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On the Mixture of Topp–Leone–G Class and Exponentiated–G Class of Distribution

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Abstract

This article introduces a new finite mixture class of generated families between Topp–Leone–G class and exponentiated–G class, called mixture of Topp–Leone exponentiated–G (MTLE–G) class. This class be more suitable for many real life situations and improve fitted results. One sub–model of this class is studied in details, called mixture of Topp–Leone exponentiated–Weibull distribution. Some statistical properties are established. The potentiality of the new class is shown via two applications to real data sets.

1. Introduction

Exponentiated distributions can be obtained by powering a positive real number β to the cumulative distribution function (CDF), i.e, if we have CDF $F(x)$ of any random variable X , then the function

$$F(x) = \left[G(x)\right]^{\beta}, \quad \beta > 0, \tag{1}$$

is called an exponentiated distribution where introduced by Gupta et al. (1998)

Sangsanit and Bodhisuwan (2016) and AL-Shomrani et al. (2016) introduced a new generating lifetime distribution, called the Topp–Leone generating (TL–G) family of distribution. In addition, the TL–G family is capable of improving fitted results and tail behavior of existing distributions.

The CDF of TL–G is defined by the following expression

$$F(x) = \left[1 - \left(\overline{G}(x)\right)^2\right]^{\beta}, \quad \beta > 0, \tag{2}$$

and the corresponding PDF is given by

$$f_2(x) = 2\beta \overline{G}(x) g(x) \left[1 - \left(\overline{G}(x)\right)^2\right]^{\beta - 1}, \quad \beta > 0.$$

A mixture between two distributions or more is another technique to introduce a new class of distributions with CDF $G(x)$ defined by the following formula

$$G(x) = \sum_{i=1}^{n} p_i F_i(x), \tag{3}$$

where $\sum_{i=1}^{n} p_i = 1$, and p_i is a ratio number.

In this article we introduce a new class of mixture distributions based on exponentiated–G and TL–G classes. We call it as the mixture of Topp–Leone exponentiated–G family. We are motivated to study a special model of this class named as mixture of Topp–Leone exponentiated weibull distribution (MTLEWD). Its hazard function can be bathtub shaped failure rates. This paper is constructed as follows: in Section 2, we introduce the new class of mixture distributions. In section 3, we consider sub–model of the proposed family. Statistical properties including moments, incomplete moments, moments of residual life, moment generating function, quantile function and order statistics are derived in Section 4. Two real data sets are analyzed in Section 5. Finally, Section 6 concludes the article.

2. New Class of Mixture Distributions

In this section, a new mixture class of distributions, called the mixture of Topp–Leone exponentiated–G (MTLE–G) class is proposed. The CDF of the MTLE–G is defined by substituting (1) and (2) in (3) as the follows

$$\begin{aligned} F(x) &= (1-p)\left[G(x)\right]^{\beta} + p\left[1-\left(\overline{G}(x)\right)^{2}\right]^{\beta} \\ &= \left[G(x)\right]^{\beta}\left[1-p+p\left(1+\overline{G}(x)\right)^{\beta}\right], \end{aligned} \tag{4}$$

where $G(x)$ and $\overline{G}(x)$ are the CDF and the survival function (SF) of any baseline distribution, respectively, p is a ratio number and β is shape parameter.

The corresponding probability density function (PDF) is given by

$$\begin{aligned} f(x) &= \beta(1-p)g(x)\left[G(x)\right]^{\beta-1} + 2p\beta\overline{G}(x)g(x)\left[1-\left(\overline{G}(x)\right)^{2}\right]^{\beta-1} \\ &= \beta g(x)\left[G(x)\right]^{\beta-1}\left[1-p+2p\overline{G}(x)\left(1+\overline{G}(x)\right)^{\beta-1}\right], \quad \beta, p > 0, \end{aligned} \tag{5}$$

where $g(x)$ is the PDF of baseline distribution. A random variable X having MTLE–G density function (5) will be denoted by $X \sim \text{MTLE-G}$.

The corresponding SF and hazard function (HF) are provided in (6), (7), respectively:

$$S(x) = 1-\left[G(x)\right]^{\beta}\left[1-p+p\left(1+\overline{G}(x)\right)^{\beta}\right], \tag{6}$$

and

$$h(x) = \frac{\beta g(x)\left[G(x)\right]^{\beta-1}\left[1-p+2p\overline{G}(x)\left(1+\overline{G}(x)\right)^{\beta-1}\right]}{1-\left[G(x)\right]^{\beta}\left[1-p+p\left(1+\overline{G}(x)\right)^{\beta}\right]}. \tag{7}$$

Note that:

- At p=0, then the mixture be exponentiated family.
- At p=1, then the mixture be TL–G family.

3. Topp–Leone Exponentiated Weibull Distribution

Let *X* has the Weibull distribution with CDF $G(x) = 1-e^{-(\lambda x)^{\theta}}$ and PDF $g(x) = \theta\lambda(\lambda x)^{\theta-1}e^{-(\lambda x)^{\theta}}$. Then the CDF of the mixture of Topp–Leone exponentiated Weibull distribution (MTLEWD) becomes

$$F(x) = \left[1-e^{-(\lambda x)^{\theta}}\right]^{\beta}\left[1-p+p\left(1+e^{-(\lambda x)^{\theta}}\right)^{\beta}\right].$$

The corresponding PDF is

$$f(x) = \beta\theta\lambda^{\theta}x^{\theta-1}e^{-(\lambda x)^{\theta}}\left[1-e^{-(\lambda x)^{\theta}}\right]^{\beta-1}\left[1-p+2pe^{-(\lambda x)^{\theta}}\left(1+e^{-(\lambda x)^{\theta}}\right)^{\beta-1}\right].$$

Figure 1 shows some of the possible shapes of PDF of MTLEW distribution using R software for selected different values of parameters with different shapes the density function of MTLEWD is decreasing, left-skewed, right-skewed when $\theta < 1$, and more symmetric as $\theta > 1$. Figure 2 displays the hazard function of MTLEW distribution with various shapes using R software. This plot has very flexible shapes such as monotonically decreasing when $\theta < 1$, bathtub shaped, monotonically increasing and upside-down bathtub features, depending on the parameter values.

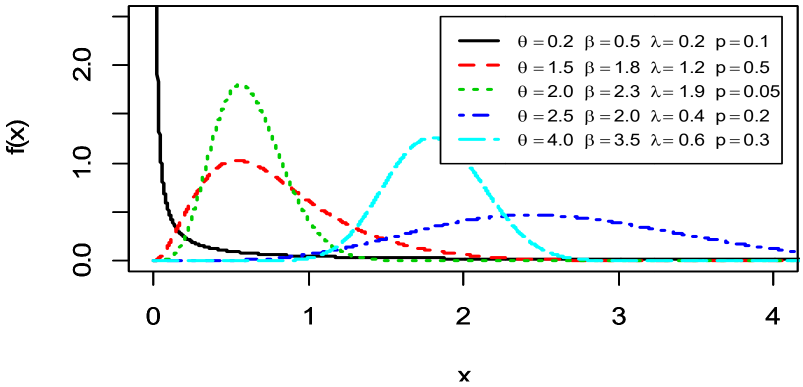

Figure 1. Plot of the PDF for some parameter values

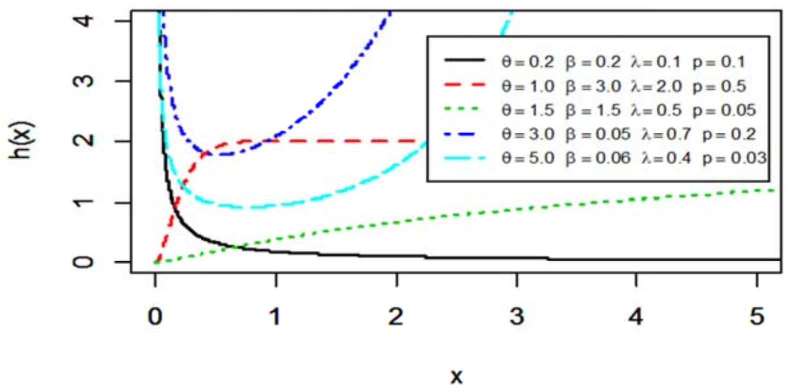

Figure 2. Plot of the HF for some parameter values

The behaviors of the PDF are

$\lim_{x\to 0} f(x) = 0,$

$\lim_{x\to \frac{1}{\lambda}} f(x) = \beta\theta\lambda e^{-1}\left[1-e^{-1}\right]^{\beta-1}\left[1-p+2pe^{-1}\left(1+e^{-1}\right)^{\beta-1}\right],$

$\lim_{x\to \infty} f(x) = 0.$

Let X follows MTLEWD, then X reduces to

1. EW distribution, if $p=0$.
2. TLW distribution, if $p=1$.
3. MTLE exponential distribution, if $\theta=1$ (new).
4. MTLE Rayleigh distribution, if $\theta=2$ (new).
5. Weibull distribution, if $p=0,\ \beta=1$.
6. Rayleigh distribution, if $p=0,\ \beta=1,\ \theta=2$.
7. Exponential distribution, if $p=0,\ \beta=1,\ \theta=1$.
8. Topp–Leone exponential distribution, if $p=1,\ \beta=1,\ \theta=1$.
9. Topp–Leone Rayleigh distribution, if $p=1,\ \beta=1,\ \theta=2$.

The corresponding SF is

$$S(x) = 1-\left[1-e^{-(\lambda x)^{\theta}}\right]^{\beta}\left[1-p+p\left(1+e^{-(\lambda x)^{\theta}}\right)^{\beta}\right], \tag{8}$$

and the HF is

$$h(x)=\frac{\beta\theta\lambda^{\theta}x^{\theta-1}e^{-(\lambda x)^{\theta}}\left[1-e^{-(\lambda x)^{\theta}}\right]^{\beta-1}\left[1-p+2pe^{-(\lambda x)^{\theta}}\left(1+e^{-(\lambda x)^{\theta}}\right)^{\beta-1}\right]}{1-\left[1-e^{-(\lambda x)^{\theta}}\right]^{\beta}\left[1-p+p\left(1+e^{-(\lambda x)^{\theta}}\right)^{\beta}\right]}. \tag{9}$$

The behaviors of the *HF* are

$$\lim_{x\to 0} h(x)=0,$$

$$\lim_{x\to\frac{1}{\lambda}} h(x)=\frac{\beta\theta\lambda e^{-1}\left[1-e^{-1}\right]^{\beta-1}\left[1-p+2pe^{-1}\left(1+e^{-1}\right)^{\beta-1}\right]}{1-\left[1-e^{-1}\right]^{\beta}\left[1-p+p\left(1+e^{-1}\right)^{\beta}\right]},$$

$$\lim_{x\to\infty} h(x)=0.$$

4. Statistical Properties

In this section, we consider some statistical properties of the proposed family.

4.1 Moments

Moments have an important role in any statistical analysis. It can be used to describe important characteristics and shapes, peakedness and study the symmetry of the shape of the distribution.

The r^{th} moment for MTLEWD about the origin is

$$\begin{aligned}\mu_r' &= \beta\theta\lambda^{\theta}\int_0^{\infty}x^{r+\theta-1}e^{-(\lambda x)^{\theta}}\left[1-e^{-(\lambda x)^{\theta}}\right]^{\beta-1}\left[1-p+2pe^{-(\lambda x)^{\theta}}\left(1+e^{-(\lambda x)^{\theta}}\right)^{\beta-1}\right]dx\\ &= \mathrm{I}_1+\mathrm{I}_2,\end{aligned} \tag{10}$$

where, I_1 and I_2 are obtained as follows

$$\mathrm{I}_1=\beta\theta\lambda^{\theta}\left(1-p\right)\int_0^{\infty}x^{r+\theta-1}e^{-(\lambda x)^{\theta}}\left[1-e^{-(\lambda x)^{\theta}}\right]^{\beta-1}dx.$$

Using binomial expansion, then I_1 can be written as follows:

$$\mathrm{I}_1=\beta\theta\lambda^{\theta}\left(1-p\right)\sum_{i=0}^{\infty}\binom{\beta-1}{i}(-1)^{i}\int_0^{\infty}x^{r+\theta-1}e^{-(i+1)(\lambda x)^{\theta}}dx=\Gamma\left(\frac{r}{\theta}+1\right)\sum_{i=0}^{\infty}\frac{A_i}{\lambda^{r}\left(1+i\right)^{\frac{r}{\theta}+1}},$$

where, $A_i=\beta\left(1-p\right)\left(-1\right)^{i}\binom{\beta-1}{i}$ and $\Gamma\left(\frac{r}{\theta}+1\right)$ is the gamma function.

and $\mathrm{I}_2=2\theta\lambda^{\theta}p\beta\int_0^{\infty}x^{r+\theta-1}e^{-2(\lambda x)^{\theta}}\left[1-e^{-2(\lambda x)^{\theta}}\right]^{\beta-1}dx.$

Using binomial expansion, then I_2 can be written as follows:

$$\mathrm{I}_2=2\theta\lambda^{\theta}p\beta\sum_{j=0}^{\infty}\binom{\beta-1}{j}(-1)^{j}\int_0^{\infty}x^{r+\theta-1}e^{-2(1+j)(\lambda x)^{\theta}}dx=\Gamma\left(\frac{r}{\theta}+1\right)\sum_{j=0}^{\infty}\frac{\lambda^{-r}B_j}{\left[2\left(1+j\right)\right]^{\frac{r}{\theta}+1}},$$

where, $B_j=2p\beta\left(-1\right)^{j}\binom{\beta-1}{j}$.

Substituting I_1 and I_2 in (10) we get

$$\mu_r' = \Gamma\left(\frac{r}{\theta}+1\right)\sum_{i=0}^{\infty}\frac{A_i}{\lambda^r\left(1+i\right)^{\frac{r}{\theta}+1}} + \Gamma\left(\frac{r}{\theta}+1\right)\sum_{j=0}^{\infty}\frac{\lambda^{-r}B_j}{\left[2\left(1+j\right)\right]^{\frac{r}{\theta}+1}}$$

$$= \lambda^{-r}\Gamma\left(\frac{r}{\theta}+1\right)\left\{\sum_{i=0}^{\infty}\frac{A_i}{\left(1+i\right)^{\frac{r}{\theta}+1}} + \sum_{j=0}^{\infty}\frac{B_j}{\left[2\left(1+j\right)\right]^{\frac{r}{\theta}+1}}\right\}.$$

The mean and the variance of MTLEWD are given as follows:

$$\mu_1' = \frac{1}{\lambda}\Gamma\left(\frac{1}{\theta}+1\right)\left\{\sum_{i=0}^{\infty}\frac{A_i}{\left(1+i\right)^{\frac{1}{\theta}+1}} + \sum_{j=0}^{\infty}\frac{B_j}{\left[2\left(1+j\right)\right]^{\frac{1}{\theta}+1}}\right\}, \tag{11}$$

and

$$\operatorname{var}(X) = \frac{1}{\lambda^2}\Gamma\left(\frac{2}{\theta}+1\right)\left\{\sum_{i=0}^{\infty}\frac{A_i}{\left(1+i\right)^{\frac{2}{\theta}+1}} + \sum_{j=0}^{\infty}\frac{B_j}{\left[2\left(1+j\right)\right]^{\frac{2}{\theta}+1}}\right\}$$

$$-\frac{1}{\lambda^2}\left[\Gamma\left(\frac{1}{\theta}+1\right)\right]^2\left\{\sum_{i=0}^{\infty}\frac{A_i}{\left(1+i\right)^{\frac{1}{\theta}+1}} + \sum_{j=0}^{\infty}\frac{B_j}{\left[2\left(1+j\right)\right]^{\frac{1}{\theta}+1}}\right\}^2. \tag{12}$$

The r^{th} central moment is defined by

$$\mu_r = E\,(X - \mu_1')^r = \sum_{i=0}^{r}(-1)^i\binom{r}{i}(\mu_1')^i\,\mu_{r-i}'.$$

The coefficient of skewness (*Sk*) and kurtosis (*Ku*) are defined by

$$\mathrm{Sk} = \frac{\mu_3}{\mu_2^{3/2}},\ \mathrm{Ku} = \frac{\mu_4}{\mu_2^2}.$$

Some measures of moments for MTLEW distribution

Table 1 contains numerical values of mean (μ_1'), variance (σ^2), *Sk*, and *Ku* of MTLEWD for some values of parameters

Table 1. μ_1', σ^2, *Sk* and *Ku* of MTLEWD

(a)

θ	β	λ	p	μ_1'	σ^2	Sk	Ku
0.4	0.5	0.2	0.9	2.275	188.632	35.089	3192.397
1.4	1.5	1.2	0.7	0.676	0.204	1.451	6.310
2.4	2.5	2.2	0.5	0.467	0.023	0.544	3.304
3.4	3.5	3.2	0.3	0.347	0.005	0.117	3.162
4.4	4.5	4.2	0.1	0.255	0.005	1.594	3.463

(b)

θ	β	λ	p	μ_1'	σ^2	Sk	Ku
0.4	0.5	0.2	0.5	5.1663	753.266	20.790	984.6328
1.4	1.5	1.2		0.7487	0.2485	1.3437	5.6045
2.4	2.5	2.2		0.4673	0.0229	0.5444	3.3039
3.4	3.5	3.2		0.3339	0.0048	0.2397	3.1860
4.4	4.5	4.2		0.2603	0.0002	0.5158	4.1288

From Table 1(a), we conclude that, as the values of θ, β and λ increase and for p decrease, then the values of μ_1', and σ^2 are decreasing and Sk and Ku are decreasing and increases. Also, it can be seen that the MTLEW distribution is right skewed and leptokurtic.

From Table 1(b), we conclude that, as the values of θ, β and λ increase and for fixed p, then the values of μ_1', and σ^2 are decreasing and Sk and Ku are decreasing and increases.

Incomplete Moments

Incomplete moments of the income distribution form natural building blocks for measuring inequality: for example, income quintiles, the Lorenz curve, which depend upon the incomplete moments of the income distribution. The s^{th} incomplete moment of *X*, denoted by $\varphi_s(t)$, is given by

$$\varphi_s = \int_0^t x^s f(x)dx = \beta\int_0^t x^s g(x)\left[G(x)\right]^{\beta-1}\left[1-p+2p\overline{G}(x)\left(1+\overline{G}(x)\right)^{\beta-1}\right]dx. \tag{13}$$

The s^{th} incomplete moment of MTLEWD is as follows:

$$\begin{aligned}\varphi_s &= \beta\theta\lambda^\theta\int_0^t x^{s+\theta-1}e^{-(\lambda x)^\theta}\left[1-e^{-(\lambda x)^\theta}\right]^{\beta-1}\left[1-p+2pe^{-(\lambda x)^\theta}\left(1+e^{-(\lambda x)^\theta}\right)^{\beta-1}\right]dx \\ &= J_1+J_2,\end{aligned} \tag{14}$$

where,

$$J_1 = \beta\theta\lambda^\theta(1-p)\int_0^t x^{s+\theta-1}e^{-(\lambda x)^\theta}\left[1-e^{-(\lambda x)^\theta}\right]^{\beta-1}dx.$$

Using binomial expansion, then J_1 can be written as follows:

$$J_1 = \beta\theta\lambda^\theta(1-p)\sum_{i=0}^\infty\binom{\beta-1}{i}(-1)^i\int_0^t x^{s+\theta-1}e^{-(i+1)(\lambda x)^\theta}dx = \sum_{i=0}^\infty A_i\,\frac{\gamma\left(\frac{s}{\theta}+1,(1+i)(\lambda t)^\theta\right)}{\lambda^s(1+i)^{1+\frac{s}{\theta}}}.$$

where, $\gamma\left(\frac{s}{\theta}+1,(1+i)(\lambda t)^{\theta}\right)$ is the incomplete lower gamma function.

and $J_2 = 2\beta\theta\lambda^{\theta} p\int_0^t x^{s+\theta-1} e^{-2(\lambda x)^{\theta}}\left[1-e^{-2(\lambda x)^{\theta}}\right]^{\beta-1} dx$.

Using binomial expansion, then J_2 can be written as follows:

$$J_2 = 2\beta\theta\lambda^{\theta} p\sum_{j=0}^{\infty}\binom{\beta-1}{j}(-1)^j\int_0^t x^{s+\theta-1} e^{-2(j+1)(\lambda x)^{\theta}} dx = \sum_{j=0}^{\infty} B_j \frac{\gamma\left(\frac{s}{\theta}+1, 2(1+j)(\lambda t)^{\theta}\right)}{\lambda^s\left[2(1+j)\right]^{1+\frac{s}{\theta}}}.$$

Substituting the values of J_1 and J_2 in (14) we get

$$\varphi_s(t) = \sum_{i=0}^{\infty} A_i \frac{\gamma\left(\frac{s}{\theta}+1,(1+i)(\lambda t)^{\theta}\right)}{\lambda^s (1+i)^{1+\frac{s}{\theta}}} + \sum_{j=0}^{\infty} B_j \frac{\gamma\left(\frac{s}{\theta}+1, 2(1+j)(\lambda t)^{\theta}\right)}{\lambda^s\left[2(1+j)\right]^{1+\frac{s}{\theta}}}. \quad (15)$$

4.2 Moments of the Residual and Reversed Residual Life

The n^{th} moment of the residual life (MRL), $m_n(x) = E\{(X-x)^n | X > x\}$, $n = 1,2,...$ uniquely determines $F(x)$, (see Navarro et al., 1998). It is given by

$$m_n(t) = \frac{1}{1-F(t)}\int_t^{\infty}(x-t)^n f(x) dx.$$

Using the binomial expansion, then

$$m_n(t) = \frac{1}{1-F(t)}\sum_{l=0}^{n}\binom{n}{l}(-t)^l\int_t^{\infty} x^{n-l} f(x) dx. \quad (16)$$

The n^{th} MRL for MTLEWD can be obtained as follows

$$m_n(t) = \frac{1}{1-F(t)}\sum_{l=0}^{n}\binom{n}{l}(-t)^l\{\omega_1+\omega_2\}, \quad (17)$$

where,

$$\omega_1 = \theta\lambda^{\theta}\beta(1-p)\int_t^{\infty} x^{n+\theta-l-1} e^{-(\lambda x)^{\theta}}\left[1-e^{-(\lambda x)^{\theta}}\right]^{\beta-1} dx .$$

Using binomial expansion, then ω_1 can be written as follows:

$$\omega_1 = \theta\lambda^{\theta}\beta(1-p)\sum_{i=0}^{\infty}\binom{\beta-1}{i}(-1)^i\int_t^{\infty} x^{n+\theta-l-1} e^{-(1+i)(\lambda x)^{\theta}} dx = \sum_{i=0}^{\infty}\frac{A_i\Gamma\left(\frac{n-l}{\theta}+1,(1+i)(\lambda t)^{\theta}\right)}{\lambda^{n-l}(1+i)^{1+\frac{n-l}{\theta}}},$$

where, $\Gamma\left(\frac{n-l}{\theta}+1,(1+i)(\lambda t)^{\theta}\right)$ is the incomplete upper gamma function.

Also, $\omega_2 = 2\theta\lambda^{\theta} p\beta\int_t^{\infty} x^{n+\theta-l-1} e^{-2(\lambda x)^{\theta}}\left[1-e^{-2(\lambda x)^{\theta}}\right]^{\beta-1} dx$.

Using binomial expansion, then ω_2 can be written as follows:

$$\omega_2 = 2\theta\lambda^\theta p\beta\sum_{j=0}^{\infty}\binom{\beta-1}{j}(-1)^j\int_t^\infty x^{n+\theta-l-1}e^{-2(1+j)(\lambda x)^\theta}dx = \sum_{j=0}^{\infty}\frac{B_j\Gamma\left(\frac{n-l}{\theta}+1,2(1+j)(\lambda t)^\theta\right)}{\lambda^{n-l}\left[2(1+j)\right]^{1+\frac{n-l}{\theta}}}.$$

Substituting the values of ω_1 and ω_2 in (17) we get

$$m_n(t) = \sum_{l=0}^{n}\binom{n}{l}\frac{(-t)^l\lambda^{l-n}}{1-F(t)}\left\{\sum_{i=0}^{\infty}\frac{A_i\Gamma\left(\frac{n-l}{\theta}+1,(1+i)(\lambda t)^\theta\right)}{(1+i)^{1+\frac{n-l}{\theta}}}+\sum_{j=0}^{\infty}\frac{B_j\Gamma\left(\frac{n-l}{\theta}+1,2(1+j)(\lambda t)^\theta\right)}{\left[2(1+j)\right]^{1+\frac{n-l}{\theta}}}\right\}.$$

4.3 Moment Generating Function

The general form of moment generating function for MTLEWD is defined as:

$$M(t) = \int_0^\infty e^{tx}f(x)dx = \eta_1+\eta_2, \tag{18}$$

where,

$$\eta_1 = \beta\theta\lambda^\theta(1-p)\int_0^\infty x^{\theta-1}e^{tx}e^{-(\lambda x)^\theta}\left[1-e^{-(\lambda x)^\theta}\right]^{\beta-1}dx.$$

η_1 can be written as follows:

$$\eta_1 = \beta\theta\lambda^\theta(1-p)\sum_{i,k=0}^{\infty}\frac{t^k(-1)^i}{k!}\binom{\beta-1}{i}\int_0^\infty x^{\theta-1+k}e^{-(i+1)(\lambda x)^\theta}dx = \sum_{i,k=0}^{\infty}\frac{A_i t^k}{\lambda^k k!(1+i)^{\frac{k}{\theta}+1}}\Gamma\left(\frac{k}{\theta}+1\right).$$

Also, $\eta_2 = 2\beta\theta p\lambda^\theta\int_0^\infty x^{\theta-1}e^{tx}e^{-2(\lambda x)^\theta}\left(1-e^{-2(\lambda x)^\theta}\right)^{\beta-1}dx.$

We apply the binomial expansion to η_2

$$\eta_2 = 2\beta\theta p\lambda^\theta\sum_{j,k=0}^{\infty}\frac{t^k(-1)^j}{k!}\binom{\beta-1}{j}\int_0^\infty x^{\theta-1+k}e^{-2(j+1)(\lambda x)^\theta}dx = \sum_{j,k=0}^{\infty}\frac{B_j t^k}{k!\lambda^k\left[2(1+j)\right]^{1+\frac{k}{\theta}}}\Gamma\left(\frac{k}{\theta}+1\right).$$

Substituting the values of η_1 and η_2 in (18) we get

$$\mathrm{M}(t) = \sum_{k=0}^{\infty}\frac{t^k}{k!\lambda^k}\Gamma\left(\frac{k}{\theta}+1\right)\left\{\sum_{i=0}^{\infty}\frac{A_i}{(1+i)^{\frac{k}{\theta}+1}}+\sum_{j=0}^{\infty}\frac{B_j}{\left[2(1+j)\right]^{1+\frac{k}{\theta}}}\right\}.$$

4.4 Quantile and Median

The quantile function x_q of MTLEWD is the real solution of the following equation

$$q = \left[1-e^{-(\lambda x_q)^\theta}\right]^\beta\left[1-p+p\left(1+e^{-(\lambda x_q)^\theta}\right)^\beta\right] \tag{19}$$

The percentage points at 25%, 50% and 75% of some specific choices of the parameters are given in Table 2.

Table 2. Percentage points for β,θ,λ and p

λ	θ	β	p	25%	50%	75%
1.2	1.5	1.8	0.5	0.464	0.714	1.048
1.9	2	2.3	0.05	0.458	0.601	0.761
0.4	2.5	2	0.2	2.012	2.566	3.173
0.6	4	3.5	0.3	1.603	1.812	2.028

We detect from Table 2 that as the values of λ are increase, then the values of percentage points are decreases.

4.5 Order Statistics

In this subsection, we drive the single order statistics for MTLE–G. Let $x_1 \ldots x_n$ denote *n* independent and identically distributed MTLE–G random variables. Further, let $x_{1:n},\ldots,x_{n:n}$ denote the order statistics from these *n* variables. Then, the PDF of the r^{th} order statistic $x_{(r:n)}$, say $f_{r:n}(x)$, the r^{th} order statistic is given by David and Nagaraja (2003)

$$f_{r:n}(x)=c_{r:n}\left[F(x)\right]^{r-1} f(x)\left[1-F(x)\right]^{n-r}.$$

Using the binomial expansion, then

$$\begin{aligned} f_{r:n}(x)&=c_{r:n}\sum_{i=0}^{n-r}\binom{n-r}{i}(-1)^i\left[F(x)\right]^{r+i-1} f(x)\\ &=c_{r:n}\sum_{i=0}^{n-r}\binom{n-r}{i}(-1)^i\,\beta g(x)\left[1-p+2p\overline{G}(x)\left(1+\overline{G}(x)\right)^{\beta-1}\right]\\ &\qquad\left[G(x)\right]^{\beta(r+i)-1}\left[1-p+p\left(1+\overline{G}(x)\right)^{\beta}\right]^{r+i-1}. \end{aligned}$$

where $c_{r:n}=\dfrac{n!}{(r-1)!(n-r)!}$.

Using the binomial expansion again, then the r^{th} order statistic for MTLE–G is as the following

$$\begin{aligned} f_{r:n}(x)=c_{r:n}\sum_{i=0}^{n-r}\sum_{v=0}^{r+i-1}\binom{n-r}{i}\binom{r+i-1}{v}(-1)^i\,\beta p^v(1-p)^{r+i-1-v} g(x)\left[G(x)\right]^{\beta(r+i)-1}\\ \left(1+\overline{G}(x)\right)^{\beta v}\left[1-p+2p\overline{G}(x)\left(1+\overline{G}(x)\right)^{\beta-1}\right]. \end{aligned}$$

The r^{th} order statistics for MTLEWD is

$$\begin{aligned} f_{r:n}(x)=c_{r:n}\sum_{i=0}^{n-r}\sum_{v=0}^{r+i-1}\binom{n-r}{i}\binom{r+i-1}{v}(-1)^i\,\beta\theta\lambda^\theta x^{\theta-1}p^v(1-p)^{r+i-1-v}e^{-(\lambda x)^\theta}\\ \left[1-e^{-(\lambda x)^\theta}\right]^{\beta(r+i)-1}\left(1+e^{-(\lambda x)^\theta}\right)^{\beta v}\left[1-p+2pe^{-(\lambda x)^\theta}\left(1+e^{-(\lambda x)^\theta}\right)^{\beta-1}\right]. \end{aligned}$$

The k^{th} moments of r^{th} order statistics for MTLE–G is

$$\mu_{r:n}^{(k)}=c_{r:n}\sum_{i=0}^{n-r}\sum_{v=0}^{r+i-1}\binom{n-r}{i}\binom{r+i-1}{v}(-1)^i\left\{y_1+y_2\right\},$$

where, $y_1=\beta p^v(1-p)^{r+i-v}\int_0^\infty x^k g(x)\left[G(x)\right]^{\beta(r+i)-1}\left(1+\overline{G}(x)\right)^{\beta v}dx$.

and $y_2 = 2\beta p^{v+1}(1-p)^{r+i-1-v}\int_0^\infty x^k g(x)\bar{G}(x)\left[G(x)\right]^{\beta(r+i)-1}\left(1+\bar{G}(x)\right)^{\beta(v+1)-1}dx$.

5. Application to Real Data

In this section, we fit the MTLEWD to real data sets and compare the fitness with Weibull – Weibull distribution (WWD) (Abouelmagd et al. (2017)), Topp – Leone Weibull distribution (TLWD), and exponentiated Weibull distribution (EWD) (Pal et al. (2006)).

The second data set is reported by Fuller et al (1994), which is related with strength data of window glass of the aircraft of 31 windows.

In order to compare distributions, we consider the Kolmogorov-Smirnov (KS) statistic, minus 2 of log likelihood (–2lnL), Akaike Information Criterion (AIC), Akaike Information Criterion Corrected (AICC), Bayesian Information Criterion (BIC), Hannan–Quinn information criterion (HQIC) and P–value. The best distribution corresponds to lower K-S, –2lnL, AIC, BIC, AICC and HQIC statistics value and high P–value.

The numerical values of the –2lnL, AIC, AICC, BIC, HQIC, SS and K–S statistics are listed in Tables 3 and 5, whereas Tables 4 and 6 list the MLEs of the model parameters.

Table 3. The statistics –2lnL, AIC, AICC, BIC, HQIC, K–S, and P–value for the first data set

Model	MTLEWD	EWD	TLWD	WWD
−2lnL	204.7422	208.0852	208.0852	211.7528
AIC	212.7422	214.0852	214.0852	219.7528
AICC	214.2807	214.9741	214.9741	214.2807
BIC	218.4781	218.3872	218.3872	225.4887
HQIC	214.612	215.4875	215.4875	221.6226
K–S	0.087062	0.12733	0.12714	0.14864
P–value	0.9567	0.65	0.6518	0.4564

Table 4. ML estimates of the model parameters for the first data set

Model	$\hat{\theta}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{p}$
MTLEWD	1.8004	91.6147	0.0661	0.5296
EWD	1.4661	0.0764	19.4026	——
TLWD	1.4732	19.0691	0.0473	——
WWD	0.3683	7.0835	0.3518	0.0011

Figure 3 shows the empirical and theoretical density and CDF of the fitted MAPEWD, and Figure 4 shows the P–P plot of the fitted MTLEWD for first data.

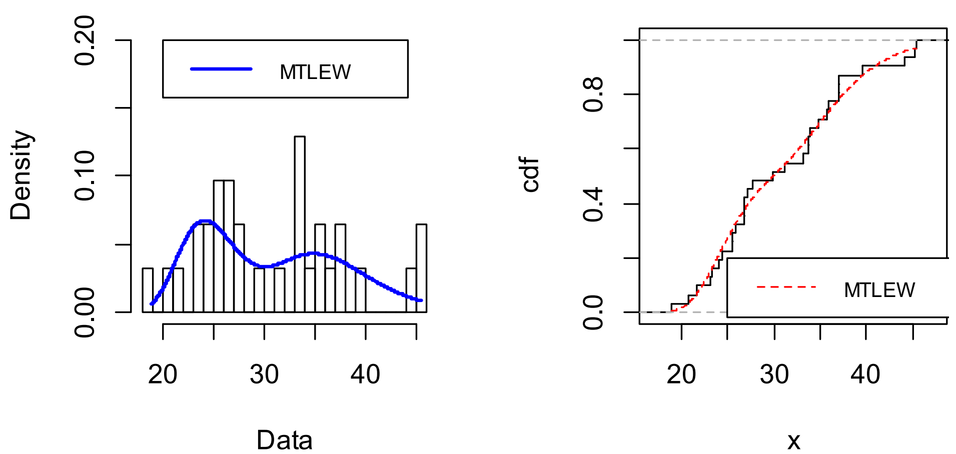

Figure 3. Empirical and theoretical density and CDF of the fitted MTLEWD for first data

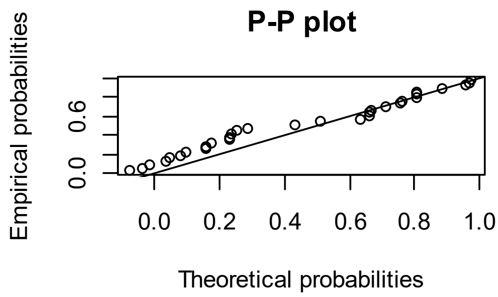

Figure 4. P–P plots of the fitted MTLEW distribution for first data

Table 5.The statistics–2lnL, AIC, AICC, BIC, HQIC, K–S and P–value for the second data set

Model	MTLEWD	EWD	TLWD	WWD
−2lnL	346.4154	359.692	354.3028	395.0764
AIC	354.4154	365.692	360.3028	403.0764
AICC	355.3043	366.2137	360.8245	355.3043
BIC	362.0635	371.4281	366.0389	410.7245
HQIC	357.3278	367.8763	362.4871	405.9888
K–S	0.084125	0.13188	0.11357	0.23178
P–value	0.8418	0.3207	0.5033	0.00762

Table 6. ML estimates of the model parameters for the second data set

Model	$\hat{\theta}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{p}$
MTLEWD	0.6831	3218.4163	0.2702	0.9632
EWD	0.5012	1.4925	735.0415	—
TLWD	0.3770	21149.66	2.4306	—
WWD	0.0275	66.7118	0.6319	0.4907

Figure 5 shows the empirical and theoretical density and CDF of the fitted MTLEWD, and Figure 6 shows the P–P plot of the fitted MTLEWD for second data.

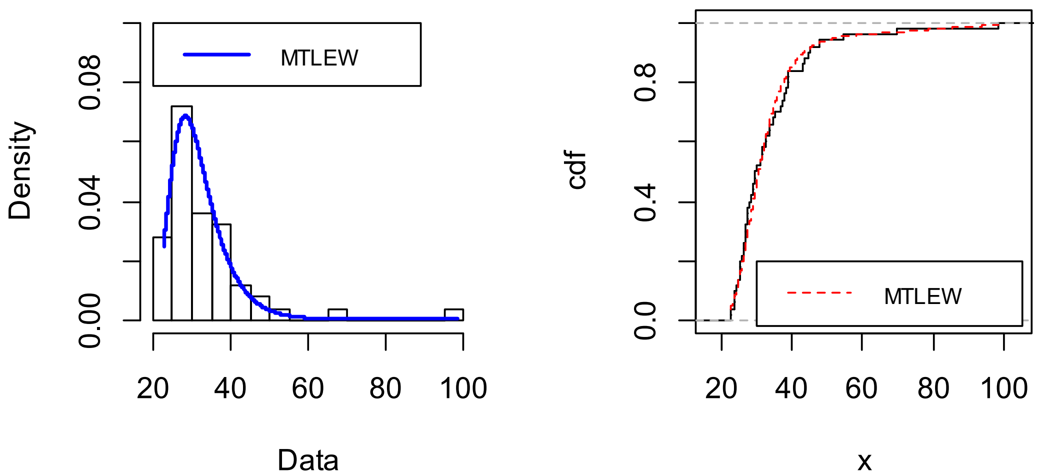

Figure 5. Empirical and theoretical density and CDF of the fitted MTLEWD for second data

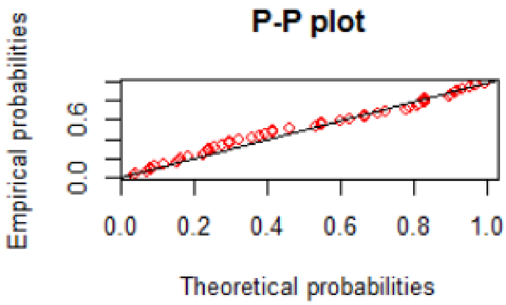

Figure 6. P–P plots of the fitted MTLEW distribution for second data

6. Concluding Remarks

In this article, a new class of mixture distributions named MTLE–G family is introduced. One model called the MTLEW distribution is studied. Some statistical properties of the new distribution are presented and discussed. The estimation of the model parameters is derived by maximum likelihood method. An application to real data sets indicates that the new model is superior to the fits than the other existing distributions.

References

Abouelmagd, T. H. M., Al-mualim, S., Elgarhy, M., Afify, A. Z., & Ahmad, M. (2017). Properties of the Four-Parameter Weibull Distribution and its Applications. *Pak. J. Statist., 33*(6), 449-466.

AL-Shomrani, A., Arif, O., Shawky, A., Hanif, S., & Shahbaz, M. Q. (2016). Topp-Leone Family of Distributions: Some properties and Applications. *Pakistan Journal of Statistics and Operation Reasearch, 12*(3), 443-451.

David, H. A., & Nagaraja, H. N. (2003). *Order Statistics.* John Wiley, New York.

Fuller, Jr, E. R., Frieman, S. W., Quinn, J. B., & Quinn, G. D., & Carter, W. C. (1994). Fracture mechanics approach to the design of glass aircraft windows: A case study. *SPIE Proc., 2286*, 419–430.

Gupta, R. C., Gupta, R. D., & Gupta, P. L. (1998). Modeling failure time data by Lehman alternatives. *Comm. Statist. Theory Methods, 27,* 887-904.

Gupta, R. D., & Kundu, D. (2001). Exponentiated exponential family: an alternative to Gamma and Wiebull distributions. *Biometrical Journal, 43*(1), 117-130.

Mudholkar, G. S., & Srivastava, D. K. (1993). Exponentiated Weibull Family for analyzing bathtub failure rate data. IEEE Trans. *Reliability, 42*(2), 299-302.

Nadarajah, S. (2006). The exponentiated Gumbel distribution with climate application. *Environmetrics, 17,* 13-23.

Nadarajah, S. (2005). Exponentiated Pareto distributions. *Statistics, 39*(3), 255 – 260.

Navarro, J., Franco, M., & Ruiz, J. M. (1998). Characterization through moments of the residual life and conditional spacing. *The Indian Journal of Statistics, Series A, 60*, 36-48.

Sangsanit, Y., & Bodhisuwan, W. (2016). The Topp-Leone Generator of Distributions: Properties and Inferences. *Songklanakarin Journal of Science and Technology, 38*(5), 537-548.

Shawky, A. I., & Abu-Zinadah, H. H. (2008). Characterization of exponentiated Pareto distribution based on record values. *Appl. Math. Sci.,* (2), 1283 - 1290.