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Zero Truncated Poisson Harris Weibull Distribution: Properties and Applications to Lifetime Datas

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Abstract

In this study, we proposed and developed a more flexible distribution with wider applications called Zero Truncated Poisson Harris Weibull (ZTPHW) distribution. Some well-known mathematical properties such as ordinary moments and incomplete moment, moment generating function, quantile function, Renyi and Tsallis entropy of ZTPHW distribution are investigated. The expressions of order statistics are derived. Parameters of the derived distribution are obtained using the maximum likelihood method and simulation studied is carried out to examine the validity of the method of estimation. The flexibility of the proposed distribution in modeling real life data is demonstrated using two lifetime data set.

Keywords: Quantile function, Moments, moment generating function, Tsallis entropy

1. Introduction

Weibull models are used to describe various types of observed failures of components and phenomena. They are mostly used in survival and reliability analysis. However, only mononically increasing and decreasing hazard functions can be obtained from the classical two-parameter Weibull distribution, and hence it cannot be used to model phenomena with non-monotone, unimodal or bathtub-shaped failure rate. Hence, there is need for extending the classical Weibull distribution in such a way that it can be used to model phenomena with different shapes of the hazard function.

A random variable X is said to follows a Weibull distribution if its cumulative distribution function is given by

$$F^{w}(x;\ \alpha,\beta) = 1 - e^{-\alpha x^{\beta}} \quad ,x > 0 \tag{1}$$

The corresponding probability density function and survival function is given respectively by

$$f^{w}(x;\ \alpha,\beta) = \alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}, \qquad x > 0 \tag{2}$$

And

$$S^{w}(x;\ \alpha,\beta) = e^{-\alpha x^{\beta}}, \qquad x > 0 \tag{3}$$

Where α is a positive scale parameter and β is a positive shape parameter

The Weibull distribution has been applied in many areas of applied statistics which includes: Reliability studies, medicine, life testing, etc. In recent years, several generalizations of the classical Weibull distribution have been developed and studied by different authors to cope with bathtub-shaped failure rates. The procedure which entails adding one or two parameters to a family of distributions to obtain more flexibility is a well-known technique in the existing literature. It includes the Marshall-Olkin-G proposed and studied by Marshall and Olkin (1997), Eugene et al. (2002) developed the beta-G, Kumaraswamy-G was developed by Cordeiro and de Castro (2011), McDonald-G by Alexander et al. (2012), gamma-G was developed by Zografos and Balakrishnan (2009), Alizadeh et al. (2015) proposed and studied the Kumaraswamy odd log-logistic-G, beta odd log-logistic generalized was studied by Cordeiro et al. (2015), transmuted exponentiated generalized-G by Yousof et al.(2015), generalized transmuted-G by Nofal et al. (2017), Afify et al. (2016a) developed and studied transmuted geometric- G, Kumaraswamy transmuted-G by Afify et al. (2016b), Afify et al. (2017) studied the beta transmuted-H, Burr X-G by Yousof et al. (2016) and Alizadeh et al (2017) developed and studied the odd-Burr generalized-G (2017) families and many others. Aly and Benkherouf (2011) who proposed and developed a new family of distributions, called the Harris extended (HE) family, by adding two new parameters to a baseline distribution. The new method is based on the probability generating function (pgf) of the Harris (1948) distribution. If $F(x)$, $\bar{F}(x)$, and $f(x)$ denote the cdf, survival function (sf) and probability density function of a parent distribution, respectively, then the survival function of the Harris Extended (HE) family of distributions is given by

$$\bar{G}(x) = \left[\frac{\theta\bar{F}(x)^{\lambda}}{1-\bar{\theta}\bar{F}(x)^{\lambda}}\right]^{1/\lambda}, x > 0, \theta > 0, \bar{\theta} = 1-\theta, \lambda > 0 \quad (4)$$

Here, the parameters $\lambda \; and \; \theta$ are additional shape parameters that aim to induce flexibility.

The HE cumulative and density function is respectively given as

$$G(x) = 1 - \left[\frac{\theta\bar{F}(x)^{\lambda}}{1-\bar{\theta}\bar{F}(x)^{\lambda}}\right]^{1/\lambda}, x > 0, \theta > 0, \bar{\theta} = 1-\theta, \lambda > 0 \quad (5)$$

And

$$g(x) = \frac{\theta^{1/\lambda} f(x)}{[1-\bar{\theta}\bar{F}(x)^{\lambda}]^{\left(1+\frac{1}{\lambda}\right)}} \quad (6)$$

When $\lambda = 1$, equation (6) reduces to Marshall-Olkin family of distributions. Hence the HE family of distributions generalizes the well-known Marshall-Olkin class of distributions.

1.1 Zero Truncated Poisson Harris Weibull Distribution

Given N let $X_1, X_2, \ldots, X_N$ be independent and identically distributed random variable from Harris G distribution. Let N be distributed according to the zero truncated Poisson distribution with pdf

$$P(N = n) = \frac{k^N e^{-k}}{n!\,(1-e^{-k})}, \;\; n = 1,2,\ldots, k > 0 \quad (7)$$

Let

$X = max\langle V_1, V_2, \ldots, V_N\rangle$, then the cdf of $X/N = n$ is given by

$$G(x) = \left\{1 - \left(\frac{\theta\bar{F}(x)^{\lambda}}{1-\bar{\theta}\bar{F}(x)^{\lambda}}\right)^{\frac{1}{\theta}}\right\}^{n} \quad (8)$$

The zero truncated Harris G distribution is the marginal cdf of X, given by

$$F^{ZTPH} = \frac{1}{(1-e^{-k})}\left(1 - exp\left[-k\left\{1 - \left(\frac{\theta\bar{F}(x)^{\lambda}}{1-\bar{\theta}\bar{F}(x)^{\lambda}}\right)^{\frac{1}{\theta}}\right\}\right]\right) \quad (9)$$

Here, the parameters $k, \lambda \; and \; \theta$ are additional shape parameters that aim to induce greater flexibility.

Putting equation (1) in (9), we have a new distribution called Zero Truncated Poisson Harris extended Weibull $(ZTPHW)$ distribution with the cdf given by

$$F(x) = \frac{1}{1-e^{\lambda}}\left(1 - e^{\left[-k\left\{1-\theta^{\frac{1}{\lambda}}e^{-\alpha x^{\beta}}\left(1-\bar{\theta}e^{-\alpha\lambda x^{\beta}}\right)^{-\frac{1}{\lambda}}\right\}\right]}\right), x > 0 \quad (10)$$

With positive shape parameters $k, \beta, \lambda, \theta$ and positive scale parameter α. The figure 1 drawn below is the graph of the distribution function of $ZTPHW$ distribution with fixed parameter values of λ and θ with different values of k, α, and β

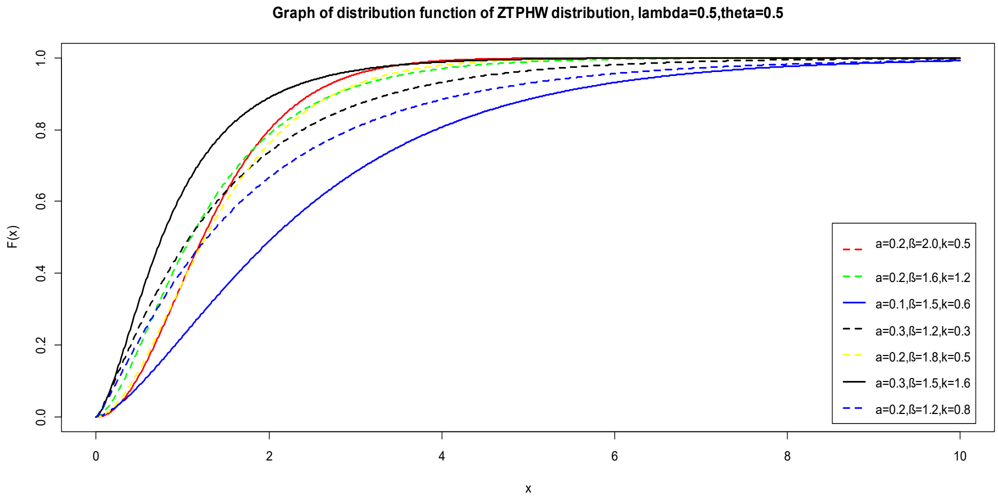

Figure 1. Graph of the distribution function of $\boldsymbol{ZTPHW}$ distribution

✓ Figure 1 drawn above clearly indicates that the $ZTPHW$ distribution has a proper density function

The associated pdf to (10) is given by

$$f(x)=\frac{\alpha\beta k\lambda\theta^{1/\lambda}}{1-e^{\lambda}}x^{\beta-1}e^{-\alpha x^{\beta}}\left[1-\bar{\theta}e^{-\alpha\lambda x^{\beta}}\right]^{-\left(1+\frac{1}{\lambda}\right)}e^{\left[-k\left\{1-\theta^{\frac{1}{\lambda}}e^{-\alpha x^{\beta}}\left(1-\bar{\theta}e^{-\alpha\lambda x^{\beta}}\right)^{-\frac{1}{\lambda}}\right\}\right]} \tag{11}$$

With positive shape parameters k,β,λ,θ and positive scale parameter α. The figure 2 drawn below is the graph of the density function of $ZTPHW$ distribution with fixed parameter values of λ and θ with different values of k, α, and β

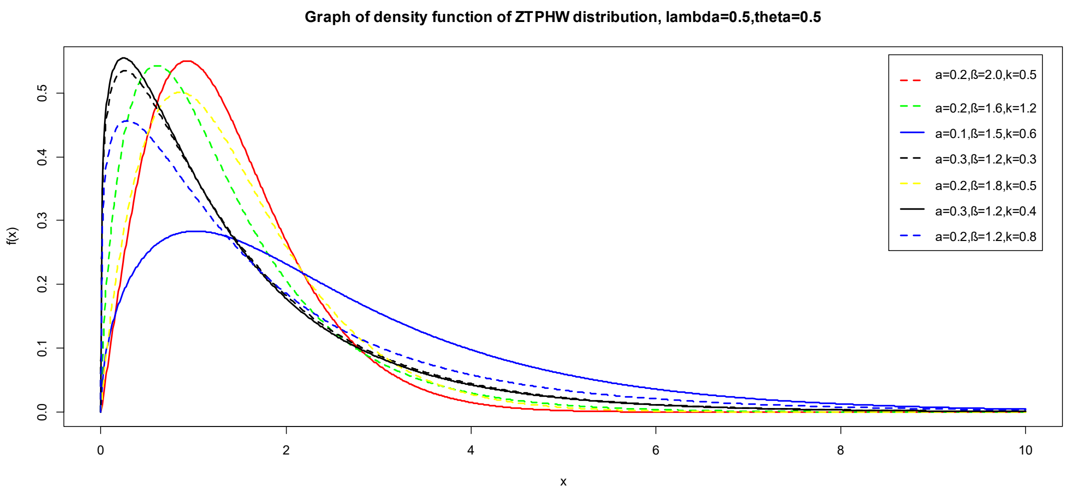

Figure 2 Graph of the distribution function of $\boldsymbol{ZTPHW}$ distribution

✓ The graph of distribution function of $ZTPHW$ distribution drawn above indicates that the distribution is unimodal

1.2 Survival Function

The survival function of $ZTPHW$ distribution is given by

$$S(x)=1-F(x) \tag{12}$$

Putting equation (10) in (12), we obtain the survival function of ZTPHW distribution as

$$S(x)=1-\frac{1}{1-e^{\lambda}}\left(1-e^{\left[-k\left\{1-\theta^{\frac{1}{\lambda}}e^{-\alpha x^{\beta}}\left(1-\bar{\theta}e^{-\alpha\lambda x^{\beta}}\right)^{-\frac{1}{\lambda}}\right\}\right]}\right) \tag{13}$$

The figure 3 drawn below is the graph of the survival function of $ZTPHW$ distribution with fixed parameter values of λ and θ with different values of k, α, and β

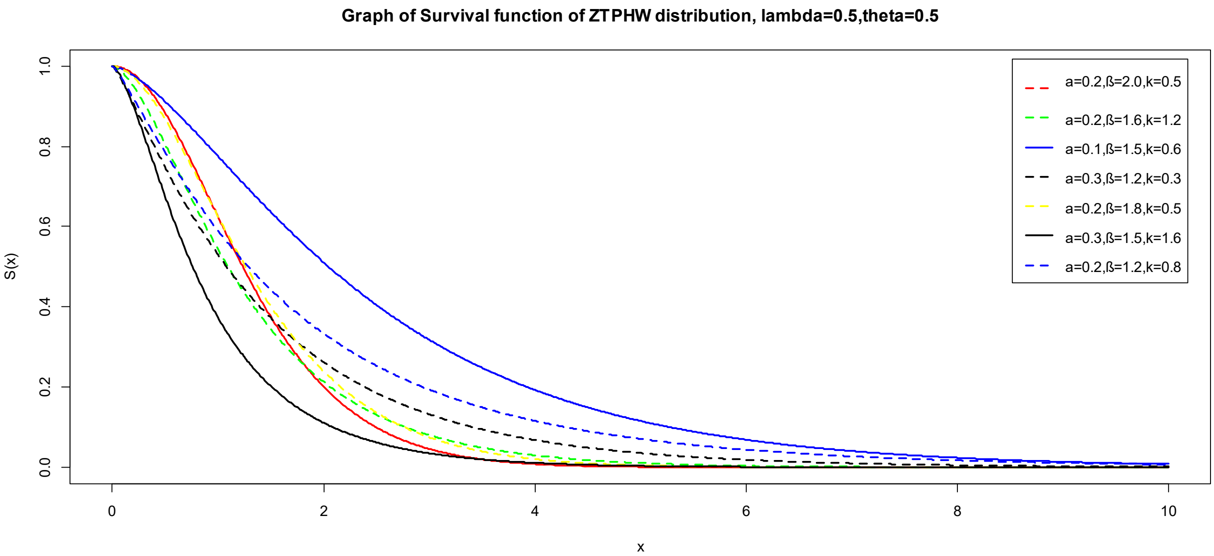

Figure 3. Graph of the distribution function of $\boldsymbol{ZTPHW}$ distribution

1.3 Hazard Function

Then hazard function is given by

$$h(x)=\frac{f(x)}{S(x)} \tag{15}$$

putting equation (10) and (11) in (15), we have the hazard model of $ZTPHW$ distribution given by

$$\boldsymbol{h(x)}=\frac{\frac{\alpha\beta k\lambda\theta^{1/\lambda}}{1-e^{\lambda}}x^{\beta-1}e^{-\alpha x^{\beta}}\left[1-\bar{\theta}e^{-\alpha\lambda x^{\beta}}\right]^{-\left(1+\frac{1}{\lambda}\right)}e^{\left[-k\left\{1-\theta^{\frac{1}{\lambda}}e^{-\alpha x^{\beta}}\left(1-\bar{\theta}e^{-\alpha\lambda x^{\beta}}\right)^{-\frac{1}{\lambda}}\right\}\right]}}{\mathbf{1}-\frac{1}{1-e^{\lambda}}\left(1-e^{\left[-k\left\{1-\theta^{\frac{1}{\lambda}}e^{-\alpha x^{\beta}}\left(1-\bar{\theta}e^{-\alpha\lambda x^{\beta}}\right)^{-\frac{1}{\lambda}}\right\}\right]}\right)} \tag{16}$$

The figure 4 drawn below is the graph of the hazard function of $ZTPHW$ model with fixed parameter values of λ and θ with different values of k, α, and β

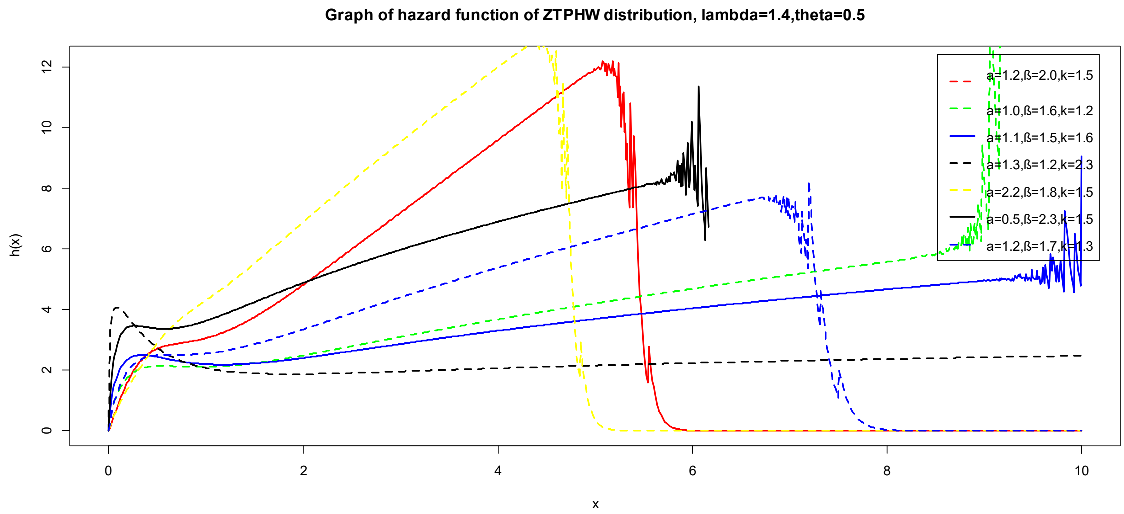

Figure 4. Graph of the hazard function of $\boldsymbol{ZTPHW}$ model

Figure 4 drawn above indicates that the $ZTPHW$ model can be used to model different shapes of the hazard function.

1.4 Quantiles of the $\boldsymbol{ZTPHW}$ *Distribution*

The quantile function of a distribution is a very important tool used in describing some important properties of a distribution. In this section, we present the quantile function of the $ZTPHW$ distribution, as well as some of its related properties, applications, and functions.

The u^{th} quantile (x_u) of the $ZTPHW$ distribution is obtained by solving equation.

$$F(x_u) = \mathrm{u},$$

Hence solving equation (10) we get

$$x_u = \left\{-\frac{1}{\alpha} ln\left[\frac{\left\{\frac{1}{\theta^{1/\lambda}}\left(1+\frac{1}{k}\{ln(1-u[1-e^{-k}])\}\right)\right\}^{\lambda}}{1+\left[\frac{1}{\theta^{1/\lambda}}\left(1+\frac{1}{k}\{ln(1-u[1-e^{-k}])\}\right)\right]^{\lambda}(1-\theta)}\right]\right\}^{1/\beta}, \tag{17}$$

the u^{th} quantile for $u \in (0,1)$

for $u = 0.25, 0.5, 0.75$, we have the lower quartile, middle quartile (median) and the upper quartile of the $ZTPHW$ distribution respectively, given by

$$x_{0.25} = \left\{-\frac{1}{\alpha} ln\left[\frac{\left\{\frac{1}{\theta^{1/\lambda}}\left(1+\frac{1}{k}\{ln(1-0.25[1-e^{-k}])\}\right)\right\}^{\lambda}}{1+\left[\frac{1}{\theta^{1/\lambda}}\left(1+\frac{1}{k}\{ln(1-0.25[1-e^{-k}])\}\right)\right]^{\lambda}(1-\theta)}\right]\right\}^{1/\beta},$$

$$x_{0.5} = \left\{-\frac{1}{\alpha} ln\left[\frac{\left\{\frac{1}{\theta^{1/\lambda}}\left(1+\frac{1}{k}\{ln(1-0.5[1-e^{-k}])\}\right)\right\}^{\lambda}}{1+\left[\frac{1}{\theta^{1/\lambda}}\left(1+\frac{1}{k}\{ln(1-0.5[1-e^{-k}])\}\right)\right]^{\lambda}(1-\theta)}\right]\right\}^{1/\beta},$$

And

$$x_{0.75} = \left\{ -\frac{1}{\alpha} ln \left[\frac{\left\{ \frac{1}{\theta^{1/\lambda}} \left(1 + \frac{1}{k} \{ ln(1 - 0.75[1 - e^{-k}]) \} \right) \right\}^{\lambda}}{1 + \left[\frac{1}{\theta^{1/\lambda}} \left(1 + \frac{1}{k} \{ ln(1 - 0.75[1 - e^{-k}]) \} \right) \right]^{\lambda} (1 - \theta)} \right] \right\}^{1/\beta},$$

Classical measures of skewness and kurtosis may be extremely difficult to obtain as a result of non-existence of higher moments in several heavy tailed distributions. When such a situation arises, the quantile measures can be considered to be a suitable measure. The Bowley skewness; Kenny and Keeping (1962) is one of the foremost measures of skewness that is based on quantile of a distribution. It is given by

$$B_s = \frac{q_{1/4} - 2q_{2/4} + q_{1/4}}{q_{3/4} - q_{2/4}}$$

Also, the coefficient of Kurtosis can be estimated using Moor's (1988) approach to estimating kurtosis based on octiles of a distribution and is given by

$$M_k = \frac{q_{7/8} - q_{3/8} - q_{5/8} + q_{1/8}}{q_{6/8} - q_{4/8}}$$

It should be noted that the two measures are more robust to outliers. Table 1 drawn below gives the Bowley skewness and Moor's kurtosis of $ZTPHW$ distribution for a fixed values of $\alpha = 0.1$ and $\beta = 0.3$ and varying the values of the parameters k, λ and θ.

Table 1. Values of Bowley Skewness and Moors Kurtosis for given values of the parameters

Quartiles	$k = 0.5,$ $\lambda = 0.4$ $\theta = 1.4$	$k = 0.6,$ $\lambda = 0.6$ $\theta = 1.0$	$k = 0.1,$ $\lambda = 0.3$ $\theta = 0.2$	$k = 0.4$ $\lambda = 0.7$ $\theta = 0.2$	$k = 1.2,$ $\lambda = 0.5$ $\theta = 1.5$
$q_{1/4}(X)$	0.0017	0.0250	0.2971	$5.9638e-05$	0.3836
$q_{2/4}(X)$	0.0418	0.6420	7.3166	0.0013	8.0809
$q_{3/4}(X)$	0.6084	11.0557	113.6864	0.0172	96.6217
$q_{1/8}(X)$	0.0001	0.0017	0.0207	$4.336e-06$	0.0283
$q_{3/8}(X)$	0.0101	0.1479	1.7269	0.0003	2.0795
$q_{5/8}(X)$	0.1554	2.5541	28.0249	0.0047	27.7049
$q_{7/8}(X)$	3.3089	70.9733	638.2892	0.0852	429.8091
B_s	1.4304	0.0805	0.0077	49.980	0.0087
M_k	5.2143	6.2170	5.3971	−2737.331	4.1995

From Table 1, we can conclude that the ZTPHW distribution can be used to model data that skewed to the right (positively skewed) with various degree of kurtosis

1.5 Expansion for the Density Function

Consider the power series given by

$$e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!} \tag{19}$$

Which holds for $|f| < 1$ and $m > 0$ is a real non-integer. Using the power series in equation (19)

$$e^{\left[-k\left\{1-\theta^{\frac{1}{\lambda}}e^{-\alpha x^{\beta}}\left(1-\bar{\theta}e^{-\alpha\lambda x^{\beta}}\right)^{-\frac{1}{\lambda}}\right\}\right]} = \sum_{i=0}^{\infty} \frac{(-1)^i (k)^i}{i!} \left\{1 - \theta^{\frac{1}{\lambda}} e^{-\alpha x^{\beta}} \left(1 - \bar{\theta} e^{-\alpha\lambda x^{\beta}}\right)^{-\frac{1}{\lambda}}\right\}^i$$

Subsequently,

$$\left\{1 - \theta^{\frac{1}{\lambda}} e^{-\alpha x^{\beta}} \left(1 - \bar{\theta} e^{-\alpha\lambda x^{\beta}}\right)^{-\frac{1}{\lambda}}\right\}^i = \sum_{j=0}^{\infty} (-1)^j \binom{i}{j} \theta^{\frac{j}{k}} e^{-\alpha j x^{\beta}} \left(1 - \bar{\theta} e^{-\alpha\lambda x^{\beta}}\right)^{-\frac{j}{\lambda}}$$

Finally, we have

$$f(x)=\alpha\beta\theta^{\frac{1}{\lambda}}\sum_{\boldsymbol{i,j,l=0}}^{\infty}\frac{(-1)^{i+j+l}(k)^{i+1}}{\big(\lambda(j+l+1)\big)i!}\binom{i}{j}\binom{l+\frac{j+1}{\lambda}}{l}\theta^{\frac{j}{k}}\bar{\theta}^{l}\lambda(j+l+1)x^{\beta-1}e^{-\alpha x^{\beta}\lambda(j+l+1)} \quad (20)$$

2. The Ordinary and Incomplete Moments of $ZTPHW$ Distribution

Moments are defined as the expected value of certain function of a random variable. The moments of different types of $ZTPHW$ distribution can be obtained by direct calculations due to its mathematical tractability. In a related manner, the first incomplete moment can be used for the computation of Bonferroni and Lorenz curves, the mean waiting time and the mean residual life which plays an important role in reliability studies. Thus, the r^{th} ordinary moment of a distribution is given by

$$\mu_r'=E(x)^r$$

Thus the r^{th} moment of ZTPHW distribution is given by

$$\mu_r'=\int_0^{\infty}x^r f(x)dx \quad (21)$$

Putting equation (20) in (21), we have

$$\mu_r'=\alpha\beta\theta^{\frac{1}{\lambda}}\sum_{\boldsymbol{i,j,l=0}}^{\infty}\frac{(-1)^{i+j+l}(k)^{i+1}}{\big(\lambda(j+l+1)\big)i!}\binom{i}{j}\binom{l+\frac{j+1}{\lambda}}{l}\theta^{\frac{j}{k}}\bar{\theta}^{l}\lambda(j+l+1)\int_0^{\infty}x^{r+\beta-1}e^{-\alpha x^{\beta}\lambda(j+l+1)}dx \quad (22)$$

Suppose, we let

$$J(x)=\int_0^{\infty}x^{r+\beta-1}e^{-\alpha x^{\beta}\lambda(j+l+1)}dx \quad (23)$$

Letting

$z=\alpha x^{\beta}\lambda(j+l+1), x=z^{1/\beta}[\alpha\lambda(j+l+1)]^{-1/\beta}, dx={1/\beta}\, z^{\frac{1}{\beta}}[\alpha\lambda(j+l+1)]^{-1/\beta}$ and putting that in equation (23), we have

$$J(x)={1/\beta}\,[\alpha\lambda(j+l+1)]^{-\left(\frac{r}{\beta}+1\right)}\int_0^{\infty}z^{r/\beta}e^{-z}dz \quad (24)$$

Then, we have

$$J(x)={1/\beta}\,[\alpha\lambda(j+l+1)]^{-\left(\frac{r}{\beta}+1\right)}\Gamma\left(\frac{r}{\beta}+1\right)$$

Finally, we have

$$\mu_r'=\alpha^{r/\beta}W_{i,j,l}^{k,\lambda,\theta}[\alpha\lambda(j+l+1)]^{-\left(\frac{r}{\beta}\right)}\Gamma\left(\frac{r}{\beta}+1\right),\beta>r \quad (25)$$

Where,

$$W_{i,j,l}^{k,\lambda,\theta}=\theta^{\frac{1}{\lambda}}\sum_{\boldsymbol{i,j,l=0}}^{\infty}\frac{(-1)^{i+j+l}(k)^{i+1}}{\big(\lambda(j+l+1)\big)i!}\binom{i}{j}\binom{l+\frac{j+1}{\lambda}}{l}\theta^{\frac{j}{k}}\bar{\theta}^{l} \quad (25.1)$$

Equation (25) is an expression for the r^{th} moment of ZTPHW distribution. For $r=1$, we obtain the first moment (mean) of ZTPHW distribution as

$$\mu_1'=\alpha^{1/\beta}W_{i,j,l}^{k,\lambda,\theta}[\alpha\lambda(j+l+1)]^{-\left(\frac{1}{\beta}\right)}\Gamma\left(\frac{1}{\beta}+1\right) \quad (26)$$

For $r = 2$, we obtain the second moment of $ZTPHW$ distribution as

$$\mu_2' = \alpha^{2/\beta} W_{i,j,l}^{k,\lambda,\theta} \bar{\theta}^l [\alpha\lambda(j+l+1)]^{-\left(\frac{2}{\beta}\right)} \Gamma\left(\frac{2}{\beta}+1\right) \tag{27}$$

And the variance (μ) can be obtained as $\mu = \mu_2' - (\mu_1')^2$

Table 2 drawn below gives the first six moments of ZTPHW distribution for a fixed parameters of $\alpha = 0.5$ and $\beta = 6.1$

Table 2. First six moments and variance of ZTPHW distribution

$moments$	$k = 0.1$, $\lambda = 0.1$, $\theta = 0.2$	$k = 0.3$, $\lambda = 1.3$, $\theta = 1.5$	$k = 0.8$, $\lambda = 2.0$, $\theta = 2.5$	$k = 2.5$, $\lambda = 3.5$, $\theta = 4.5$	$k = 4.5$, $\lambda = 5.5$, $\theta = 6.5$	$k = 10.8$, $\lambda = 10.5$, $\theta = 15.2$
μ_1'	0.8242	1.0648	1.0751	1.0230	0.9635	0.8870
μ_2'	0.7122	1.1712	1.1889	1.0726	0.9480	0.7974
μ_3'	0.6416	1.3234	1.3459	1.1487	0.9500	0.7250
μ_4'	0.6000	1.5301	1.5551	1.2540	0.9678	0.6656
μ_5'	0.5804	1.8049	1.8294	1.3909	1.0013	0.6165
μ_6'	0.5793	2.1672	2.1875	1.5732	1.0514	0.5756
μ	0.0329	0.0374	0.0331	0.0261	0.0197	0.0106

2.1 Incomplete Moment of $ZTPHW$ Distribution

The incomplete moment of $ZTPHW$ distribution can be obtained using

$$\phi(x) = \int_0^t x^r f(x)dx \tag{28}$$

Putting equation (12) in (28), we have

$$\mu_r' = \alpha\beta\theta^{\frac{1}{\lambda}} \sum_{i,j,l=0}^{\infty} \frac{(-1)^{i+j+l}(k)^{i+1}}{\left(\lambda(j+l+1)\right)i!} \binom{i}{j} \binom{l+\frac{j+1}{\lambda}}{l} \theta^{\frac{j}{k}} \bar{\theta}^l \lambda(j+l+1)$$

$$\times \int_0^t x^{r+\beta-1} e^{-\alpha x^\beta \lambda(j+l+1)} dx$$

Suppose, we let

$$H(t) = \int_0^t x^{r+\beta-1} e^{-\alpha x^\beta \lambda(j+l+1)} dx \tag{29}$$

Letting,

$z = \alpha x^\beta \lambda(j+l+1), x = z^{1/\beta}[\alpha\lambda(j+l+1)]^{-1/\beta}, dx = {1/\beta}\, z^{\frac{1}{\beta}}[\alpha\lambda(j+l+1)]^{-1/\beta}$ and putting that in equation (),

we have

$$H(t) = {1/\beta}\, [\alpha\lambda(j+l+1)]^{-\left(\frac{r}{\beta}+1\right)} \int_0^t z^{r/\beta} e^{-z} dz \tag{30}$$

Then, we have

$$H(t) = {}^{1}/_{\beta}\,[\alpha\lambda(j+l+1)]^{-\left(\frac{r}{\beta}+1\right)}\Gamma\left\{\left(\frac{r}{\beta}+1\right); \alpha t^{\beta}\lambda(j+l+1)\right\}$$

Finally we have

$$\phi(x) = \alpha^{r/\beta} W_{i,j,l}^{k,\lambda,\theta}\Gamma\left\{\left(\frac{r}{\beta}+1\right); \alpha t^{\beta}\lambda(j+l+1)\right\} \tag{31}$$

2.2 Moment Generating Function $ZTPHW$ Distribution

Moment generating function is a very useful function that can be useful in describing certain properties of the distribution. It can be used to obtain moments of a distribution. The moment generating function of $ZTPHW$ distribution is obtained as follows: The moment generating function of a random variable X is given by

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tX} f(x)dx \tag{32}$$

Where $f(x)$ is given in (12). Using series expansion for e^{tX} given by

$$e^{tX} = \sum_{r=0}^{\infty}\frac{t^r}{r!}x^r \tag{33}$$

Using (33), we can re-write equation (32) as follows

$$M_X(t) = \int_{-\infty}^{\infty}\sum_{r=0}^{\infty}\frac{t^r}{r!}x^r f(x)dt = \sum_{r=0}^{\infty}\frac{t^r}{r!}E(X^r) \tag{34}$$

Putting equation (25) in (34), we have an expression for the moment generating function of $ZTPHW$ distribution as

$$M_X(t) = \sum_{r=0}^{\infty}\frac{t^r\alpha^{r/\beta}}{(r)!}W_{i,j,l}^{k,\lambda,\theta}[\alpha\lambda(j+l+1)]^{-\left(\frac{r}{\beta}\right)}\Gamma\left(\frac{r}{\beta}+1\right) \tag{35}$$

2.3 Bonferroni and Lorenz Curves of $ZTPHW$ Distribution

The Bonferroni and Lorenz curves have been found suitable to have applications not only in economics to study income and poverty, but also in other field like demography, insurance, medicine, and reliability. The Bonferroni and Lorenz curves are defined by

$$B(p) = \frac{1}{p\mu}\int_0^q xf(x;v,w,\lambda)dx \tag{36}$$

and

$$L(p) = \frac{1}{\mu}\int_0^q xf(x;v,w,\lambda)dx \tag{37}$$

Respectively, where $\mu = E(X)$ and $q = F^{-1}(u)$. In the case of $ZTPHW$ distribution, we obtain

$$B(p) = \frac{1}{p\mu}\alpha^{r/\beta}W_{i,j,l}^{k,\lambda,\theta}\Gamma\left\{\left(\frac{r}{\beta}+1\right); \alpha q^{\beta}\lambda(j+l+1)\right\} \tag{38}$$

And

$$L(p) = \frac{1}{\mu}\alpha^{r/\beta} W_{i,j,l}^{k,\lambda,\theta} \Gamma\left\{\left(\frac{r}{\beta}+1\right); \alpha q^{\beta}\lambda(j+l+1)\right\} \tag{39}$$

Where $W_{i,j,l}^{k,\lambda,\theta}$ is as define in equation (25.1)

3. Information Measures

In this section some information measures of the $ZTPHW$ distribution are considered. Namely, Renyi entropy and the Tsallis entropy measures, both measures the variation or uncertainty that may exist in the distribution considered.

3.1 Renyi Entropy

Renyi (1961), gave a useful mathematical expression that can be used to measure the entropy of a $ZTPHW$ distribution given by

$$I_R^{(v)} = \frac{1}{1-v} log\left[\int_0^{\infty} f_{\mathrm{ZTPHW}}(x;\zeta)^{v}\right], \quad v > 0, v \neq 1 \tag{39}$$

And

$$\int_0^{\infty} f^{v}(dx) = \int_0^{\infty}\left\{\frac{\alpha\beta k\lambda\theta^{1/\lambda}}{1-e^{\lambda}} x^{\beta-1}e^{-\alpha x^{\beta}}\left[1-\bar{\theta}e^{-\alpha\lambda x^{\beta}}\right]^{-\left(1+\frac{1}{\lambda}\right)} e^{\left[-k\left\{1-\theta^{\frac{1}{\lambda}}e^{-\alpha x^{\beta}}\left(1-\bar{\theta}e^{-\alpha\lambda x^{\beta}}\right)^{-\frac{1}{\lambda}}\right\}\right]}\right\}^{v} dx \tag{40}$$

$$\int_0^{\infty} f(x;\zeta)^{v} = \sum_{i,j,l=0}^{\infty} \frac{(-1)^{i+j}}{i!} k^{i+v}(\alpha\beta)^{v}\binom{i}{j}\binom{v\left(1+\frac{1}{\lambda}\right)+\frac{j}{\lambda}-1}{l}\theta^{\frac{j+v}{\lambda}}\bar{\theta}^{l}$$

$$\times\int_0^{\infty} x^{v(\beta-1)}e^{-\alpha x^{\beta}(j+v+\lambda l)}\,dx \tag{41}$$

From equation (41), letting

$$H(x) = \int_0^{\infty} x^{v(\beta-1)}e^{-\alpha x^{\beta}(j+v+\lambda l)}\,dx \tag{42}$$

Then substitute,

$w = \alpha x^{\beta}(j+v+\lambda l), x = w^{1/\beta}[\alpha(j+v+\lambda l)]^{-1/\beta}, dx = {}^{1}/_{\beta}\, w^{1/\beta-1}[\alpha(j+v+\lambda l)]^{-1/\beta}$ in equation (42), we

have

$$H(x) = {}^{1}/_{\beta}\,[\alpha(j+v+\lambda l)]^{-\left(\frac{v(\beta-1)+1}{\beta}\right)}\int_0^{\infty} w^{\left(\frac{v(\beta-1)-\beta+1}{\beta}\right)}e^{-w}\,dw$$

Subsequently,

$$H(x) = {}^{1}/_{\beta}\,[\alpha(j+v+\lambda l)]^{-\left(\frac{v(\beta-1)+1}{\beta}\right)}\Gamma\left(\frac{v(\beta-1)+1}{\beta}\right)$$

Therefore,

$$\int_0^{\infty} f(x;\zeta)^{v} = \phi_{i,j,l}^{v,\beta,\theta} k^{i+v}\Gamma\left(\frac{v(\beta-1)+1}{\beta}\right)$$

where

$$\phi_{i,j,l}^{v,\beta,\theta} = \sum_{i,j,l=0}^{\infty} \frac{(-1)^{i+j}}{i!}\binom{i}{j}\binom{v\left(1+\frac{1}{\lambda}\right)+\frac{j}{\lambda}-1}{l}\theta^{\frac{j+v}{\lambda}}\bar{\theta}^{l}[\alpha(j+v+\lambda l)]^{-\left(\frac{v(\beta-1)+1}{\beta}\right)}\alpha^{v}\beta^{v-1} \tag{44}$$

Finally,

$$I_R^{(v)} = \frac{1}{1-v} log\left\{\phi_{i,j,l}^{v,\beta,\theta} k^{i+v}\Gamma\left(\frac{v(\beta-1)+1}{\beta}\right)\right\} \tag{45}$$

3.2 Tsallis Entropy

The Tsallis entropy was first discovered by Havrda and Charvat (1967) and later developed by Tsallis (1988). The Tsallis entropy of the $ZTPHW$ distribution can be defined as

$$I_T^{(v)} = \frac{1}{v-1}\left[1-\int_0^{\infty} f(x;\zeta)^v\right], \quad v>0, v\neq 1 \tag{46}$$

Since,

$$\int_0^{\infty} f(x;\zeta)^v = \int_0^{\infty} f(x;\zeta)^v = \phi_{i,j,l}^{v,\beta,\theta} k^{i+v}\Gamma\left(\frac{v(\beta-1)+1}{\beta}\right)$$

Where $\phi_{i,j,l}^{v,\beta,\theta}$ is as define is equation (44)

Therefore,

$$I_T^{(v)} = \frac{1}{v-1}\left[1-\phi_{i,j,l}^{v,\beta,\theta} k^{i+v}\Gamma\left(\frac{v(\beta-1)+1}{\beta}\right)\right] \tag{47}$$

4. Simulation Study

We perform the simulation study to evaluate the performance of MLEs of $ZTPHW$ distribution. The random number generation is obtained with its quantile function (qf). We note that the u^{th} qf of the $ZTPHW$ is given in equation (17). Hence, if U has a uniform random variable on interval 0 and 1, then X_u has the $ZTPHW$ random variable.

We generated $N = 1000$ samples of sizes 50, 100, 200, 300 and 500 from PHW distribution with its qf. Afterward, we computed the empirical means, standard deviation (SE), absolute bias (AB), mean square errors (MSE) of the MLEs with $AB_{\hat{f}} = \left|\frac{1}{N}\sum_{i=1}^{N}(\hat{f}-f)\right|$ and $MSE_{\hat{f}} = \frac{1}{N}\sum_{i=1}^{N}(\hat{f}-f)^2$, where $f = \alpha,\beta,\theta,\lambda,k$. All results were obtained by using optim's CG routine in the R program. The results obtained from simulation are reported in Table 3. The result shows that as the sample sizes increases the mean square error decreases as expected.

Table 3. The empirical means, AB, SE and MSE of ZTPHW distribution' parameters

$par.$	$sample$	$Mean$	AB	SE	MSE
α	$n = 50$	0.4830	0.6170	0.3840	0.5281
	$n = 100$	0.5116	0.5884	0.3378	0.4563
	$n = 200$	0.5385	0.5615	0.4202	0.4919
	$n = 300$	0.7383	0.3617	0.5963	0.4864
	$n = 400$	0.6636	0.4364	0.4817	0.4225
	$n = 500$	0.7121	0.3879	0.2503	0.2131
k	$n = 50$	2.0104	0.8104	2.9177	9.1697
	$n = 100$	1.7560	0.5560	1.8964	3.9055
	$n = 200$	1.1950	0.0050	1.5067	2.2702
	$n = 300$	0.5975	0.6025	0.2566	0.4288
	$n = 400$	0.6790	0.5210	0.2163	0.3182
	$n = 500$	1.3023	0.1023	0.1667	0.0383
β	$n = 50$	1.1848	0.1152	0.2606	0.0812
	$n = 100$	1.1263	0.1737	0.1837	0.0639
	$n = 200$	1.1769	0.1231	0.1692	0.0438

	$n = 300$	1.1748	0.1252	0.1800	0.0481
	$n = 400$	1.1490	2.1510	0.1017	0.0331
	$n = 500$	1.2399	0.0601	0.0777	0.0096
θ	$n = 50$	0.1111	0.0111	0.1497	0.0225
	$n = 100$	0.1551	0.0551	0.1696	0.0318
	$n = 200$	0.2352	0.1352	0.2312	0.0717
	$n = 300$	0.0598	0.0402	0.0541	0.0045
	$n = 400$	0.0487	0.0513	0.0333	0.0037
	$n = 500$	0.1194	0.0194	0.0098	0.0005
λ	$n = 50$	2.1961	1.6961	3.9891	18.7897
	$n = 100$	2.7017	2.2017	2.5334	11.2656
	$n = 200$	2.6531	2.1531	2.3870	10.3336
	$n = 300$	−2.2189	2.7189	1.6396	10.0807
	$n = 400$	−1.9947	2.4947	1.5155	8.5203
	$n = 500$	1.0453	0.5453	1.3382	2.0881

4.1 Maximum Likelihood Estimates of the Parameters

The maximum likelihood approach is used to estimates the unknown parameters of the distribution. Let $\underline{x} = x_1 \dots, x_n$ represent a random sample drawn from the $ZTPHW$ distribution with parameters $(\zeta = k, \alpha, \beta, \lambda, \theta)$. Then the likelihood function $L(x;\zeta)$ and the log-likelihood function $logL(x;\zeta) = l(x;\zeta)$ corresponding to (48) are respectively given as

$$L(x) = \prod_{i=1}^{n} \frac{\alpha\beta k\lambda\theta^{1/\lambda}}{1-e^{\lambda}} x^{\beta-1} e^{-\alpha x^{\beta}} \left[1-\bar{\theta}e^{-\alpha\lambda x^{\beta}}\right]^{-\left(1+\frac{1}{\lambda}\right)} e^{\left[-k\left\{1-\theta^{\frac{1}{\lambda}}e^{-\alpha x^{\beta}}\left(1-\bar{\theta}e^{-\alpha\lambda x^{\beta}}\right)^{-\frac{1}{\lambda}}\right\}\right]} \tag{48}$$

and

$$l(x) = nlog\left(\frac{\alpha\beta k\lambda\theta^{1/\lambda}}{1-e^{\lambda}}\right) + (\beta-1)\sum_{i=i}^{n} x_i - \left(1+\frac{1}{\lambda}\right)\sum_{i=1}^{n} log\left[1-(1-\theta)e^{-\alpha\lambda x^{\beta}}\right]$$

$$-\alpha\sum_{i=1}^{n} x_i^{\beta} - k\sum_{i=1}^{n}\left\{1-\theta^{\frac{1}{\lambda}}e^{-\alpha x^{\beta}}\left(1-(1-\theta)e^{-\alpha\lambda x^{\beta}}\right)^{-\frac{1}{\lambda}}\right\} \tag{49}$$

The maximum likelihood (ML) method and its procedures can be obtained from literature with details.

4.2 Applications

In this subsection, we evaluate the performance of the $ZTPHW$ model by fitting the distribution to two reliability data sets with other competing distributions namely: Kumaraswamy Lomax (KL) model, Kumaraswamy power Lomax (KPL) model and the Weibull (W) model under the estimated log-likelihood $(\hat{l})$ value, Akaike information criterion (AIC), Consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC), and Hannan-Quinn information criterion (HQIC), Cramer-von Misses (W) and Kolmogorov-Smirnov (KS) statistic with its p-value (P) are used to compare these distributions, where the smaller values of these statistics and larger p-value give the best fit to the data.

The first data set (data set 1) represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal et al. (1960). The observations are as follows: 0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63,1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55. The exploratory data analysis of the data is given below in Table 4 and Table 5 and 6 gives the estimates of the parameters and measures of goodness of fit of ZTPHW distribution. We observe that the data is positively skewed and mesokurtic. Also since the value of variance is less than the value of mean for the pig data it shows that the data is under-dispersed. The total time on test plot in figure 5 shows a concave transform, indicating that the hazard rate function is possibly increasing. The graph of the kernel density for the data is drawn in figure 6 which shows that the data is moderately skewed to the right.

The second data set (data set 2) consists of data of cancer patients. The data represents the remission times (in months) of a random sample of 128 bladder cancer patients from Lee and Wang (2003). The data point is given as: 0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69. The exploratory data analysis of the data is given below in Table7, Table 8 and 9 gives the estimates of the parameters and measures of goodness of fit of $ZTPHW$ distribution. We observe that the data is positively skewed and kleptokurtic. Also, Since the value of mean is less than the value of variance for cancer data, it can be concluded that the data is over-dispersed. The total time on test plot in figure 7 shows a concave transform, indicating that the hazard rate function is possibly increasing. The graph of the kernel density for the data is drawn in figure 8 which shows that the data is moderately skewed to the right. Estimates of the parameters $ZTPHW$, KPL, KL and W distribution; AIC, BIC, CAIC, W and KS and P values

Table 4. Exploratory Data Analysis (EDA) of survival time of pigs

min	Q_1	$median$	$mean$	Q_3	max	$kurtosis$	$skew.$	$var.$	$range$
0.10	1.08	1.495	1.768	2.240	5.550	2.225	1.371	1.070	5.45

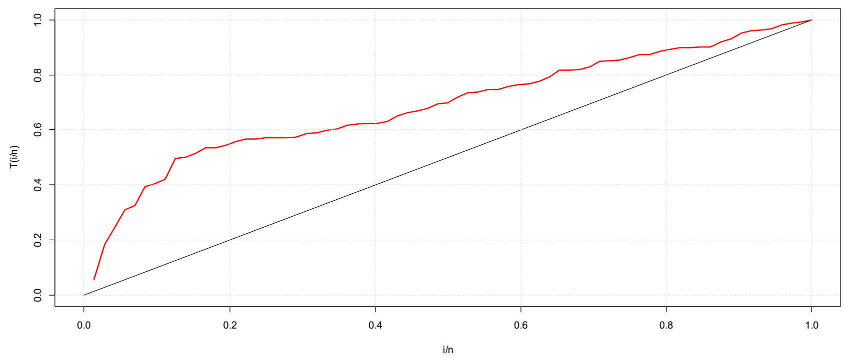

Figure 5. TTT plot for pig data

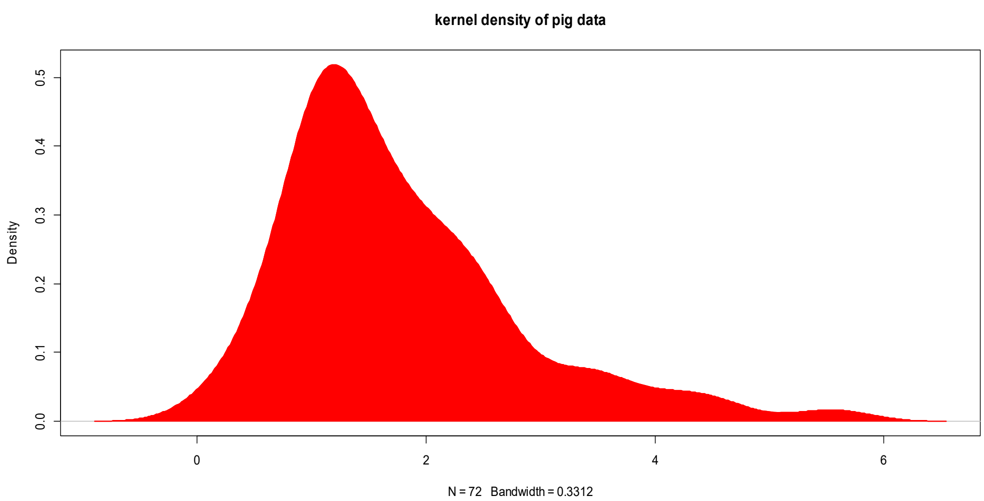

Figure 6. Kernel density plot for pig data

Table 5. Estimates of the parameters of ZTPHW model for pig data

$model$	α	β	λ	θ	k
$ZTPHW$	6.7415 (3.3138)	0.4579 (0.2487)	11.9713 (12.7845)	1.3733 (0.2743)	1.0471 (1.2408)

KPL	10.9947 (1.1577)	0.6672 (0.9233)	2.5119 (2.8190)	2.8679 (0.0114)	7.0135 (6.9522)
KL	3.0086 (0.9339)	2.0675 (8.0888)	6.7747 (2.6373)	4.4996 (1.7179)	− (−)
W	0.2835 (0.0542)	1.8253 (0.1588)	− (−)	− (−)	− (−)

Table 6. Measures of goodness of fit for $ZTPHW$ model for the pig data

$model$	l	AIC	$HQIC$	$CAIC$	BIC	W	K	P
$ZTPHW$	91.725	193.504	198.036	194.413	204.888	0.0520	0.0779	0.7753
KPL	93.556	197.111	201.642	198.020	208.494	0.0611	0.7555	2.2e-16
KL	93.9379	195.876	199.591	196.473	204.982	0.0823	0.9863	2.2e-16
W	95.7898	195.580	197.392	195.754	200.133	0.1649	0.1050	0.4055

Table 7. Exploratory Data Analysis of the Cancer data

min	Q_1	$median$	$mean$	Q_3	max	$kurtosis$	$skew.$	$var.$	$range$
0.08	3.348	6.395	9.366	11.840	79.05	16.154	3.326	110.425	78.85

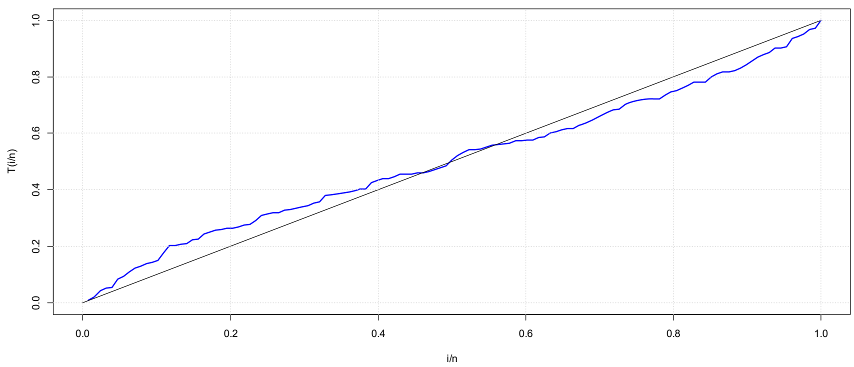

Figure 7. TTT plot for cancer data

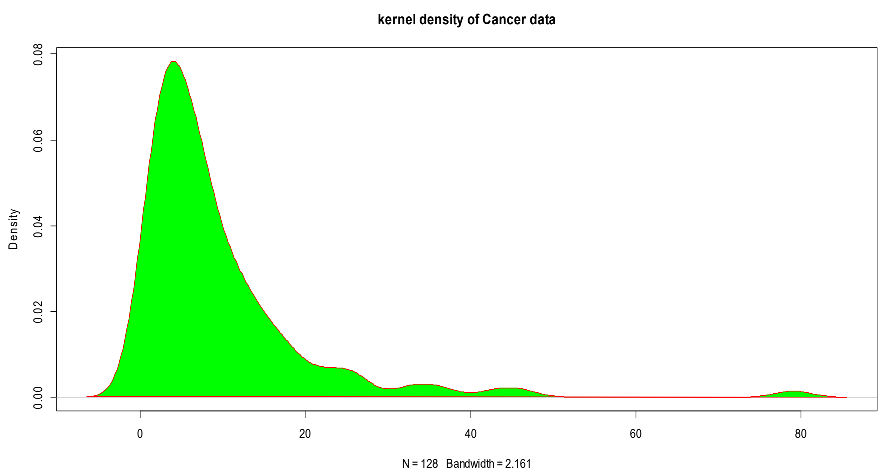

Figure 8. Kernel density plot for cancer data

Table 8. Estimates of the parameters of ZTPHW model for Cancer data

$model$	α	β	λ	θ	k
$ZTPHW$	4.0375 (1.4410)	0.2810 (0.1575)	8.5689 (7.1957)	0.7445 (0.1176)	0.3370 (1.5338)

KPL	1.1325 (0.5450)	0.2288 (0.3261)	1.3093 (0.5305)	10.5116 (1.5057)	22.2195 (1.1052)
KL	0.2678 (0.2678)	0.5600 (3.2362)	7.9419 (1.9495)	13.3280 (0.0289)	− (−)
W	0.0932 (0.0932)	1.0502 (0.0675)	− (−)	− (−)	− (−)

Table 9. Measures of goodness of fit of ZTPHW model for cancer data

$model$	l	AIC	$HQIC$	$CAIC$	BIC	W	K	P
$ZTPHW$	409.691	829.382	835.176	829.874	843.642	0.0201	0.0401	0.9864
KPL	409.802	829.604	835.398	830.096	843.864	0.0923	0.9999	2.2e-16
KL	411.312	830.624	835.729	830.949	842.032	0.0939	0.9931	2.2e-16
W	414.088	832.175	834.493	832.271	837.879	0.1320	0.0771	0.5437

5. Conclusion

In this work, some properties of the newly developed Zero truncated Poisson Harris Weibull model is proposed and studied. The Zero truncated Poisson Harris Weibull model is a generalization of Weibull model. Various properties of the newly developed model are investigated, including ordinary and incomplete moment, Bonferroni and Lorenz curve, Renyl and Tsallis Entropy, moment generating function and hazard function. Simulation study is carried out to investigate the reliability of the method of estimated which shows the adequacy of the method adopted. Two real data sets are fitted to the Zero truncated Poisson Harris Weibull model and compared with other known competing distributions and its sub-model. The results show that the Zero Truncated Harris Weibull model provides a good fit to each data set indicating its flexibility and adaptability in modeling data of various forms of the shape of the hazard function.

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