} \leq R^*. \]

If a function \( g \) belongs to \( B^s_p(R) \)

\[ \sup_{j \geq \tau} \sum_{k \in \Lambda_j} \beta^2_{j,k} \leq R^*. \]

We set \( \beta_{r-1,k} = \alpha_{r,k} \). \( p \) and \( r \) are norm parameters and \( s \) is a smoothness parameter. According to Meyer (1990), for a specific choice of \( s, p, \) and \( r, B^s_p(R) \) contain the Holder and Sobolev balls.

**Almost Periodically Correlated Processes**

Before defining an almost periodically correlated process, let us introduce the notion of almost periodic functions. Other definitions equivalent to the following can be found in Corduneanu (1968).

**Definition 2.1.** (Dehay.D , Monsan.V (2007)).

Consider a complex-valued function \( C : \mathbb{S} \times \mathbb{T} \rightarrow \mathbb{C} \) with \( \mathbb{S} = \mathbb{Z} \) or \( \mathbb{R} \), and \( \mathbb{T} = \mathbb{Z} \) or \( \mathbb{R} \). The function \( C(s, \tau) \) is said to be almost periodic in \( s \) uniformly with respect to \( \tau \) varying in \( \mathbb{T} \), whenever for any \( \epsilon > 0 \) there exists \( L_\epsilon > 0 \) such that for any interval \( I \) with length greater than \( L_\epsilon \), there is \( p_\epsilon \in I \cap \mathbb{S} \) with

\[ \sup_{\tau \in \mathbb{T}} \sup_{s \in \mathbb{S}} |C(s + p_\epsilon, \tau) - C(s, \tau)| < \epsilon. \]
Moreover, whenever \( S = \mathbb{R} \) for simplicity, we assume that the function \((s, \tau) \mapsto C(s, \tau)\) is continuous on \( \mathbb{R} \times \mathbb{T} \), any function \( C : \mathbb{R} \times \mathbb{T} \to \mathbb{C} \) which is almost periodic uniformly on \( \mathbb{T} \), satisfies the following uniform continuity property (Corduneau, 1968)

\[
\lim_{h \to 0} \sup_{\tau \in \mathbb{T}} |C(t, \tau) - C(s, \tau)| = 0.
\]

Next, we define the almost periodically correlated processes (Gladyshev (1963); Hurd (1991)).

**Definition 2.2.** A real-valued process with zero mean \( X = \{X_t : t \in S\} \) is uniformly almost periodically correlated when \( \mathbb{E}(X_t^2) < \infty \), for any \( s \in S \), the (shifted) covariance function \( C(s, \tau) = \text{Cov}(X_s, X_{s+\tau}) = \mathbb{E}(X_sX_{s+\tau}) \) is almost periodic in \( s \) uniformly in \( \tau \) varying in \( \mathbb{T} = \mathbb{S} \).

3. **Assumptions**

In this section we start by introducing additional assumptions to the assumptions of section 1 then we provide notations.

**(A1) Assumptions on \( X_1, ..., X_n \).**

\( X_1, ..., X_n \) are pairwise positively quadrant dependent (PPQD) i.e. for any \( i, l \in \{1, ..., n\} \) with \( i \neq l \) and any \((y, z) \in [0, 1]^2\)

\[
\mathbb{P}(X_i > y, X_l > z) \geq \mathbb{P}(X_i > y)\mathbb{P}(X_l > z)
\]

This form of dependence was introduced by Lehmann (1966) and it generalizes other forms of dependence.

We recall \( X_1, ..., X_n \) is a discrete-time process extracted from the continuous time process \( X \in \mathbb{R} \) uniformly almost periodically correlated, of square-integrable and zero means.

- For any \( i, l \in \{1, ..., n\} \)

\[
\text{Cov}(X_i, X_l) = \mathbb{E}(X_iX_l) \leq \sup_{t \in \mathbb{R}} \mathbb{E}(X_t^2) = b_0 < \infty,
\]

**(A2) Assumptions on \( f_1, ..., f_m \).**

Without losing all generality, for any \( d \in \{1, ..., m\} \), we assume that the support of \( f_d \) is \([0, 1]\).

**(A3) Assumptions on the weights of the mixture.**

We assume that the matrix \( \Gamma_n \)

\[
\Gamma_n = \left( \frac{1}{n} \sum_{i=1}^{n} \Pi_k(i)\Pi_l(i) \right)_{k,l \in \{1, ..., m\}}
\]

is non-singular i.e. \( \text{det}(\Gamma_n) > 0 \). For fixed \( \nu \) (the one that refers to our estimate of \( f_n \)) and any \( i \in \{1, ..., n\} \) we set

\[
a_\nu(i) = \frac{1}{\text{det}(\Gamma_n)} \sum_{k=1}^{m} (-1)^{k+i} y_{\nu,k}^n \Pi_k(i),
\]

where \( y_{\nu,k}^n \) represents the determinant of the minor \((\nu, k)\) of the matrix \( \Gamma_n \). \((a_\nu(1), ..., a_\nu(n))\) is unique solution of the following quadratic objective with linear constraints (quadratic optimization)

\[
\arg\min_{(b_1, ..., b_n) \in \mathbb{R}^n} \frac{1}{n} \sum_{i=1}^{n} b_i^2,
\]

such that for any \( d \in \{1, ..., m\} \)

\[
\frac{1}{n} \sum_{i=1}^{n} b_i \Pi_d(i) = \delta_{\nu,d},
\]

where \( \delta_{\nu,d} \) represents the Kronecker delta.

For technical details see Maiboroda (1996) and Pokhyl’ko (2005).

We set

\[
z_n = \frac{1}{n} \sum_{i=1}^{n} a_\nu^2(i).
\]

We assume that \( z_n < n \) for technical reasons.
4. Linear Estimator

Suppose that \( f_v \in B_{p,r}(R) \) with \( R > 0, s > 0, r \geq 1 \) and \( p \geq 2 \), we define the estimator \( \hat{f} \) by

\[
\hat{f}(x) = \sum_{k=0}^{2^j_0-1} \hat{a}_{j_0,k} \phi_{j_0,k}(x), \quad x \in [0,1]
\]

where

\[
\hat{a}_{j_0,k} = \frac{1}{n} \sum_{i=1}^{n} a_v(i) \phi_{j_0,k}(X_i), \quad \text{with} \quad \phi_{j_0,k}(x) = 2^j \phi(2^j x - k),
\]

with odd integer \( \omega > 0 \). \( j_0 \) is the integer satisfying

\[
\frac{1}{2} \left( \frac{n}{z_n} \right)^{1/(2s+\omega+3)} < j_0 \leq \left( \frac{n}{z_n} \right)^{1/(2s+\omega+3)}.
\]

\( \hat{a}_{j_0,k} \) is defined taking into account the pairwise positive quadrant dependence (PPQD) case, \( j_0 \) is chosen to minimize the mean integrated squared error (MISE) of \( \hat{f} \).

5. Result

We formulate the main result of the article, the upper bound of the estimation error of \( \hat{f} \) is given in Theorem 5.1 belongs.

**Theorem 5.1.** Let \( X_1, \ldots, X_n \) be \( n \) random variables as described in Section 1 under the assumptions \((A_1), (A_2) \) and \((A_3)\). Suppose that \( f_v \in B_{p,r}(R) \) with \( p \geq 2, s > 0, r \geq 1 \) and \( \omega > 0 \) a odd integer. Then there exists a constant \( C > 0 \) such that

\[
\mathbb{E}\left( \int_0^1 (\hat{f}(x) - f_v(x))^2 \, dx \right) \leq C \left( \frac{n}{z_n} \right)^{2s/(2s+\omega+3)}.
\]

In the case where \( m = 1, \Pi_1(1) = \ldots = \Pi_1(n) = 1, z_n = 1 \) and \( f_v = f = f_1 \), the rate of convergence attained by \( \hat{f} \) becomes \( \left( \frac{1}{n} \right)^{2s/(2s+\omega+3)} \).

The case of independent random variables has been studied by Pokhyl’ko who provides an optimal convergence rate i.e. \( \left( \frac{1}{n} \right)^{2s/(2s+1)} \).

6. Proofs

In the sequel, \( C \) denotes a constant that may differ from one term to another. Its value can depend on \( \omega, \phi \) or \( \psi \) but does not depend on \( j, k \) and \( n \).

To establish the proof of theorem 5.1 we use an inequality of moments on a particular decomposition of the mean integrated squared error (MISE).

**Proposition 6.1.** Let \( X_1, \ldots, X_n \) be \( n \) random variables as described in Section 1 under the assumptions \((A_1), (A_2) \) and \((A_3)\). For any \( k \in \{0, \ldots, 2^j_0 - 1\} \). Then there exists a constant \( C > 0 \) such that

\[
\mathbb{E}\left( (\hat{a}_{j_0,k} - \alpha_{j_0,k})^2 \right) \leq C 2^{j_0(\omega+2)} \frac{z_n}{n}
\]

**Proof.** Proof of Proposition 6.1.

\[
\mathbb{E}(\hat{a}_{j_0,k}) = \frac{1}{n} \sum_{i=1}^{n} a_v(i) \mathbb{E}(\phi_{j_0,k}(X_i))
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} a_v(i) \left( \sum_{d=1}^{n} \Pi_d(i) \int_0^1 f_\omega(x) \phi_{j_0,k}(x) \, dx \right)
\]

\[
= \sum_{d=1}^{n} \int_0^1 f_\omega(x) \phi_{j_0,k}(x) \, dx \left( \frac{1}{n} \sum_{i=1}^{n} a_v(i) \Pi_d(i) \right)
\]

\[
= \int_0^1 f_\omega(x) \phi_{j_0,k}(x) \, dx = \int_0^1 f_\omega(x) \phi_{j_0,k}(x) \, dx, \quad (\phi_\omega = \phi)
\]

\[
= \alpha_{j_0,k}.
\]
Therefore
\[ \mathbb{E}\left( (\hat{\alpha}_{j,k} - \alpha_{j,k})^2 \right) = \text{Var}(\hat{\alpha}_{j,k}) \]
\[ \leq \frac{1}{n^2} \sum_{i=1}^{n} \sum_{l=1}^{n} |a_i(r)| |a_l(r)| |\text{Cov}(\phi_{j,k}(X_i), \phi_{j,k}(X_l))| . \]

From the assumption (A1), \( X_1, ..., X_n \) are pairwise positive quadrant dependence (PPQD). It arises from (Newman 1980, Lemma 3) that, for any \( i, l \in \{1, ..., n\} \) with \( i \neq l \),
\[ |\text{Cov}(\phi_{j,k}(X_i), \phi_{j,k}(X_l))| \leq \left( \sup_{x \in [0,1]} |(\phi_{j,k}(x))'\right)^2 \text{Cov}(X_i, X_l). \]

Therefore
\[ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{l=1}^{n} |a_i(r)| |a_l(r)| |\text{Cov}(\phi_{j,k}(X_i), \phi_{j,k}(X_l))| \leq \frac{1}{n^2} AB \]
where
\[ A = \left( \sup_{x \in [0,1]} |(\phi_{j,k}(x))'\right)^2 , \]
\[ B = \sum_{i=1}^{n} \sum_{l=1}^{n} |a_i(r)| |a_l(r)| \sup_{x \in \mathbb{R}} \mathbb{E}(X_i^2) , \]
with \( \sup_{x \in [0,1]} \mathbb{E}(X_i^2) = b_0 < \infty \).

Now let's find a bound for \( A \) and \( B \) in turn.

Upper bound for \( A \). Since \( \phi \in C^1([0, 1]) \), we have \( (\phi_{j,k}(x))' = 2^{j/(2 \omega)} \omega x. \phi^{(\omega-1)}(2^j x - k), \phi'(2^j x - k) \), so
\[ \sup_{x \in [0,1]} |(\phi_{j,k}(x))'| = 2^{j/(\omega(\omega+2))} \sup_{x \in [0,1]} |\omega \phi'(x)\phi^{(\omega-1)}(x)| = C 2^{j/(\omega(\omega+2))} . \]
\( \omega > 0 \) a odd integer. Hence
\[ A \leq C 2^{j/(\omega(\omega+2))} . \]

Determine an upper bound for \( B \). We have
\[ B \leq b_0 n z_n + 2 \sum_{i=2}^{n} \sum_{l=1}^{n-1} |a_i(r)| |a_l(r)| b_0 \]
\[ \leq b_0 n z_n + \sum_{i=2}^{n} \sum_{l=1}^{n-1} (a_i^2(r) + a_l^2(l)) \]
\[ \leq b_0 n z_n + \sum_{i=2}^{n} a_i^2(r) + i-1 \sum_{l=1}^{n-1} a_l^2(l) \]
\[ \leq b_0 n z_n + n z_n + n z_n = 3 n z_n b_0 . \]

Hence
\[ B \leq C n z_n . \]

We obtain
\[ \mathbb{E}\left( (\hat{\alpha}_{j,k} - \alpha_{j,k})^2 \right) \leq C 2^{j/(\omega(\omega+2))} \frac{z_n}{n} . \]

This completes the proof of Proposition 6.1.

\[ \square \]

**Proof.** Proof of Theorem 5.1.

We expand the function \( f_r \) on \( B \) as
\[ f_r(x) = \sum_{k=0}^{2^j - 1} \alpha_{j,k} \phi_{j,k}(x) + \sum_{j=j_0}^{j} \sum_{k=0}^{2^j - 1} \beta_{j,k} \psi_{j,k}(x) . \]
where $\alpha_{j_0,k}$ and $\beta_{j,k}$ are the wavelet coefficients of $f_r$ defined by

$$\alpha_{j,k} = \int_0^1 f_r(x)\phi_{j_0,k}(x)dx, \quad \beta_{j,k} = \int_0^1 f_r(x)\psi_{j,k}(x)dx.$$ 

We have

$$\hat{f}(x) - f_r(x) = \sum_{k=0}^{2^{j_0}-1} (\hat{\alpha}_{j_0,k} - \alpha_{j_0,k})\phi_{j,k}(x) - \sum_{j=j_0}^{\infty} \sum_{k=0}^{2^j-1} \beta_{j,k}\psi_{j,k}(x).$$ 

Since $B$ is an orthonormal basis of $L^2([0,1])$, we obtain

$$E\left(\int_0^1 (\hat{f}(x) - f_r(x))^2dx\right) \leq F + G$$

where

$$F = \sum_{k=0}^{2^{j_0}-1} E((\hat{\alpha}_{j_0,k} - \alpha_{j_0,k})^2)$$

and

$$G = \sum_{j=j_0}^{\infty} \sum_{k=0}^{2^j-1} \beta_{j,k}^2.$$ 

Using the definition of $j_0$ and Proposition 6.1, we have

$$F \leq C2^{j_0}2^{j_0(\omega+2)}\frac{\varepsilon_n}{n} \leq C\left(\frac{\varepsilon_n}{n}\right)^{2\varepsilon/(2\varepsilon+\omega+3)}.$$ 

A deterministic calculation makes it possible to obtain $G$. Since $p \geq 2$, we have $B_{p,r}(R) \subset B_{2,\infty}(R)$. Hence

$$G \leq C2^{-2j_0} \leq C\left(\frac{\varepsilon_n}{n}\right)^{2\varepsilon/(2\varepsilon+\omega+3)}.$$ 

Therefore

$$E\left(\int_0^1 (\hat{f}(x) - f_r(x))^2dx\right) \leq C\left(\frac{\varepsilon_n}{n}\right)^{2\varepsilon/(2\varepsilon+\omega+3)}.$$ 

This completes the proof of Theorem 5.1.

\[\square\]

References


Copyrights

Copyright for this article is retained by the author(s), with first publication rights granted to the journal.

This is an open-access article distributed under the terms and conditions of the Creative Commons Attribution license (http://creativecommons.org/licenses/by/4.0/).