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E–Bayesian Estimation for the Inverted Topp – Leone Distribution Based on Type-II Censored Data

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Abstract

The objective of this paper is to estimate the unknown parameter for inverted Topp – Leone (ITL) distribution using the E-Bayesian estimation method under type-II censored sampling. The squared error loss function and the LINEX loss function are used to estimate the unknown parameter. The effectiveness of the E-Bayesian estimators in comparison to the Bayesian and maximum likelihood estimators based on mean square error is investigated using a Monte Carlo simulation.

Keywords: Inverted Topp – Leone distribution, E–Bayesian estimation, Type-II censored sampling, Squared error loss function, LINEX loss function, Monte Carlo simulation

1. Introduction

Hassan et al. have proposed the Inverted Topp-Leone (ITL) distribution (2020). Since this distribution is more adaptable to modelling positive real data, several literatures have taken an interest in it. For instance, problems relating to lifetime phenomena, medical and engineering research, econometrics, etc. Muhammed and Muhammed (2021) studied the Bayesian and non – Bayesian estimation of the inverted Topp – Leone distribution. Al – Dayian et al (2022) added a new parameter to the inverted Topp – Leone distribution by exponentiated it and studied the properties of the new resulted distribution. Hassan et al (2021) introduced a new two parameter mode related to the inverted Topp – Leone distribution and developed the acceptance sampling plans for it. Bayesian estimation of the shape parameter of the inverted Topp – Leone distribution under different loss functions is studied by Aijaz et al (2021).

The probability density function (pdf) of the ITL distribution is given by:

$$f(x;\beta)=2\beta x\left(1+x\right)^{-2\beta-1}\left(1+2x\right)^{\beta-1}; \qquad x,\beta>0, \tag{1}$$

And the cumulative distribution function (cdf) corresponding to (1) is given by:

$$F(x;\beta)=1-\left[\frac{\left(1+2x\right)^{\beta}}{\left(1+x\right)^{2\beta}}\right]; \qquad x,\beta>0. \tag{2}$$

The reliability function and the hazard rate function of the ITL distribution are given by

$$R(x;\beta)=\left[\frac{\left(1+2x\right)^{\beta}}{\left(1+x\right)^{2\beta}}\right],$$

and

$$h(x;\beta)=2\beta x\left[\left(1+x\right)\left(1+2x\right)\right]^{-1},$$

where β is a shape parameter.

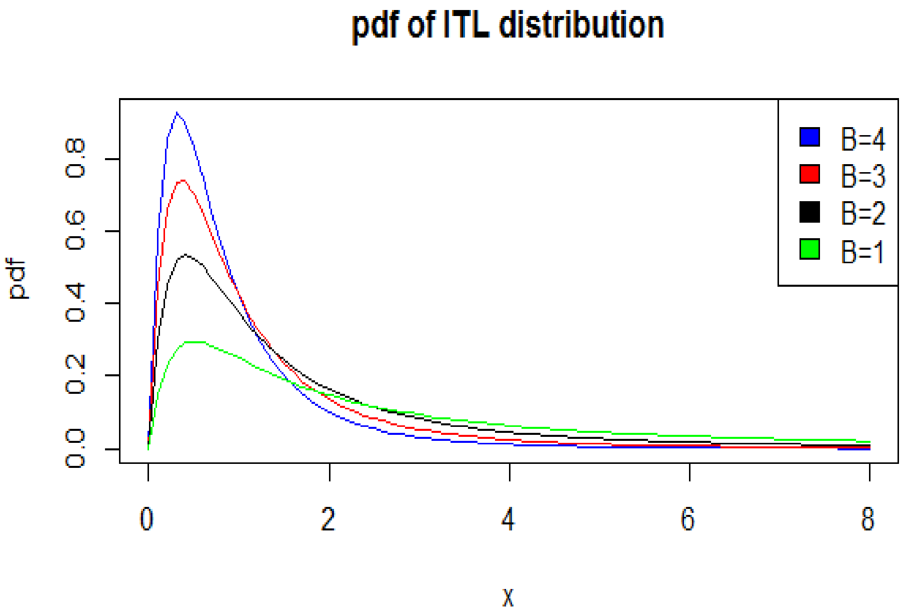

Figure 1. The pdf of the ITL distribution at various values of β

Han (2006) introduced a new method of estimation called expected Bayesian (E-Bayesian) estimation to estimate the reliability parameter. Jaheen and Okasha (2011) estimated the parameter and reliability function of the Burr XII distribution using E- Bayesian method based on squared error and LINEX loss function under type II censored sampling. Okasha (2014) considered the E- Bayesian method to estimate the unknown parameter and some survival time parameters based on censored type II for lomax distribution. Algarni and Almarashi (2020) presented the E- Bayesian estimation of the scale parameter for the chen distribution based on censored type II with squared error loss function and gamma distribution as a conjugate prior distribution. Also, Basheer et al (2021) obtained the E- Bayesian estimation of the parameter and reliability function of the inverse Weibull distribution in closed forms. In this paper, the E-Bayesian estimation method is used to estimate the unknown parameter for the inverted Topp – Leone (ITL) distribution under type-II censored.

The remainder of this paper is structured as follows: The maximum likelihood estimator is given in Section (2). The Bayesian estimators under squared error loss function and LINEX loss function are given in Section (3). Also, the E-Bayesian estimators are obtained based on a conjugate prior for the unknown parameter and the two loss functions in Section (4). Finally, a comparison between the maximum likelihood, Bayesian and E-Bayesian estimation methods are performed using Monte Carlo simulation in Section (5).

2. Maximum Likelihood Estimation (MLE)

Suppose that a random sample of n units from the ITL distribution is tested until the r first ordered observations are available, where r is determined before the experiment is performed. Then, the likelihood function under type–II censoring can be written as

$$L(\beta)=\frac{n!}{(n-r)!}\,2^r\beta^r\prod_{i=1}^{r}x_{(i)}\left[\frac{\left(1+2x_{(i)}\right)^{\beta-1}}{\left(1+x_{(i)}\right)^{2\beta+1}}\right]\left[\frac{\left(1+2x_{(r)}\right)^{\beta}}{\left(1+x_{(r)}\right)^{2\beta}}\right]^{n-r}. \tag{3}$$

The MLE of the parameter β under type-II censored sampling can be shown has the form

$$\hat{\beta}_{ML}=\frac{r}{T}\ \text{, where } T=\sum_{i=1}^{r}\ln\frac{\left(1+x_{(i)}\right)^2}{\left(1+2x_{(i)}\right)}+(n-r)\ln\frac{\left(1+x_{(r)}\right)^2}{\left(1+2x_{(r)}\right)}. \tag{4}$$

3. Bayesian Estimation

Assume that a censored samples of size r are obtained from a life test of n units from the ITL distribution based on type-II censoring. Under the assumption that the parameter β is unknown, then the Bayes estimator of β can be obtained by combining the prior density for the parameter β and the likelihood function. We use the following gamma conjugate prior density for the parameter β

$$\pi\left(\beta\right)=\frac{b^{a}}{\Gamma a}\beta^{a-1}e^{-b\beta}, \qquad \beta>0 \ \ and \ \ a,b>0, \tag{5}$$

where $\Gamma a=\int_0^\infty t^{a-1}e^{-t}\,dt$. The posterior density function of β given the data denoted by $\pi\left(\beta/\underline{x}\right)$, can be obtained from (3) and (5) and written as

$$\pi\left(\beta/\underline{x}\right)=\frac{\left(T+b\right)^{r+a}}{\Gamma r+a}\,\beta^{r+a-1}\,e^{-\beta(T+b)}. \tag{6}$$

So the posterior density function of β has the gamma distribution with parameters $\left(r+a,T+b\right)$.

3.1 Bayesian Estimation Under Squared Error Loss Function

From a decision-theoretic view point, in order to select a single value as representing our estimator of β, one must first specify a loss function, $L\left(\beta,\hat{\beta}\right)$. This loss function represents the cost involved in using the estimate $\hat{\beta}$ when the true value is β. A commonly used loss function for estimating β is the squared error loss, $L(\hat{\beta},\beta)=(\hat{\beta}-\beta)^2$.

Under this loss function, the Bayes estimator of β, denoted by $\hat{\beta}_{BS}$, is the mean of the posterior density function. Then, the Bayes estimator of β under squared error loss function has the form

$$\hat{\beta}_{BS}=\frac{r+a}{T+b}. \tag{7}$$

3.2 Bayesian Estimation Under LINEX Loss Function

The use of symmetric loss functions is incorrect in many situations, especially when positive and negative errors have different effects. So, Varian (1975) developed the asymmetric LINEX (linear exponential) loss function to overcome this problem. The LINEX loss function has the form

$$L\left(\hat{\beta},\beta\right)\propto e^{-c\Delta}-c\Delta-1, \qquad \text{where } \ c\neq 0 \ and \ \Delta=\hat{\beta}-\beta.$$

Based on the LINEX loss function, the Bayes estimate of β is given by

$$\hat{\beta}_{BL}=\frac{-1}{c}\ln\left[E_{\beta}\left(e^{-c\beta}\right)\right], \qquad c\neq 0 \tag{8}$$

Using equation (6) and equation (8), the Bayes estimator of β under LINEX loss function has the form

$$\hat{\beta}_{BL}=\frac{-\left(r+a\right)}{c}\ln\left(\frac{T+b}{T+b+c}\right). \tag{9}$$

4. E-Bayesian Estimation

Han (1997) suggested that, the hyperparameters a and b should be selected to guarantee that $\pi\left(\beta/a,b\right)$ in (5) is a decreasing function of β. The derivative of $\pi\left(\beta/a,b\right)$ with respect to β is

$$\frac{d\,\pi\left(\beta\right)}{d\,\beta}=\frac{b^{a}}{\Gamma a}\beta^{a-2}e^{-b\beta}\left[\left(a-1\right)-b\beta\right].$$

In order to ensure that the function $\pi\left(\beta/a,b\right)$ is a decreasing function of β, we must choose $0<a<1, b>$. Assuming that the hyperparameters a and b are independent random variables with density functions $\pi_1\left(a\right)$ and

$\pi_2(b)$, respectively. Then the joint density function of a and b is given by

$$\pi(a,b)=\pi_1(a)\pi_2(b).$$

Thereby, the E-Bayesian estimate of β can be expressed as

$$\hat{\beta}_{EB}=\iint_D \hat{\beta}_B(a,b)\pi(a,b)da\ db, \tag{10}$$

where D is the domain of a and b for which the prior density is decreasing in β, and $\hat{\beta}_B(a,b)$ is the Bayes estimate of β given by equation (7). For more details [see Han (2009)].

As mentioned earlier, the joint density function of a and b, $\pi(a,b)$, resulted from the density function of a, $\pi_1(a)$, and the density function of b, $\pi_2(b)$. In order to investigate the E-Bayesian estimate of β, two density functions of a and three density functions of b are used. So six joint density functions of a and b are resulted.

The two density functions of a are

(i) $\pi_{11}(a)=\dfrac{1}{B(u,v)}a^{u-1}(1-a)^{v-1}, \quad 0<a<1$

(ii) $\pi_{12}(a)=2\theta a^{\theta-1}(1-a)(2-a)^{\theta-1}, \quad 0<a<1$ [Topp – Leone distribution]

The three density functions of b are

(i) $\pi_{21}(b)=\dfrac{1}{s}, \quad 0<b<s$

(ii) $\pi_{22}(b)=\dfrac{2(s-b)}{s^2}, \quad 0<b<s$

(iii) $\pi_{23}(b)=\dfrac{2b}{s^2}, \quad 0<b<s$

Therefore, the six joint density functions of a and b are

$$\begin{aligned}
\pi_1(a,b)&=\frac{1}{s\,B(u,v)}a^{u-1}(1-a)^{v-1},\\
\pi_2(a,b)&=\frac{2(s-b)}{s^2B(u,v)}a^{u-1}(1-a)^{v-1},\\
\pi_3(a,b)&=\frac{2b}{s^2B(u,v)}a^{u-1}(1-a)^{v-1},\\
\pi_4(a,b)&=\frac{2\theta}{s}a^{\theta-1}(1-a)(2-a)^{\theta-1},\\
\pi_5(a,b)&=\frac{4\theta(s-b)}{s^2}a^{\theta-1}(1-a)(2-a)^{\theta-1},\\
\pi_6(a,b)&=\frac{4\theta b}{s^2}a^{\theta-1}(1-a)(2-a)^{\theta-1}.
\end{aligned}$$

(11)

4.1 E-Bayesian Estimation Under Squared Error Loss Function

Based on the squared error loss function, the Bayesian estimate of β under $\pi_1(a,b)$ can be given by using equations (7) and (11) in (10) as following

$$\hat{\beta}_{EBS1}=\frac{1}{s\,B(u,v)}\int_0^s\int_0^1\left(\frac{r+a}{T+b}\right)a^{u-1}(1-a)^{v-1}da\,db=\frac{1}{s}\left(r+\frac{u}{u+v}\right)\ln\left(\frac{T+s}{T}\right), \tag{12}$$

Similarly,

$$\hat{\beta}_{EBS\,2} = \frac{2}{s^2}\left(r + \frac{u}{u+v}\right)\left[(T+s)\ln\left(\frac{T+s}{T}\right) - s\right], \tag{13}$$

$$\hat{\beta}_{EBS\,3} = \frac{2}{s^2}\left(r + \frac{u}{u+v}\right)\left[s - T\ln\left(\frac{T+s}{T}\right)\right], \tag{14}$$

$$\hat{\beta}_{EBS\,4} = \frac{1}{s}\left(r + 1 - \frac{\sqrt{\pi}\,\Gamma(\theta+1)}{2\,\Gamma\left(\theta+\frac{3}{2}\right)}\right)\ln\left(\frac{T+s}{T}\right), \tag{15}$$

$$\hat{\beta}_{EBS\,5} = \frac{2}{s^2}\left(r + 1 - \frac{\sqrt{\pi}\,\Gamma(\theta+1)}{2\,\Gamma\left(\theta+\frac{3}{2}\right)}\right)\left[(T+s)\ln\left(\frac{T+s}{T}\right) - s\right], \tag{16}$$

$$\hat{\beta}_{EBS\,6} = \frac{2}{s^2}\left(r + 1 - \frac{\sqrt{\pi}\,\Gamma(\theta+1)}{2\,\Gamma\left(\theta+\frac{3}{2}\right)}\right)\left[s - T\ln\left(\frac{T+s}{T}\right)\right]. \tag{17}$$

4.2 E-Bayesian Estimation Under LINEX Loss Function

The Bayesian estimate of β relative to $\pi_1(a,b)$ under LINEX loss function can be obtained by using equations (9) and (11) in (10) as following

$$\hat{\beta}_{EBL\,1} = \frac{-1}{s\,B(u,v)}\int_0^s\int_0^1 \frac{(r+a)}{c}\ln\left(\frac{T+b}{T+b+c}\right) a^{u-1}(1-a)^{v-1}\,da\,db$$

$$= \frac{-\left(r + \frac{u}{u+v}\right)}{s\,c}\left[T\ln\left(\frac{(T+s)(T+c)}{T(T+s+c)}\right) + s\ln\left(\frac{T+s}{T+s+c}\right) + c\ln\left(\frac{T+c}{T+s+c}\right)\right]. \tag{18}$$

Similarly,

$$\hat{\beta}_{EBL2} = 2\hat{\beta}_{EBL1} + \frac{\left(r+\frac{u}{u+v}\right)}{s^2\,c}K(c,s), \tag{19}$$

where

$$K(c,s) = \left(s^2 - T^2\right)\ln(T+s) + T^2\ln T + \left[(T+c)^2 - s^2\right]\ln(T+s+c)$$
$$-(T+c)^2\ln(T+c) - cs$$

$$\hat{\beta}_{EBL\,3} = \frac{-\left(r + \frac{u}{u+v}\right)}{s^2 c}K(c,s), \tag{20}$$

$$\hat{\beta}_{EBL\,4} = \frac{-1}{s\,c}\left(r + 1 - \frac{\sqrt{\pi}\,\Gamma(\theta+1)}{2\,\Gamma\left(\theta+\frac{3}{2}\right)}\right)\left[T\ln\left(\frac{(T+s)(T+c)}{T(T+s+c)}\right) + s\ln\left(\frac{T+s}{T+s+c}\right) + c\ln\left(\frac{T+c}{T+s+c}\right)\right], \tag{21}$$

$$\hat{\beta}_{EBL5} = 2\hat{\beta}_{EBL4} + \frac{1}{s^2 c}\left(r + 1 - \frac{\sqrt{\pi}\,\Gamma(\theta+1)}{2\,\Gamma\left(\theta+\frac{3}{2}\right)}\right)K(c,s), \tag{22}$$

$$\hat{\beta}_{EBL6} = \frac{-1}{s^2 c}\left(r + 1 - \frac{\sqrt{\pi}\,\Gamma(\theta+1)}{2\,\Gamma\left(\theta+\frac{3}{2}\right)}\right)K(c,s). \tag{23}$$

5. Monte Carlo Simulation Study

This **Section** deals with the comparison between the E-Bayesian, Bayesian and maximum likelihood estimators under type-II censored sampling based on the risk functions. The following steps are used to perform this comparison numerically using a Monte Carlo simulation:

(i) select different values of n and r for $\beta = 2$.

(ii) determine the values $a = 0.5, b = 0.25, s = 0.5, c = 1.5, u = 2 \text{ and } v = 2$.

(iii) generate $U \sim UNIF(0,1)$.

(iv) generate $x_i = (1-u_i)^{-1/\beta}\left[1-(1-u_i)^{1/\beta} + \sqrt{1-(1-u_i)^{1/\beta}}\right]$.

(v) Compute $\hat{\beta}_{ML}$ from equation (4).

(vi) Under the squared error loss function compute $\hat{\beta}_{BS}$ and $\hat{\beta}_{EBSj}$, $j = 1, 2, ..., 6$ from equation (7), (12), (13), (14), (15), (16) and (17).

(vii) Under the LINEX loss function compute $\hat{\beta}_{BL}$ and $\hat{\beta}_{EBLj}$, $j = 1, 2, ..., 6$ from equation (9), (18), (19), (20), (21), (22) and (23).

(viii) Steps (iii) to (vii) are repeated 10000 times

(ix) Compute the mean square error (MSE) by

$$MSE(\hat{\beta}) = \frac{1}{10000}\sum_{i=1}^{10000}(\hat{\beta}_i - \beta)^2.$$

The simulation results are shown in table (1).

Table (1). MSEs for different Bayesian estimates of the parameter β

n	r	$\hat{\beta}_{ML}$	$\hat{\beta}_{BS}$ $\hat{\beta}_{BL}$	$\hat{\beta}_{EBS1}$ $\hat{\beta}_{EBL1}$	$\hat{\beta}_{EBS2}$ $\hat{\beta}_{EBL2}$	$\hat{\beta}_{EBS3}$ $\hat{\beta}_{EBL3}$	$\hat{\beta}_{EBS4}$ $\hat{\beta}_{EBL4}$	$\hat{\beta}_{EBS5}$ $\hat{\beta}_{EBL5}$	$\hat{\beta}_{EBS6}$ $\hat{\beta}_{EBL6}$
20	12	0.5334	0.4716 0.2926	0.4288 0.2662	0.4663 0.2455	0.3945 0.2600	0.4644 0.2671	0.5074 0.2480	0.4248 0.2578
	15	0.3643	0.3338 0.2292	0.3061 0.2051	0.3277 0.1925	0.2863 0.2009	0.3290 0.2068	0.3537 0.1954	0.3062 0.2006
	20	0.2576	0.2432 0.1799	0.2301 0.1747	0.2419 0.1663	0.2192 0.1722	0.2411 0.1754	0.2542 0.1678	0.2288 0.1720
40	30	0.1622	0.1494 0.1281	0.1447 0.1237	0.1491 0.1199	0.1407 0.1223	0.1488 0.1241	0.1537 0.1206	0.1442 0.1222
	35	0.1371	0.1288 0.1119	0.1245 0.1090	0.1282 0.1061	0.1210 0.1079	0.1281 0.1095	0.1322 0.1068	0.1242 0.1081
	40	0.1050	0.1211 0.0879	0.1175 0.0866	0.1205 0.0845	0.1146 0.0860	0.1206 0.0867	0.1239 0.0848	0.1175 0.0859
60	40	0.1152	0.0892 0.0968	0.1112 0.0900	0.1133 0.0882	0.1093 0.0889	0.1142 0.0909	0.1165 0.0893	0.1120 0.0895
	50	0.0926	0.0677 0.0802	0.0806 0.0779	0.0818 0.0766	0.0795 0.0773	0.0824 0.0783	0.0837 0.0770	0.0811 0.0775
	60	0.0654	0.0643 0.0581	0.0740 0.0578	0.0750 0.0568	0.0730 0.0575	0.0758 0.0578	0.0769 0.0569	0.0746 0.0574

The results in Table (1) show the following

(i) For all methods of estimation, the MSE is decreasing when n and r are increasing.

(ii) For all sample sizes n, the MLE has the biggest MSE.

(iii) The E-Bayesian estimation of β has a smallest MSE for all sample sizes n and all cases.

(iv) The LINEX loss function performs better than the square error loss function for Bayesian and E-Bayesian estimation methods.

The research presented in this study generally demonstrated that the E-Bayesian estimation approach outperforms the ML and Bayesian estimation methods in terms of efficiency.

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