International Journal of Statistics and Probability; Vol. 11, No. 6; November 2022
ISSN 1927-7032 E-ISSN 1927-7040
Published by Canadian Center of Science and Education

# Negative Binomial and Geometric; Bivariate and Difference Distributions

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Received: September 17, 2022 Accepted: October 18, 2022 Online Published: October 30, 2022

doi:10.5539/ijsp.v11n6p60 URL: https://doi.org/10.5539/ijsp.v11n6p60

**Abstract**

A similarity and a difference between bivariate negative binomial distribution and bivariate geometric distribution is presented. The distribution of negative binomial difference and geometric difference and the corresponding characteristic function are presented.

**Keywords:** Negative Binomial, Geometric, Bivariate, Difference

## 1. Introduction

As a bivariate extension of two exponential distributions, Freund (1961) created his model. A family of bivariate distributions produced by the bivariate Bernoulli distributions were explored by Marshall and Olkin (1985). Bivariate exponential and geometric distributions were explored by Nair and Nair (1988). In Basu and Dhar (1995) presented the BGD (B&D) bivariate geometric model, which is comparable to the Marshall and Olkin (1985) bivariate distribution. A new discrete analog of Freund's model, called BGD (F), was developed by Dhar (1998).

In their 2008 study, Ong et al. studied at the distribution of two discrete random variables from the Panjer family. A skewed distribution known as the generalized discrete Laplace distribution was introduced by Lekshmi and Sebastian (2014). In their 2014 study, Nastic et al. presented the negative binomial difference distribution with an equal chance of success using the INAR model with discrete Laplace marginal distribution. The difference between two independent negative binomial random variables with various parameters was taken into consideration by Song and Smith (2011).

The distribution of $Z = X_1 - X_2$ when $X_1$ and $X_2$ are drawn from one of the following bivariate negative binomial distributions or one of the following bivariate geometric distributions is what we are examining in this paper.

## 2. Bivariate Negative Binomial Distributions

### *2.1 Double Negative Binomial*

The probability density function for the bivariate negative binomial distribution of $X_1$ and $X_2$ is given by

$$f(x_1, x_2) = \binom{x_1 + r_1 - 1}{x_1}\binom{x_2 + r_2 - 1}{x_2}(1 - p_1)^{r_1}(1 - p_2)^{r_2}{p_1}^{x_1}{p_2}^{x_2};\ \ x_1, x_2 = 0,1, ...$$

where $r_1, r_2 > 0, 0 \le p_1, p_2 \le 1$.

where the probability distribution of $X_i$ is given by

$$f(x_i) = \binom{x_i + r_i - 1}{x_i} p_i^{x_i}(1 - p_i)^{r_i}; x_i = 0,1,2, ...$$

and $X_1$ ,$X_2$ be two independent random variables with negative binomial distributions.

The characteristic function provided by $\varphi_{X_1,X_2}(t, s) = \left(\frac{1-p_1}{1-p_1e^{it}}\right)^{r_1}\left(\frac{1-p_2}{1-p_2e^{is}}\right)^{r_2}$

### *2.2 Chou Bivariate Negative Binomial*

A bivariate negative binomial distribution is proposed by Chou et al. (2011) as a combination of bivariate Poisson and Gamma distributions. Given by is the joint probability density function.

$$f(x_1, x_2) = \frac{\Gamma(x_1 + x_2 + r)}{x_1!\, x_2!\, \Gamma(r)} \frac{r^r {p_1}^{x_1} {p_2}^{x_2}}{(r + p_1 + p_2)^{r + x_1 + x_2}};\ x_1, x_2 = 0,1,2, \dots$$

where $r, p_1, p_2 \geq 0$. The marginal mass function is given by

$$f(x_i) = \binom{r + x_i - 1}{x_i} \left(\frac{p_i}{r + p_i}\right)^{x_i} \left(\frac{r}{r + p_i}\right)^{r}$$

with correlation coefficient

$$corr(x_1, x_2) = \sqrt{\frac{p_1 p_2}{p_1 p_2 + r(1 + p_1 + p_2)}}$$

and the characteristic function given by $\varphi_{X_1,X_2}(t,s) = \left(\frac{r}{r + p_1 + p_2 - p_1 e^{it} - p_2 e^{is}}\right)^r$

*2.3 Dependent Bivariate Negative Binomial*

Dependent two-variate negative binomial distribution with $\rho = \frac{1 - p_1 - p_2}{\sqrt{p_1 p_2}}$ correlation. Given is the probability density function.

$$f(x_1, x_2) = \binom{x_1 + x_2 + r - 1}{x_1, x_2} (1 - p_1 - p_2)^r {p_1}^{x_1} {p_2}^{x_2};\ \ x_1, x_2 = 0,1, \dots$$

where $r > 0, 0 \leq p_1, p_2 \leq 1,\ p_1 + p_2 < 1$.

with the characteristic function presented by$\varphi_{X_1,X_2}(t,s) = \left(\frac{1 - p_1 - p_2}{1 - p_1 e^{it} - p_2 e^{is}}\right)^r$

*2.4 Arbous and Sichel Bivariate Negative Binomial*

A symmetric bivariate negative binomial distribution with a probability mass function was first introduced by Arbous and Sichel (1954)

$$f_{X_1,X_2}(x_1, x_2) = \frac{(x_1 + x_2 + r - 1)!}{x_1!\, x_2!\, (r - 1)!} \left(\frac{r}{r + 2\theta}\right)^r \left(\frac{\theta}{r + 2\theta}\right)^{x_1 + x_2};\ x_1, x_2 = 0,1, \dots$$

where $r, \theta > 0$.

The characteristic function defined by $\varphi_{X_1,X_2}(t,s) = \left(\frac{r}{r + 2\theta - \theta e^{it} - \theta e^{is}}\right)^r$

and the marginal probability mass function of $X_i$

$$f_{X_i}(x_i) = \binom{x_i + r - 1}{x_i} \left(\frac{r}{r + \theta}\right)^r \left(\frac{\theta}{r + \theta}\right)^{x_i};\ x_i = 0,1, \dots$$

*2.5 Lundberg's Bivariate Negative Binomial*

The bivariate negative binomial distributions created by Arbous and Sichel (1954) are a special case of those created by Lundberg (1940), where $\rho = \frac{r}{r + \theta}$, represents for the bivariate negative binomial distributions with the probability mass function

$$f_{X_1,X_2}(x_1,x_2)=\frac{(x_1+x_2+r-1)!}{x_1!\,x_2!\,(r-1)!}\left(\frac{1-\rho}{1+\rho}\right)^r\left(\frac{\rho}{1+\rho}\right)^{x_1+x_2};x_1,x_2=0,1,\ldots$$

where $r>0, 0<\rho<1$.

and the characteristic function $\varphi_{X_1,X_2}(t,s)=\left(\frac{1-\rho}{1+\rho-\rho e^{it}-\rho e^{is}}\right)^r$.

*2.6 Rao Bivariate Negative Binomial*

Rao et al. (1973) gave a bivariate negative binomial distribution with probability mass function $f_{X_1,X_2}(x_1,x_2)=\binom{x_1+x_2+r-1}{x_1+x_2}\binom{x_1+x_2}{x_1}w^r\left(\frac{1-w}{2}\right)^{x_1+x_2};x_1,x_2=0,1,\ldots$

where $r>0, 0<w<1$.

With the characteristic function given by $\varphi_{X_1,X_2}(t,s)=\left(\frac{2w}{2-(1-w)e^{it}-(1-w)e^{is}}\right)^r$

*2.7 Bivariate Negative Binomial by Redaction Method*

Suppose that $Y_1=X_0+X_1$ and $Y_2=X_0+X_2$ have a negative binomial distribution, where $X_i\sim NB(r_i,p)$, and $X_i, i=0,1,2$ are independent. The joint probability mass function is given by

$$f_{Y_1,Y_2}(y_1,y_2)=\frac{(r_0+r_1+y_1-1)!\,(r_0+r_2+y_2-1)!}{y_1!\,y_2!\,(r_0+r_1-1)!\,(r_0+r_2-1)!}(1-p)^{2r_0+r_1+r_2}p^{y_1+y_2}$$

for $y_1,y_2=0,1,\ldots$, where $r_0,r_1,r_2>0$ and $0<p<1$.

with the characteristic function given by $\varphi_{Y_1,Y_2}(t,s)=\left(\frac{1-p}{1-pe^{it}}\right)^{r_0+r_1}\left(\frac{1-p}{1-pe^{is}}\right)^{r_0+r_2}$

**Conclusion 1.** When we compare the characteristic function, we find that there are only differences between double negative binomial distributions and dependent bivariate negative binomial distributions. We can find the other bivariate distributions by reparametrized double negative binomial distributions or dependent bivariate negative binomial distributions.

**Proof**. The characteristic function for each bivariate is given by:

Chou bivariate negative binomial distribution: $\varphi_{X_1,X_2}(t,s)=\left(\frac{r}{r+p_{12}+p_{22}-p_{12}e^{it}-p_{22}e^{is}}\right)^r$

Dependent bivariate negative binomial distribution: $\varphi_{X_1,X_2}(t,s)=\left(\frac{1-p_{13}-p_{23}}{1-p_{13}e^{it}-p_{23}e^{is}}\right)^r$

Arbous and Sichel bivariate negative binomial distribution: $\varphi_{X_1,X_2}(t,s)=\left(\frac{r}{r+2\theta-\theta e^{it}-\theta e^{is}}\right)^r$

Lundberg bivariate negative binomial distribution: $\varphi_{X_1,X_2}(t,s)=\left(\frac{1-\rho}{1+\rho-\rho e^{it}-\rho e^{is}}\right)^r$

Rao bivariate negative binomial distribution: $\varphi_{X_1,X_2}(t,s)=\left(\frac{2w}{2-(1-w)e^{it}-(1-w)e^{is}}\right)^r$

By comparing characteristic functions, we find that:

If $p_{12}=\frac{rp_{13}}{1-p_{13}-p_{23}}$ and $p_{22}=\frac{rp_{23}}{1-p_{13}-p_{23}}$, then Chou $\Leftrightarrow$ dependent

If $p_{12}=p_{22}=\theta$, then Chou $\Leftrightarrow$ Arbous and Sichel

If $p_{12}=p_{22}=\frac{r\rho}{(1-\rho)}$, then Chou $\Leftrightarrow$ Lundberg's

If $p_{12}=p_{22}=\frac{r(1-w)}{2w}$, then Chou $\Leftrightarrow$ *Rao*

Thus, the joint distributions according to dependent, Chou, Arbous and Sichel, Lundberg and Rao bivariate negative binomial distributions are corresponding distributions.

The characteristic function for the independent bivariate negative binomial distribution and the bivariate one using the redaction method given by

$$\varphi_{X_1,X_2}(t,s)=\left(\frac{1-p_1}{1-p_1e^{it}}\right)^{r_{11}}\left(\frac{1-p_2}{1-p_2e^{is}}\right)^{r_{21}}$$

$$\varphi_{X_1,X_2}(t,s)=\left(\frac{1-p}{1-pe^{it}}\right)^{r_{07}+r_{17}}\left(\frac{1-p}{1-pe^{is}}\right)^{r_{07}+r_{27}}$$

we find that, if $p_1=p_2=p$, $r_{11}=r_{07}+r_{17}$ and $r_{21}=r_{07}+r_{27}$, then the independent bivariate negative binomial distribution and the bivariate with the redaction method are corresponding distributions.

Then we only need to define two different distributions for the negative binomial difference distribution.

## 3. Negative Binomial Difference Distributions

### *3.1 Independent Negative Binomial Difference*

If $X_1$ and $X_2$ are jointly distributed by double negative binomial distribution, then the random variable $Z=X_1-X_2$ has the negative binomial difference distribution. The probability distribution is given by Ong, et. Al (2008):

$$f_z(z)=\binom{r_1+z-1}{z}(1-p_1)^{r_1}(1-p_2)^{r_2}\,p_1^z\;\;{}_2F_1(r_2,r_1+z;1+z;p_1p_2)\,;\;z=0,1,2,\ldots$$

and $f(z;r_1,p_1,r_2,p_2)=f(-z;r_2,p_2,r_1,p_1)$

or

$$f_{X_1-X_2}(z)=\begin{cases}\binom{r_1+z-1}{z}(1-p_1)^{r_1}(1-p_2)^{r_2}\,p_1^z\;\;{}_2F_1(r_2,r_1+z;z+1;p_1p_2) & ;\;z=0,1,2,\ldots\\ \binom{r_2-z-1}{-z}(1-p_1)^{r_1}(1-p_2)^{r_2}p_2^{-z}\;\;{}_2F_1(r_1,r_2-z;1-z;p_1p_2) & ;\;z=-1,-2,\ldots\end{cases}$$

The characteristic function is given by $\varphi_Z(t)=\left(\frac{1-p_1}{1-p_1e^{it}}\right)^{r_1}\left(\frac{1-p_2}{1-p_2e^{-it}}\right)^{r_2}$.

The expected value is $E(Z)=\frac{r_1p_1}{1-p_1}-\frac{r_2p_2}{1-p_2}$, while the variance is $V(Z)=\frac{r_1p_1}{(1-p_1)^2}+\frac{r_2p_2}{(1-p_2)^2}$.

If $r_1=r_2=r$

$$f(z)=\binom{r+|z|-1}{|z|}[(1-p_1)(1-p_2)]^r\;\;{}_2F_1(r,r+|z|;1+|z|;p_1p_2)\begin{cases}p_1^z;z=0,1,2,\ldots\\ p_2^{|z|};z=-1,-2,\ldots\end{cases}$$

$$r_1,r_2\geq 0,0\leq p_1,p_2\leq 1$$

$$\varphi_Z(t)=\left(\frac{(1-p_1)(1-p_2)}{(1-p_1e^{it})(1-p_2e^{-it})}\right)^r$$

### *3.2 Dependent Negative Binomial Difference*

Let $X_1$ and $X_2$ be jointly distributed dependent bivariate negative binomial distribution, then the probability distribution for the difference $z=x_1-x_2$ random variable be given by

$$f_Z(z)=\binom{r+|z|-1}{|z|}(1-p_1-p_2)^r*\;{}_2F_1\left(\frac{r+|z|}{2},\frac{r+|z|+1}{2};1+|z|;4p_1p_2\right)\begin{cases}p_1^z;z=0,1,2,\ldots\\ p_2^{|z|};z=-1,-2,\ldots\end{cases}$$

The characteristic function is given by $\varphi_Z(t)=\left(\frac{1-p_1-p_2}{1-p_1e^{it}-p_2e^{-it}}\right)^r$. The expected value is $E(Z)=\frac{r(p_1-p_2)}{1-p_1-p_2}$, and the variance is $V(Z)=\frac{r(p_1+p_2-2p_1p_2)}{(1-p_1-p_2)^2}$.

**Conclusion 2.** The negative binomial difference between $X_1$ and $X_2$ is the same for any bivariate negative binomial distribution.

**Proof.** The characteristic function from both negative binomial differences is compared, and we discover that, for every $0 \le p_1, p_2 \le 1,\ p_1 + p_2 < 1$, or $0 \le q_i \le 1$, there are $p_i = \frac{q_i}{1+q_1 q_2}$, then, $\varphi_Z(t) = \left(\frac{1-p_1-p_2}{1-p_1 e^{it} - p_2 e^{-it}}\right)^r \Leftrightarrow \varphi_Z(t) = \left(\frac{(1-q_1)(1-q_2)}{(1-q_1 e^{it})(1-q_2 e^{-it})}\right)^r$, where $q_i = \frac{1-\sqrt{1-4p_1 p_2}}{2p_j}, i \ne j$.

## 4. Bivariate Geometric Distributions

*4.1 Independent Bivariate Geometric*

Let X and Y be independent, bivariate geometric distributions, and

$$f(x,y) = (1-p_1)(1-p_2){p_1}^x {p_2}^y; x, y = 0,1,2,\dots$$

be their probability density function.

The $\varphi_{X,Y}(t,s) = \frac{(1-p_1)(1-p_2)}{(1-p_1 e^{it})(1-p_2 e^{is})}$ provided characteristic function.

*4.2 Dependent Bivariate Geometric*

Let X and Y be dependent bivariate geometric distributions, where

$$f(x_1, x_2) = \binom{x_1 + x_2}{x_1}(1-p_1-p_2){p_1}^x {p_2}^y;\ x, y = 0,1,\dots$$

where $0 \le p_1, p_2 \le 1,\ p_1 + p_2 < 1$.

denotes the probability density function and $\varphi_{X,Y}(t,s) = \frac{1-p_1-p_2}{1-p_1 e^{it} - p_2 e^{is}}$ denotes the characteristic function.

*4.3 Omey and Minkova Bivariate Geometric*

A bivariate geometric distribution with a probability density function supplied by

$$f(x,y) = \begin{cases} p_1 p_2 q^{x-1}(1-p_2)^{y-x-1}; y > x \ge 1 \\ 0 \qquad ; x = y \\ p_1 p_2 q^{y-1}(1-p_1)^{x-y-1}; x > y \ge 1 \end{cases}$$

where $p_1, p_2, q \ge 0,\ p_1 + p_2 + q = 1$

and a characteristic function defined by $\varphi_{X,Y}(t,s) = \frac{p_1 p_2 e^{it+is}}{1-qe^{it+is}}\left(\frac{e^{it}}{1-(1-p_1)e^{it}} + \frac{e^{is}}{1-(1-p_2)e^{is}}\right)$ was proposed by Omey and Minkova (2013).

*4.4 Bao Bivariate Geometric*

Bao (2011) suggested a bivariate geometric distribution with the characteristic function denoted by $\varphi_{X,Y}(t,s) = \frac{e^{it+is}}{(1-qe^{it+is})}\left(p_{12} + \frac{p_2(p_2+p_{12})e^{it}}{1-(1-p_2-p_{12})e^{it}} + \frac{p_1(p_1+p_{12})e^{is}}{1-(1-p_1-p_{12})e^{is}}\right)$ and a probability density function denoted by

$$f(x,y) = \begin{cases} p_1(p_1+p_{12})q^{x-1}(1-p_1-p_{12})^{y-x-1}; y > x, x, y = 1,2,\dots \\ p_{12}q^{x-1} \qquad ; x = y, x, y = 1,2,\dots \\ p_2(p_2+p_{12})q^{y-1}(1-p_2-p_{12})^{x-y-1}; x > yx, y = 1,2,\dots \end{cases}$$

where $q = 1 - p_1 - p_2 - p_{12}, 0 \le p_1, p_2, p_{12} \le 1$.

*4.5 Basu and Dhar Bivariate Geometric*

A bivariate geometric model (BGD (B&D)) similar to Marshall and Olkin's (1967) bivariate distribution with the pmf

$$f(x,y) = \begin{cases} q_1(1-p_2 p_{12}){p_1}^{x-1}(p_2 p_{12})^{y-1}; y > x \\ (p_1 p_2 p_{12})^{x-1}(1 - p_1 p_{12} - p_2 p_{12} + p_1 p_2 p_{12}); x = y \\ q_2(1-p_1 p_{12}){p_2}^{y-1}(p_1 p_{12})^{x-1}; x > y \end{cases}$$

where $1 \le x, y \in Z^+, q_i = 1 - p_i; i = 1,2, 0 \le p_1, p_2 \le 1$

and characteristic function given by

$\varphi_{X,Y}(t,s) = \frac{e^{it+is}}{(1-p_1p_2p_{12}e^{it+is})}\left((1-p_1p_{12}-p_2p_{12}+p_1p_2p_{12}) + \frac{p_1p_{12}q_2(1-p_1p_{12})e^{it}}{1-p_1p_{12}e^{it}} + \frac{p_2p_{12}q_1(1-p_2p_{12})e^{is}}{1-p_2p_{12}e^{is}}\right)$ was proposed by Basu and Dhar (1995).

**Conclusion 3.** There are only two different bivariate geometric distributions, and by reparametrizing these two, we can identify the remaining bivariate distributions.

**Proof.** According to separately, Omey and Minkova, Bao and Basu, and Dhar, the joint distributions are the equivalent distributions.

By displaying the distinctive properties of each distribution, which include

$$\varphi_{X,Y}(t,s) = \frac{(1-p_1)(1-p_2)}{(1-p_1e^{it})(1-p_2e^{is})} = \frac{(1-p_1)(1-p_2)}{1-p_1p_2e^{it+is}}\left(1 + \frac{e^{it}p_1}{1-p_1e^{it}} + \frac{e^{is}p_2}{1-p_2e^{is}}\right)$$

at $X, Y \neq 0$, then

$$\varphi_{X,Y}(t,s) = \frac{(1-p_1)(1-p_2)e^{it+is}}{1-p_1p_2e^{it+is}}\left(1 + \frac{e^{it}p_1}{1-p_1e^{it}} + \frac{e^{is}p_2}{1-p_2e^{is}}\right)$$

$$\varphi_{X,Y}(t,s) = \frac{p_{13}p_{23}e^{it+is}}{1-q_3e^{it+is}}\left(\frac{e^{it}}{1-q_{13}e^{it}} + \frac{e^{is}}{1-q_{23}e^{is}}\right)$$
$$= \frac{(1-(q_{13}+q_{23})e^{it+is})p_{13}p_{23}e^{it+is}}{(1-q_3e^{it+is})(1-q_{13}e^{it})(1-q_{23}e^{is})} - \frac{p_{13}p_{23}e^{it+is}(1-e^{it}-e^{is})}{(1-q_3e^{it+is})(1-q_{13}e^{it})(1-q_{23}e^{is})}$$

Bao: $\varphi_{X,Y}(t,s) = \frac{e^{it+is}}{(1-q_4e^{it+is})}\left(p_{124} + \frac{p_{24}(p_{24}+p_{124})e^{it}}{1-(1-p_{24}-p_{124})e^{it}} + \frac{p_{14}(p_{14}+p_{124})e^{is}}{1-(1-p_{14}-p_{124})e^{is}}\right)$

If $p_{124} = (1-p_1)(1-p_2)$, $p_{14} = p_2(1-p_1)$, $p_{24} = p_1(1-p_2)$, and $q_4 = p_1p_2$, then $independent$ at $X, Y \neq 0 \Leftrightarrow$ Bao.

Basu and Dhar Bivariate Geometric: $\varphi_{X,Y}(t,s) = \frac{e^{it+is}}{(1-p_{15}p_{25}p_{125}e^{it+is})}\left((1-p_{15}p_{125}-p_{25}p_{125}+p_{15}p_{25}p_{125}) + \frac{p_{15}p_{125}q_{25}(1-p_{15}p_{125})e^{it}}{1-p_{15}p_{125}e^{it}} + \frac{p_{25}p_{125}q_{15}(1-p_{25}p_{125})e^{is}}{1-p_{25}p_{125}e^{is}}\right)$

If $p_{125} = 1$, then $independent$ at $X, Y \neq 0 \Leftrightarrow$ Basu and Dhar.

## 5. Geometric Difference Distributions

### *5.1 Independent Geometric Difference*

The probability distribution for the difference $Z = X - Y$ is a random variable if X and Y are simultaneously distributed according to an independent bivariate geometric distribution.

$$f(z) = \frac{(1-p_1)(1-p_2)}{1-p_1p_2}\begin{cases} {p_2}^{-z}; z \le 0 \\ {p_1}^{z}; z > 0 \end{cases}$$

The characteristic function is $\varphi_Z(t) = \left(\frac{(1-p_1)(1-p_2)}{(1-p_1e^{it})(1-p_2e^{-it})}\right)$. The expected value is $E(Z) = \frac{p_1}{1-p_1} - \frac{p_2}{1-p_2}$, while the variance is $V(Z) = \frac{p_1}{(1-p_1)^2} + \frac{p_2}{(1-p_2)^2}$.

which corresponds to the Laplace distribution.

*5.2 Dependent Geometric Difference*

If $X$ and $Y$ are jointly distributed according to a dependent bivariate geometric distribution.

The probability distribution for the difference $Z = X - Y$ random variable is given by

$$f_{X_1-X_2}(z) = (1-p_1-p_2) * {}_2F_1\left(\frac{1+|z|}{2}, \frac{2+|z|}{2}; 1+|z|; 4p_1p_2\right)\begin{cases} p_1^z; z = 0,1,2,\dots \\ p_2^{|z|}; z = -1,-2,\dots \end{cases}$$

The characteristic function is given by $\varphi_Z(t) = \left(\frac{1-p_1-p_2}{1-p_1e^{it}-p_2e^{-it}}\right)$. The expected value is $E(Z) = \frac{p_1-p_2}{1-p_1-p_2}$, while the variance is $V(Z) = \frac{p_1+p_2-2p_1p_2}{(1-p_1-p_2)^2}$.

**Conclusion 4.** The Laplace distribution is the same geometric difference between $X_1$ and $X_2$ if they come from any bivariate geometric distribution.

**Proof.** For any $0 \le p_1, p_2 \le 1$, $p_1 + p_2 < 1$, or $0 \le q_i \le 1$, there are $p_i = \frac{q_i}{1+q_1q_2}$,

then, $\varphi_Z(t) = \left(\frac{1-p_1-p_2}{1-p_1e^{it}-p_2e^{-it}}\right) \Leftrightarrow \varphi_Z(t) = \left(\frac{(1-q_1)(1-q_2)}{(1-q_1e^{it})(1-q_2e^{-it})}\right)$,

where $q_i = \frac{1-\sqrt{1-4p_1p_2}}{2p_j}, i \neq j$.

## 6. New formula of ${}_2F_1$ Hypergeometric Function

For the hypergeometric function ${}_2F_1(.,.;.;.)$ that is readily obtained from the negative binomial distribution, the following theorem provides additional relations.

For any $a > 0$, $b > 0$ and $0 \le p_1, p_2, p \le 1$.

1. $\sum_{n=0}^{\infty}\left(\frac{(a)_{(n)}(b)_{(n)}}{n!^2}(p_1p_2)^n\right)\left[{}_2F_1(n+a,1;n+1;p_1) + {}_2F_1(n+b,1;n+1;p_2)\right] = \left[\frac{1}{(1-p_1)^a(1-p_2)^b} + {}_2F_1(a,b;1;p_1p_2)\right]$

   i. $\sum_{n=0}^{\infty}\left(\frac{(a)_{(n)}}{n!}p^n\right)^2 {}_2F_1(n+a,1;n+1;p) = \frac{1}{2}\left[(1-p)^{-2a} + {}_2F_1(a,a;1;p^2)\right]$

   ii. $\sum_{n=0}^{\infty}\left(\frac{(a)_{(n)}}{n!}\right)^2 (p_1p_2)^n\left[{}_2F_1(n+a,1;n+1;p_1) + {}_2F_1(n+a,1;n+1;p_2)\right] = \left[\frac{1}{[(1-p_1)(1-p_2)]^a} + {}_2F_1(a,a;1;p_1p_2)\right]$

   iii. $\sum_{n=-\infty}^{\infty}\int_0^{2\pi}\frac{cos(|n|t)}{(1-2pcos(t)+p^2)^a}dt = 2\pi(1-p)^{-2a}$

2. $\sum_{n=0}^{\infty}\frac{(a)_{(n)}(b)_{(n)}}{n!(n-1)!}(p_1p_2)^n\left[p_1(n+a)\,{}_2F_1(n+a,2;n+2;p_1) - p_2(n+b)\,{}_2F_1(n+b,2;n+2;p_2)\right] = \frac{1}{(1-p_1)^a(1-p_2)^b}\left(\frac{ap_1}{1-p_1} - \frac{bp_2}{1-p_2}\right)$

*or*

$$\sum_{n=0}^{\infty}\frac{\Gamma(n+a+1)\Gamma(n+b)}{n!\,(n-1)!}(p_1)^{n+1}(p_2)^n\ {}_2F_1(n+a,2;n+2;p_1)-\frac{\Gamma(a+1)\Gamma(b)p_1}{(1-p_1)^{a+1}(1-p_2)^b}$$

$$=\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+1+n)}{n!\,\Gamma(n)}(p_1)^n(p_2)^{n+1}\ {}_2F_1(n+b,2;n+2;p_2)$$

$$-\frac{\Gamma(a)\Gamma(b+1)p_2}{(1-p_1)^a(1-p_2)^{b+1}}$$

i. $\sum_{n=0}^{\infty}\frac{(a)_{(n)}(b)_{(n)}}{n!(n-1)!}p^{2n}\left[(a+n)\ {}_2F_1(a+n,2;n+2;p)-(b+n)\ {}_2F_1(b+n,2;n+2;p)\right]=\frac{(a-b)}{(1-p)^{a+b+1}}$

ii. $\sum_{n=0}^{\infty}\frac{(a+n)(a)_{(n)}{}^2}{n!\Gamma(n)}(p_1p_2)^n\left[p_1\ {}_2F_1(n+a,2;n+2;p_1)-p_2\ {}_2F_1(n+a,2;n+2;p_2)\right]=$

$\frac{a}{[(1-p_1)(1-p_2)]^a}\left(\frac{p_1}{1-p_1}-\frac{p_2}{1-p_2}\right)$

3. $(1-p_1)^a(1-p_2)^b\sum_{n=0}^{\infty}\frac{n^2}{n!}\left[p_1{}^n(a)_{(n)}\ {}_2F_1(b,a+n;n+1;p_1p_2)+p_2{}^n(b)_{(n)}\ {}_2F_1(a,b+n;n+1;p_1p_2)\right]=$

$\frac{ap_1}{(1-p_1)^2}(1+ap_1)+\frac{bp_2}{(1-p_2)^2}(1+bp_2)-\frac{2abp_1p_2}{(1-p_1)(1-p_2)}$

*or*

$$(1-p_1)^a(1-p_2)^b\sum_{n=0}^{\infty}\frac{n^2(a)_{(n)}}{n!}p_1{}^n\ {}_2F_1(b,a+n;n+1;p_1p_2)-\frac{ap_1}{(1-p_1)^2}(1+ap_1)+\frac{abp_1p_2}{(1-p_1)(1-p_2)}$$

$$=-(1-p_1)^a(1-p_2)^b\sum_{n=0}^{\infty}\frac{n^2(b)_{(n)}p_2{}^n}{n!}\ {}_2F_1(a,b+n;n+1;p_1p_2)+\frac{bp_2}{(1-p_2)^2}(1+bp_2)$$

$$-\frac{abp_1p_2}{(1-p_1)(1-p_2)}$$

i. $(1-p)^{a+b}\sum_{n=0}^{\infty}\frac{n^2p^n}{n!}\left[(a)_{(n)}\ {}_2F_1(b,a+n;1+n;p^2)+(b)_{(n)}\ {}_2F_1(a,b+n;n+1;p^2)\right]=$

$\frac{p}{(1-p)^2}[a+b+p(a-b)^2]$

ii. $\sum_{n=0}^{\infty}\frac{n^2p^n(a)_{(n)}}{n!}\ {}_2F_1(n+a,a;n+1;p^2)=\frac{ap}{(1-p)^{2(1+a)}}$

iii. $\sum_{n=0}^{\infty}\frac{n^2}{2\pi}\int_0^{2\pi}\frac{cos(nt)}{(1-2pcos(t)+p^2)^a}dt=\frac{ap}{(1-p)^{2(1+a)}}$

iv. $[(1-p_1)(1-p_2)]^a\sum_{n=0}^{\infty}\frac{n^2(a)_{(n)}}{n!a}\ {}_2F_1(n+a,a;n+1;p_1p_2)[p_1{}^n+p_2{}^n]=\frac{p_1}{(1-p_1)^2}(1+ap_1)+$

$\frac{p_2}{(1-p_2)^2}(1+ap_2)-\frac{2ap_1p_2}{(1-p_1)(1-p_2)}$

4. $\sum_{n=0}^{\infty}\frac{(a)_{(n)}}{n!}(p_1e^{it})^n\ {}_2F_1(n+a,b;n+1;p_1p_2)+\sum_{n=0}^{\infty}\frac{(b)_{(n)}}{n!}(p_2e^{-it})^n\ {}_2F_1(a,n+b;n+1;p_1p_2)-$

${}_2F_1(a,b;1;p_1p_2)=\frac{1}{\left(1-p_1e^{it}\right)^a\left(1-p_2e^{-it}\right)^b}$

i. $\sum_{n=0}^{\infty}\frac{(a)_{(n)}}{n!}\ {}_2F_1(a,a+n;1+n;p_1p_2)\,[(p_1e^{it})^n+(p_2e^{-it})^n]-\ {}_2F_1(a,a;1;p_1p_2)=$

$\frac{1}{[(1-p_1e^{it})(1-p_2e^{-it})]^a}$

ii. $\sum_{n=-\infty}^{\infty}e^{itn}\int_0^{2\pi}\frac{cos(|n|t)}{(1-2pcos(t)+p^2)^a}dt=\frac{2\pi}{[(1-pe^{it})(1-pe^{-it})]^a}$

iii. $\sum_{n=0}^{\infty}\frac{(a)_{(n)}}{n!}\ {}_2F_1(a,a+n;n+1;p^2)\,[(pe^{it})^n+(pe^{-it})^n]-\ {}_2F_1(a,a;1;p^2)=\frac{1}{[(1-pe^{it})(1-pe^{-it})]^a}$

iv. $\sum_{n=-\infty}^{\infty}\frac{(a)_{(|n|)}}{|n|!}p^{|n|}e^{itn}\ {}_2F_1(|n|+a,a;|n|+1;p^2)=\frac{1}{[(1-pe^{it})(1-pe^{-it})]^a}$

v. $\sum_{n=0}^{\infty}\frac{(a)_{(n)}}{n!}(pe^{it})^n\ {}_2F_1(n+a,b;n+1;p^2)+\sum_{n=0}^{\infty}\frac{(b)_{(n)}}{n!}(pe^{-it})^n\ {}_2F_1(a,n+b;n+1;p^2)-$

${}_2F_1(a,b;1;p^2)=\frac{1}{(1-pe^{it})^a(1-pe^{-it})^b}$

5. ${}_2F_1(a,b;b-c+1;z)=\frac{\Gamma(b+n)\Gamma(c)}{\Gamma(c+n)\Gamma(b)}\left(1-\sqrt{z}\right)^{2n}\ {}_2F_1(a+n,b+n;b-c+1;z)$

6. $\sum_{x=-\infty}^{\infty}\frac{(a)_{(|x|)}(b)_{(|z-x|)}}{|z-x|!|x|!}p^{|x|+|z-x|}\ {}_2F_1(|x|+a,a;|x|+1;p^2)\ {}_2F_1(|z-x|+b,b;|z-x|+1;p^2)=$

$\frac{(a+b)_{(|z|)}}{|z|!}p^{|z|}\ {}_2F_1(|z|+a+b,a+b;|z|+1;p^2)$

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