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A Comparison of Vine and Hierarchical Copulas as Discriminants

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Abstract

Here, we are investigating the possible association among three stochastic variables X, Y, Z. We compare the performance under two separate variants of the copula models; Vine based copula and the hierarchical copula in the context of discrimination.

Keywords: vine, hierarchical, discriminant, copula

1. Introduction

We can use the Copula models when there is a reasonable correlation among the stochastic variables. This is to uncover the structural relationships among the variables in the probabilistic sense. The use of Copulas and its applications began after the pioneering work of Sklar (1959). There are several different types of copulas and each differ based on the strength of the dependence and the direction of association. The Copulas help us to model the dependence structure based on the margins. For additional information, the interested readers are referred to Nelsen (2006), Joe (1993, 1996, 2014), Durante and Sempi (2016), and Mai and Scherer (2017). Here in this paper, we consider four copulas; Gaussian Copula, Clayton Copula, Frank Copula, and Gumbel Copula. The main purpose of this study is to investigate the use of the Vine and the Hierarchical versions of these copulas as discriminants.

The pairwise Vine Copulas and the 'Hierarchical' Copulas are mathematical transformations of the Copulas. In fact, the Vine Copulas are two dimensional representations of higher dimensional copulas. The Vine is a nested structure of connected trees. There are three different Vine structures; R Vine, C Vine, and D Vine. The Vine formulation is very useful in many applications in areas such as Economics, Finance, Actuarial Science, and Engineering. On the other hand, in the case of 'Hierarchical ' Copulas of higher dimensions, we place emphasis in the order of importance. This paper is devoted to the comparison of the Vine Copulas and the Hierarchical Copulas. Shi and Lu (2007) used hierarchical copulas to model two-level clustered data. Prenen et al (2017) tried hierarchical copula on one- level clustered data. Andersen (2004) used hierarchical copula to model familial data. Familial data was further studied by Zhao and Joe (2005) and Othus and Li (2010).

In this paper, we investigate the properties of the Vine Copula and the 'Hierarchical' Copula for the purpose of comparison in the context of discriminant analysis. In the case of higher dimensional multivariate distributions, there is room for singularities when the covariance matrix is nearly singular. However, these transformations such as Vine and 'Hierarchical' Copulas are helpful to avoid such scenarios. We divide the paper into four sections; introduction, methodology, numerical computations, and conclusion.

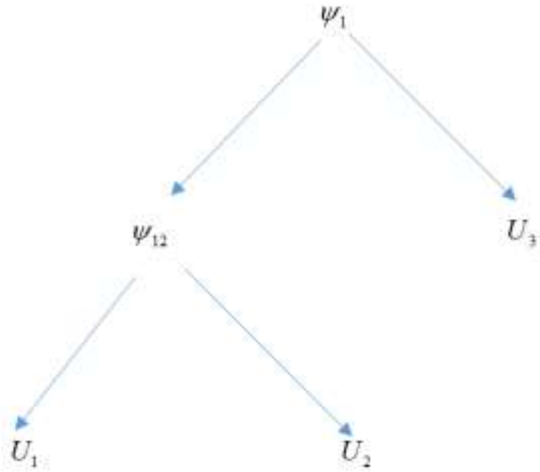
2. Methodology

We aim to compare the performance of the Vine and Hierarchical Copulas based on the discriminant properties.

2.1 Three Variate Hierarchical Copula

In this hierarchical investigation, we assume that the stochastic variable Z is more important than the other two variables. As seen from the accompanying hierarchical copula diagram; at the top level, the generator function is ψ_1. At the next level, the generator is ψ_{12}. There is a hierarchy in the level arrangement. Note that α_1 and α_{12} are the dependence parameters at the first level and second level respectively. The variables that exhibit the higher order of correlation are placed at the higher level.

Three variate Hierarchical Copula Diagram



where $U_1 = F_1(X), \quad U_2 = F_2(Y), \quad U_3 = F_3(Z)$ *are the respective marginal distributions.*

The above diagram can be written as follows by using an Archimedean Copula.

$$C(u_1,u_2,u_3) = \psi_1^{-1}(\psi_1(C_2(u_1,u_2)) + \psi_1(u_3)) \tag{1}$$

In the case of Clayton Copula

$$= [1 + \psi_1(C_2(u_1,u_2)) + \psi_1(u_3)]^{-\frac{1}{\alpha_1}} \tag{2}$$

$$= \left[1 + \psi_1\left(\psi_{12}^{-1}(\psi_{12}(u_1) + \psi_{12}(u_2))\right) + \psi_1(u_3)\right]^{-\frac{1}{\alpha_1}}$$

$$= \left[1 + \left[\psi_{12}^{-1}(\psi_{12}(u_1) + \psi_{12}(u_2))\right]^{-\alpha_1} - 1 + \psi_1(u_3)\right]^{-\frac{1}{\alpha_1}1}$$

$$= \left[\left[\psi_{12}^{-1}(\psi_{12}(u_1) + \psi_{12}(u_2))\right]^{-\alpha_1} + u_3^{-\alpha_1} - 1\right]^{-\frac{1}{\alpha_1}}$$

$$= \left[(1 + \psi_{12}(u_1) + \psi_{12}(u_2))^{\frac{\alpha_1}{\alpha_{12}}} + u_3^{-\alpha_1} - 1\right]^{-\frac{1}{\alpha_1}}$$

$$= \left[\left(1 + u_1^{-\alpha_{12}} + u_2^{-\alpha_{12}} - 2\right)^{\frac{\alpha_1}{\alpha_{12}}} + u_3^{-\alpha_1} - 1\right]^{-\frac{1}{\alpha_1}}$$

$$= \left[\left(u_1^{-\alpha_{12}} + u_2^{-\alpha_{12}} - 1\right)^{\frac{\alpha_1}{\alpha_{12}}} + u_3^{-\alpha_1} - 1\right]^{-\frac{1}{\alpha_1}} \tag{3}$$

When $\alpha_1 = \alpha_{12}$ then this three-variate hierarchical Clayton Copula will become a regular three-variate Clayton Copula.

$$C(u_1,u_2,u_3) = \left[\left(u_1^{-\alpha_1} + u_2^{-\alpha_1} - 1\right) + u_3^{-\alpha_1} - 1\right]^{-\frac{1}{\alpha_1}} \tag{4}$$

In the case of Gumbel Copula,

$$C(u_1,u_2,u_3) = \psi_1^{-1}(\psi_1(C_2(u_1,u_2)) + \psi_1(u_3)) \tag{5}$$

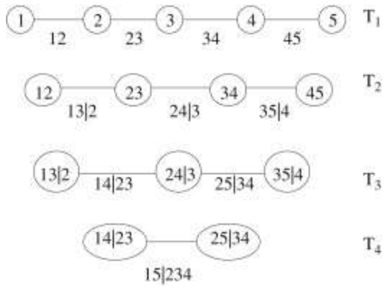
$$= \psi_1^{-1}\left(\psi_1\left(\psi_{12}^{-1}\left(\psi_{12}(u_1)+\psi_{12}(u_2)\right)\right)+\psi_1(u_3)\right)$$

$$= e^{-\left\{\left[(-\ln u_1)^{\alpha_{12}}+(-\ln u_2)^{\alpha_{12}}\right]^{\frac{\alpha_1}{\alpha_{12}}}+(-\ln u_3)^{\alpha_1}\right\}^{\frac{1}{\alpha_1}}} \quad (6)$$

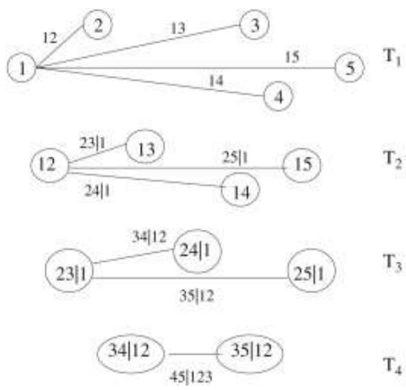
Again, note that when $\alpha_1 = \alpha_{12}$ then this three-variate hierarchical Gumbel Copula will become a regular three-variate Gumbel Copula.

$$C(u_1,u_2,u_3) = e^{-\left\{\left[(-\ln u_1)^{\alpha_{12}}+(-\ln u_2)^{\alpha_{12}}\right]+(-\ln u_3)^{\alpha_1}\right\}^{\frac{1}{\alpha_1}}} \quad (7)$$

D-Vine Copula (5 dimensional) description



C-Vine Copula (5 dimensional) description



Note: In the case of 3 dimensional Copulas, C-Vine = D-Vine

Comparison of Vine based Copulas

Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be a three variable vector and $f(x,y,z)$ be the density function. Then, by using the properties of the Vine copulas, one can write

$$f(x,y,z) = f_1(x).f_2(y).f_3(z).c_{12}.c_{23}.c_{13\backslash 2}$$

where, $f_1(x)$ is the marginal density of X

$f_2(y)$ is the marginal density of Y

$f_3(z)$ is the marginal density of Z

c_{12} is the pairwise copula density of X and Y

c_{23} is the pairwise copula density of Y and Z

$c_{13\backslash 2}$ is the pairwise copula density of X and Z **given** Y

Let us assume that the marginal densities are normal.

Then, $u_1 = \Phi\left(\frac{x-\mu_1}{\sigma_1}\right)$, $u_2 = \Phi\left(\frac{y-\mu_2}{\sigma_2}\right)$, $u_3 = \Phi\left(\frac{z-\mu_3}{\sigma_3}\right)$

Vine based on the Gaussian Copula:

Here we investigate the construction of the vine based on the Gaussian Copula. The Gaussian Copula densities are given as follows.

$$c_{12} = \frac{1}{\sqrt{1-\rho_{12}^{2}}} \cdot e^{-\frac{1}{2.(1-\rho_{12}^2)}\left\{\rho_{12}^2.\left(\frac{x-\mu_1}{\sigma_1}\right)^2+\rho_{12}^2.\left(\frac{y-\mu_2}{\sigma_2}\right)^2-2.\rho_{12}.\left(\frac{x-\mu_1}{\sigma_1}\right).\left(\frac{y-\mu_2}{\sigma_2}\right)\right\}} \tag{8}$$

$$c_{23} = \frac{1}{\sqrt{1-\rho_{23}^{2}}} \cdot e^{-\frac{1}{2.(1-\rho_{23}^2)}\left\{\rho_{23}^2.\left(\frac{y-\mu_2}{\sigma_2}\right)^2+\rho_{23}^2.\left(\frac{z-\mu_3}{\sigma_3}\right)^2-2.\rho_{23}.\left(\frac{y-\mu_2}{\sigma_2}\right).\left(\frac{z-\mu_3}{\sigma_3}\right)\right\}} \tag{9}$$

$$c_{13\backslash 2} = \frac{1}{\sqrt{1-\rho_{13\backslash 2}^{2}}} \cdot e^{-\frac{1}{2.(1-\rho_{13\backslash 2}^2)}\left\{\rho_{13\backslash 2}^2.\left(\frac{x-\mu_{1\backslash 2}}{\sigma_{1\backslash 2}}\right)^2+\rho_{13\backslash 2}^2.\left(\frac{z-\mu_{3\backslash 2}}{\sigma_{3\backslash 2}}\right)^2-2.\rho_{13\backslash 2}.\left(\frac{x-\mu_{1\backslash 2}}{\sigma_{1\backslash 2}}\right).\left(\frac{z-\mu_{3\backslash 2}}{\sigma_{3\backslash 2}}\right)\right\}} \tag{10}$$

For an arbitrary tri-variate normal population, the conditional density

$$f_{13\backslash 2}(x,z \backslash y) = \frac{f(x,y,z)}{f_2(y)}$$

$$= \frac{\sqrt{\sigma_{22}}}{2\pi.|\Sigma|^{1/2}} \cdot e^{-\frac{1}{2}\left\{a_{11}.(x-\mu_1)^2+\left(a_{22}-\frac{1}{\sigma_{22}}\right).(y-\mu_2)^2+a_{33}.(z-\mu_3)^2+2.a_{12}.(x-\mu_1).(y-\mu_2)+2.a_{13}.(x-\mu_1).(z-\mu_3)+2.a_{23}.(y-\mu_2).(z-\mu_3)\right\}} \tag{11}$$

Where is the $(i,j)^{th}$ entry of the inverse covariance matrix of Σ, and σ_{ij} is the $(i,j)^{th}$ entry of the covariance matrix Σ.

Note that,

$$E(X.Z \backslash Y) = \mu_1.\mu_3 + \frac{(a_{12}.a_{13}-a_{11}.a_{23})}{(a_{11}.a_{33}-a_{13}^2)}.\mu_1.(y-\mu_2) - \frac{(a_{12}.a_{33}-a_{13}.a_{23})}{(a_{11}.a_{33}-a_{13}^2)}.(y-\mu_2)$$
$$+\frac{(a_{12}.a_{33}-a_{13}.a_{23}).(a_{11}.a_{23}-a_{12}.a_{13}).(y-\mu_2)^2}{(a_{11}.a_{33}-a_{13}^2)^2} - \frac{a_{13}}{(a_{11}.a_{33}-a_{13}^2)} \tag{12}$$

Also, note that $E(X \backslash Y) = \mu_1 + \rho_{12}.\frac{\sqrt{\sigma_{11}}}{\sqrt{\sigma_{22}}}.(y-\mu_2)$

$$= \mu_1 + \frac{(a_{13}.a_{23}-a_{22}.a_{12})}{(a_{11}.a_{33}-a_{13}^2)}.(y-\mu_2) \tag{13}$$

and $E(Z \backslash Y) = \mu_3 + \rho_{23}.\frac{\sqrt{\sigma_{33}}}{\sqrt{\sigma_{22}}}.(y-\mu_2)$

$$= \mu_3 + \frac{(a_{13}.a_{12}-a_{11}.a_{23})}{(a_{11}.a_{33}-a_{13}^2)}.(y-\mu_2) \tag{14}$$

Also,

$$Cov(X, Z \backslash Y) = \frac{-a_{13}}{(a_{11}.a_{33} - a_{13}^{\;2})}$$

$$= \sigma_{13} - \frac{\sigma_{12}.\sigma_{23}}{\sigma_{22}} \tag{15}$$

$$Var(X \backslash Y) = \sigma_{11} - \frac{\sigma_{12}^{\;2}}{\sigma_{22}} \tag{16}$$

$$Var(Z \backslash Y) = \sigma_{33} - \frac{\sigma_{23}^{\;2}}{\sigma_{22}} \tag{17}$$

Note that $\rho_{13\backslash 2} = \dfrac{\operatorname{cov}(X, Z \backslash Y)}{\sqrt{Var(X \backslash Y)}.\sqrt{Var(Z \backslash Y)}}$ (18)

For the Vine Copula based approach, the likelihood ratio which depends on the Gaussian model is

$$L_G = \frac{f_1^{(2)}(x).f_2^{(2)}(y).f_3^{(2)}(z).c_{12}^{\;(2)}.c_{23}^{\;(2)}.c_{13\backslash 2}^{\;(2)}}{f_1^{(1)}(x).f_2^{(1)}(y).f_3^{(1)}(z).c_{12}^{\;(1)}.c_{23}^{\;(1)}.c_{13\backslash 2}^{\;(1)}} \tag{19}$$

The log-likelihood ratio is given by

$$\ln(L_G) = \ln\left(\frac{f_1^{(2)}(x)}{f_1^{(1)}(x)}\right) + \ln\left(\frac{f_2^{(2)}(y)}{f_2^{(1)}(y)}\right) + \ln\left(\frac{f_3^{(2)}(z)}{f_3^{(1)}(z)}\right) + \ln\left(\frac{c_{12}^{\;(2)}}{c_{12}^{\;(1)}}\right) + \ln\left(\frac{c_{23}^{\;(2)}}{c_{23}^{\;(1)}}\right) + \ln\left(\frac{c_{13\backslash 2}^{\;(2)}}{c_{13\backslash 2}^{\;(1)}}\right) \tag{20}$$

where the superscripts (1) and (2) represent populations (1) and (2) respectively.

Vine based on Clayton Copula

So for the Clayton Copula based on the Vine,

$$c_{12} = (1+\alpha_{12}).\left\{\Phi\left(\frac{x-\mu_1}{\sigma_1}\right).\Phi\left(\frac{y-\mu_2}{\sigma_2}\right)\right\}^{-(1+\alpha_{12})} .\left\{\left(\Phi\left(\frac{x-\mu_1}{\sigma_1}\right)^{-\alpha_{12}} + \Phi\left(\frac{y-\mu_2}{\sigma_2}\right)^{-\alpha_{12}} - 1\right)^{-\left(2+\frac{1}{\alpha_{12}}\right)}\right\} \tag{21}$$

$$c_{23} = (1+\alpha_{23}).\left\{\Phi\left(\frac{y-\mu_2}{\sigma_2}\right).\Phi\left(\frac{z-\mu_3}{\sigma_3}\right)\right\}^{-(1+\alpha_{23})} .\left\{\left(\Phi\left(\frac{y-\mu_2}{\sigma_2}\right)^{-\alpha_{23}} + \Phi\left(\frac{z-\mu_3}{\sigma_3}\right)^{-\alpha_{23}} - 1\right)^{-\left(2+\frac{1}{\alpha_{23}}\right)}\right\} \tag{22}$$

$$c_{13\backslash 2} = (1+\alpha_{13\backslash 2}).\left\{\Phi\left(\frac{x-\mu_{1\backslash 2}}{\sigma_{1\backslash 2}}\right).\Phi\left(\frac{z-\mu_{3\backslash 2}}{\sigma_{3\backslash 2}}\right)\right\}^{-(1+\alpha_{13\backslash 2})} .\left\{\left(\Phi\left(\frac{x-\mu_{1\backslash 2}}{\sigma_{1\backslash 2}}\right)^{-\alpha_{13\backslash 2}} + \Phi\left(\frac{z-\mu_{3\backslash 2}}{\sigma_{3\backslash 2}}\right)^{-\alpha_{13\backslash 2}} - 1\right)^{-\left(2+\frac{1}{\alpha_{13\backslash 2}}\right)}\right\} \tag{23}$$

So for the Vine based Clayton Copula, the joint density function

$$f(x,y,z)=(1+\alpha_{12}).\left\{\Phi\left(\frac{x-\mu_1}{\sigma_1}\right).\Phi\left(\frac{y-\mu_2}{\sigma_2}\right)\right\}^{-(1+\alpha_{12})}.\left\{\left(\Phi\left(\frac{x-\mu_1}{\sigma_1}\right)^{-\alpha_{12}}+\Phi\left(\frac{y-\mu_2}{\sigma_2}\right)^{-\alpha_{12}}-1\right)^{-\left(2+\frac{1}{\alpha_{12}}\right)}\right\}$$

$$.\ (1+\alpha_{23}).\left\{\Phi\left(\frac{y-\mu_2}{\sigma_2}\right).\Phi\left(\frac{z-\mu_3}{\sigma_3}\right)\right\}^{-(1+\alpha_{23})}.\left\{\left(\Phi\left(\frac{y-\mu_2}{\sigma_2}\right)^{-\alpha_{23}}+\Phi\left(\frac{z-\mu_3}{\sigma_3}\right)^{-\alpha_{23}}-1\right)^{-\left(2+\frac{1}{\alpha_{23}}\right)}\right\}$$

$$.\ (1+\alpha_{13\backslash 2}).\left\{\Phi\left(\frac{x-\mu_{1\backslash 2}}{\sigma_{1\backslash 2}}\right).\Phi\left(\frac{z-\mu_{3\backslash 2}}{\sigma_{3\backslash 2}}\right)\right\}^{-(1+\alpha_{13\backslash 2})}.\left\{\left(\Phi\left(\frac{x-\mu_{1\backslash 2}}{\sigma_{1\backslash 2}}\right)^{-\alpha_{13\backslash 2}}+\Phi\left(\frac{z-\mu_{3\backslash 2}}{\sigma_{3\backslash 2}}\right)^{-\alpha_{13\backslash 2}}-1\right)^{-\left(2+\frac{1}{\alpha_{13\backslash 2}}\right)}\right\}$$

$$.\ f_1(x).f_2(y).f_3(z) \tag{24}$$

Note that,

$$f_1(x)=\phi\left(\frac{x-\mu_1}{\sigma_1}\right),\qquad f_2(y)=\phi\left(\frac{y-\mu_2}{\sigma_2}\right),\qquad f_3(z)=\phi\left(\frac{z-\mu_3}{\sigma_3}\right)$$

$$\mu_{3\backslash 2} = \mu_3+\frac{(a_{13}.a_{12}-a_{11}.a_{23})}{\left(a_{11}.a_{33}-a_{13}{}^2\right)}.(y-\mu_2) \tag{25}$$

$$\mu_{1\backslash 2} = \mu_1+\frac{(a_{13}.a_{23}-a_{22}.a_{12})}{\left(a_{11}.a_{33}-a_{13}{}^2\right)}.(y-\mu_2) \tag{26}$$

$$\sigma_{1\backslash 2}^2=\sigma_{11}-\frac{\sigma_{12}{}^2}{\sigma_{22}} \tag{27}$$

$$\sigma_{3\backslash 2}^2=\sigma_{33}-\frac{\sigma_{23}{}^2}{\sigma_{22}} \tag{28}$$

Vine based on the Gumbel Copula

$$C(u_1,u_2)=\psi^{-1}(\psi(u_1)+\psi(u_2))$$

$$=e^{-\left\{(-\ln(u_1))^{\alpha}+(-\ln(u_2))^{\alpha}\right\}^{\frac{1}{\alpha}}} \tag{29}$$

The Copula density is given by $\frac{\partial^2 C}{\partial u_1 \partial u_2}$

$$c_{12}=\frac{1}{u_1u_2}(-\ln(u_1))^{\alpha-1}(-\ln(u_2))^{\alpha-1}\ e^{-\left\{(-\ln(u_1))^{\alpha}+(-\ln(u_2))^{\alpha}\right\}^{\frac{1}{\alpha}}}.$$

$$. \left\{\left[(-\ln(u_1))^{\alpha} + (-\ln(u_2))^{\alpha}\right]^{\frac{2}{\alpha}-2} + (\alpha-1)\left[(-\ln(u_1))^{\alpha} + (-\ln(u_2))^{\alpha}\right]^{\frac{1}{\alpha}-2}\right\} \quad (30)$$

$$c_{23} = \frac{1}{u_2 u_3}(-\ln(u_2))^{\alpha-1}(-\ln(u_3))^{\alpha-1} e^{-\left\{(-\ln(u_2))^{\alpha}+(-\ln(u_3))^{\alpha}\right\}^{\frac{1}{\alpha}}}.$$

$$. \left\{\left[(-\ln(u_2))^{\alpha} + (-\ln(u_3))^{\alpha}\right]^{\frac{2}{\alpha}-2} + (\alpha-1)\left[(-\ln(u_2))^{\alpha} + (-\ln(u_3))^{\alpha}\right]^{\frac{1}{\alpha}-2}\right\} \quad (31)$$

$$c_{13\backslash 2} = \frac{1}{u_{1\backslash 2} u_{3\backslash 2}}(-\ln(u_{1\backslash 2}))^{\alpha-1}(-\ln(u_{3\backslash 2}))^{\alpha-1} e^{-\left\{(-\ln(u_{1\backslash 2}))^{\alpha}+(-\ln(u_{3\backslash 2}))^{\alpha}\right\}^{\frac{1}{\alpha}}}.$$

$$. \left\{\left[(-\ln(u_{1\backslash 2}))^{\alpha} + (-\ln(u_{3\backslash 2}))^{\alpha}\right]^{\frac{2}{\alpha}-2} + (\alpha-1)\left[(-\ln(u_{1\backslash 2}))^{\alpha} + (-\ln(u_{3\backslash 2}))^{\alpha}\right]^{\frac{1}{\alpha}-2}\right\} \quad (32)$$

So, the joint density function for the Vine based Gumbel Copula is

$$f(x,y,z) = \frac{1}{\Phi\left(\frac{x-\mu_1}{\sigma_1}\right)\Phi\left(\frac{y-\mu_2}{\sigma_2}\right)}\left(-\ln\left(\Phi\left(\frac{x-\mu_1}{\sigma_1}\right)\right)\right)^{\alpha-1}\left(-\ln\left(\Phi\left(\frac{y-\mu_2}{\sigma_2}\right)\right)\right)^{\alpha-1}$$

$$e^{-\left\{\left(-\ln\left(\Phi\left(\frac{x-\mu_1}{\sigma_1}\right)\right)\right)^{\alpha}+\left(-\ln\left(\Phi\left(\frac{y-\mu_2}{\sigma_2}\right)\right)\right)^{\alpha}\right\}^{\frac{1}{\alpha}}}.$$

.

$$\left\{\left[\left(-\ln\left(\Phi\left(\frac{x-\mu_1}{\sigma_1}\right)\right)\right)^{\alpha}+\left(-\ln\left(\frac{y-\mu_2}{\sigma_2}\right)\right)^{\alpha}\right]^{\frac{2}{\alpha}-2}+(\alpha-1)\left[\left(-\ln\left(\Phi\left(\frac{x-\mu_1}{\sigma_1}\right)\right)\right)^{\alpha}+\left(-\ln\left(\Phi\left(\frac{y-\mu_2}{\sigma_2}\right)\right)\right)^{\alpha}\right]^{\frac{1}{\alpha}-2}\right\}$$

$$. \frac{1}{\Phi\left(\frac{y-\mu_2}{\sigma_2}\right)\Phi\left(\frac{z-\mu_3}{\sigma_3}\right)}\left(-\ln\left(\Phi\left(\frac{y-\mu_2}{\sigma_2}\right)\right)\right)^{\alpha-1}\left(-\ln\left(\Phi\left(\frac{z-\mu_3}{\sigma_3}\right)\right)\right)^{\alpha-1}$$

$$e^{-\left\{\left(-\ln\left(\Phi\left(\frac{y-\mu_2}{\sigma_2}\right)\right)\right)^{\alpha}+\left(-\ln\left(\Phi\left(\frac{z-\mu_3}{\sigma_3}\right)\right)\right)^{\alpha}\right\}^{\frac{1}{\alpha}}}$$

$$. \left\{\left[\left(-\ln\left(\Phi\left(\frac{y-\mu_2}{\sigma_2}\right)\right)\right)^{\alpha}+\left(-\ln\left(\Phi\left(\frac{z-\mu_3}{\sigma_3}\right)\right)\right)^{\alpha}\right]^{\frac{2}{\alpha}-2}+(\alpha-1)\left[\left(-\ln\left(\Phi\left(\frac{y-\mu_2}{\sigma_2}\right)\right)\right)^{\alpha}+\left(-\ln\left(\Phi\left(\frac{z-\mu_3}{\sigma_3}\right)\right)\right)^{\alpha}\right]^{\frac{1}{\alpha}-2}\right\}.$$

$$\frac{1}{\Phi\left(\frac{x-\mu_{1\backslash 2}}{\sigma_{1\backslash 2}}\right)\Phi\left(\frac{z-\mu_{3\backslash 2}}{\sigma_{3\backslash 2}}\right)}\left(-\ln\left(\Phi\left(\frac{x-\mu_{1\backslash 2}}{\sigma_{1\backslash 2}}\right)\right)\right)^{\alpha-1}\left(-\ln\left(\Phi\left(\frac{z-\mu_{3\backslash 2}}{\sigma_{3\backslash 2}}\right)\right)\right)^{\alpha-1}$$

$$e^{-\left\{\left(-\ln\left(\Phi\left(\frac{x-\mu_{1\backslash 2}}{\sigma_{1\backslash 2}}\right)\right)\right)^{\alpha}+\left(-\ln\left(\Phi\left(\frac{z-\mu_{3\backslash 2}}{\sigma_{3\backslash 2}}\right)\right)\right)^{\alpha}\right\}^{\frac{1}{\alpha}}}$$

$$\left\{\left[\left(-\ln\left(\Phi\left(\frac{x-\mu_{1\backslash 2}}{\sigma_{1\backslash 2}}\right)\right)\right)^{\alpha}+\left(-\ln\left(\Phi\left(\frac{z-\mu_{3\backslash 2}}{\sigma_{3\backslash 2}}\right)\right)\right)^{\alpha}\right]^{\frac{2}{\alpha}-2}+(\alpha-1)\left[\left(-\ln\left(\Phi\left(\frac{x-\mu_{1\backslash 2}}{\sigma_{1\backslash 2}}\right)\right)\right)^{\alpha}+\left(-\ln\left(\Phi\left(\frac{z-\mu_{3\backslash 2}}{\sigma_{3\backslash 2}}\right)\right)\right)^{\alpha}\right]^{\frac{1}{\alpha}-2}\right\}$$

$$.\ f_1(x).\ f_2(y).\ f_3(z) \tag{33}$$

Hierarchical Clayton Copula

From equation (3), it follows that the Hierarchical Clayton Copula is

$$C(u_1,u_2,u_3)=\left[\left(u_1^{-\alpha_{12}}+u_2^{-\alpha_{12}}-1\right)^{\frac{\alpha_1}{\alpha_{12}}}+u_3^{-\alpha_1}-1\right]^{-\frac{1}{\alpha_1}} \tag{34}$$

So, the mixed partial derivative,

$$\frac{\partial^3 C(u_1,u_2,u_3)}{\partial u_1\partial u_2\partial u_3}.=$$

$$(1+\alpha_1).(\alpha_{12}-\alpha_1).u_1^{-\alpha_{12}-1}.u_2^{-\alpha_{12}-1}.u_3^{-\alpha_1-1}.\left(\left(u_1^{-\alpha_{12}}+u_2^{-\alpha_{12}}-1\right)\right)^{\frac{\alpha_1}{\alpha_{12}}-2}.\left[\left(u_1^{-\alpha_{12}}+u_2^{-\alpha_{12}}-1\right)^{\frac{\alpha_1}{\alpha_{12}}}+u_3^{-\alpha_1}\right]^{-\frac{1}{\alpha_1}-1}+$$

$$\left(\alpha_1^2-1\right).\frac{\alpha_1}{\alpha_{12}}.u_1^{-\alpha_{12}-1}.u_2^{-\alpha_{12}-1}.u_3^{-\alpha_1-1}.\left(\left(u_1^{-\alpha_{12}}+u_2^{-\alpha_{12}}-1\right)\right)^{\frac{2.\alpha_1}{\alpha_{12}}-2}.\left[\left(u_1^{-\alpha_{12}}+u_2^{-\alpha_{12}}-1\right)^{\frac{\alpha_1}{\alpha_{12}}}+u_3^{-\alpha_1}\right]^{-\frac{1}{\alpha_1}-2} \tag{35}$$

For the hierarchical copula, the Clayton Copula based joint density function

$$g(x,y,z)=\left\{\begin{array}{l}(1+\alpha_1).(\alpha_{12}-\alpha_1).u_1^{-\alpha_{12}-1}.u_2^{-\alpha_{12}-1}.u_3^{-\alpha_1-1}\left(u_1^{-\alpha_{12}}+u_2^{-\alpha_{12}}-1\right)^{\frac{\alpha_1}{\alpha_{12}}-2}.\left[\left(u_1^{-\alpha_{12}}+u_2^{-\alpha_{12}}-1\right)^{\frac{\alpha_1}{\alpha_{12}}}+u_3^{-\alpha_1}\right]^{-\frac{1}{\alpha_1}-1}\\ +\left(\alpha_1^2+1\right).\frac{\alpha_1}{\alpha_{12}}.u_1^{-\alpha_{12}-1}.u_2^{-\alpha_{12}-1}.u_3^{-\alpha_1-1}.\left(u_1^{-\alpha_{12}}+u_2^{-\alpha_{12}}-1\right)^{\frac{2.\alpha_1}{\alpha_{12}}-2}.\left[\left(u_1^{-\alpha_{12}}+u_2^{-\alpha_{12}}-1\right)^{\frac{\alpha_1}{\alpha_{12}}}+u_3^{-\alpha_1}\right]^{-\frac{1}{\alpha_1}-2}\end{array}\right\}$$

$$.\ f_1(x).f_2(y).f_3(z) \tag{36}$$

This in turn means,

$$g(x,y,z)=\left\{\begin{array}{l}(1+\alpha_1).(\alpha_{12}-\alpha_1).\Phi\left(\frac{x-\mu_1}{\sigma_1}\right)^{-\alpha_{12}-1}.\Phi\left(\frac{y-\mu_2}{\sigma_2}\right)^{-\alpha_{12}-1}.\Phi\left(\frac{z-\mu_3}{\sigma_3}\right)^{-\alpha_3-1}\\ .\left[\Phi\left(\frac{x-\mu_1}{\sigma_1}\right)^{-\alpha_{12}}+\Phi\left(\frac{y-\mu_2}{\sigma_2}\right)^{-\alpha_{12}}-1\right]^{\frac{\alpha_1}{\alpha_{12}}-2}\\ .\left[\left(\Phi\left(\frac{x-\mu_1}{\sigma_1}\right)^{-\alpha_{12}}+\Phi\left(\frac{y-\mu_2}{\sigma_2}\right)^{-\alpha_{12}}-1\right)^{\frac{\alpha_1}{\alpha_{12}}}+\Phi\left(\frac{z-\mu_3}{\sigma_3}\right)^{-\alpha_3}\right]^{\frac{-1}{\alpha_1}-1}\\ +\left(\alpha_1^2+1\right).\frac{\alpha_1}{\alpha_{12}}.\Phi\left(\frac{x-\mu_1}{\sigma_1}\right)^{-\alpha_{12}-1}.\Phi\left(\frac{y-\mu_2}{\sigma_2}\right)^{-\alpha_{12}-1}.\Phi\left(\frac{z-\mu_3}{\sigma_3}\right)^{-\alpha_1-1}\\ .\left[\Phi\left(\frac{x-\mu_1}{\sigma_1}\right)^{-\alpha_{12}}+\Phi\left(\frac{y-\mu_2}{\sigma_2}\right)^{-\alpha_{12}}-1\right]^{\frac{2.\alpha_2}{\alpha_{12}}-2}\\ .\left[\left(\Phi\left(\frac{x-\mu_1}{\sigma_1}\right)^{-\alpha_{12}}+\Phi\left(\frac{y-\mu_2}{\sigma_2}\right)^{-\alpha_{12}}-1\right)^{\frac{\alpha_1}{\alpha_{12}}}+\Phi\left(\frac{z-\mu_3}{\sigma_3}\right)^{-\alpha_1}\right]^{\frac{-1}{\alpha_1}-2}\end{array}\right\}$$

$$f_1(x).f_2(y).f_3(z) \tag{37}$$

Hierarchical Gumbel Copula

From equation (6), it follows that the Hierarchical Gumbel Copula is

$$C(u_1,u_2,u_3) = e^{-\left\{\left[(-\ln u_1)^{\alpha_{12}}+(-\ln u_2)^{\alpha_{12}}\right]^{\frac{\alpha_1}{\alpha_{12}}}+(-\ln u_3)^{\alpha_1}\right\}^{\frac{1}{\alpha_1}}} \tag{38}$$

$$\frac{\partial^3 C(u_1,u_2,u_3)}{\partial u_1\partial u_2\partial u_3} = -e^{-\left\{\left[(-\ln u_1)^{\alpha_{12}}+(-\ln u_2)^{\alpha_{12}}\right]^{\frac{\alpha_1}{\alpha_{12}}}+(-\ln u_3)^{\alpha_1}\right\}^{\frac{1}{\alpha_1}}}.\left[(-\ln u_1)^{\alpha_{12}}+(-\ln u_2)^{\alpha_{12}}\right]^{\frac{2\,\alpha_1}{\alpha_{12}}-2}$$

$$\left\{\left[(-\ln u_1)^{\alpha_{12}}+(-\ln u_2)^{\alpha_{12}}\right]^{\frac{\alpha_1}{\alpha_{12}}}+(-\ln u_3)^{\alpha_1}\right\}^{\frac{2}{\alpha_1}-3}.(-\ln u_1)^{\alpha_{12}-1}.(-\ln u_2)^{\alpha_{12}-1}.(-\ln u_3)^{\alpha_1-1}.\frac{1}{u_1\,u_2\,u_3}$$

$$+(2\alpha_1-1)\,e^{-\left\{\left[(-\ln u_1)^{\alpha_{12}}+(-\ln u_2)^{\alpha_{12}}\right]^{\frac{\alpha_1}{\alpha_{12}}}+(-\ln u_3)^{\alpha_1}\right\}^{\frac{1}{\alpha_1}}}.\left[(-\ln u_1)^{\alpha_{12}}+(-\ln u_2)^{\alpha_{12}}\right]^{\frac{2\,\alpha_1}{\alpha_{12}}-2}$$

$$\cdot\left\{\left[(-\ln u_1)^{\alpha_{12}}+(-\ln u_2)^{\alpha_{12}}\right]^{\frac{\alpha_1}{\alpha_{12}}}+(-\ln u_3)^{\alpha_1}\right\}^{\frac{1}{\alpha_1}-3}.(-\ln u_1)^{\alpha_{12}-1}.(-\ln u_2)^{\alpha_{12}-1}.(-\ln u_3)^{\alpha_1-1}.\frac{1}{u_1\,u_2\,u_3}$$

$$. \left\{ \alpha_1 (\alpha_1 - 1) + \left[\left[(-\ln u_1)^{\alpha_{12}} + (-\ln u_2)^{\alpha_{12}} \right]^{\frac{\alpha_1}{\alpha_{12}}} + (-\ln u_3)^{\alpha_1} \right]^{\frac{1}{\alpha_1}} \right\}$$

$$+ \ (\alpha_{12} - \alpha_1) \ e^{-\left\{ \left[(-\ln u_1)^{\alpha_{12}} + (-\ln u_2)^{\alpha_{12}} \right]^{\frac{\alpha_1}{\alpha_{12}}} + (-\ln u_3)^{\alpha_1} \right\}^{\frac{1}{\alpha_1}}} . \left[(-\ln u_1)^{\alpha_{12}} + (-\ln u_2)^{\alpha_{12}} \right]^{\frac{\alpha_1}{\alpha_{12}} - 2}$$

$$\cdot \left\{ \left[(-\ln u_1)^{\alpha_{12}} + (-\ln u_2)^{\alpha_{12}} \right]^{\frac{\alpha_1}{\alpha_{12}}} + (-\ln u_3)^{\alpha_1} \right\}^{\frac{1}{\alpha_1} - 2} . (-\ln u_1)^{\alpha_{12} - 1} . (-\ln u_2)^{\alpha_{12} - 1} . (-\ln u_3)^{\alpha_1 - 1} . \frac{1}{u_1 \ u_2 \ u_3}$$

$$. \left\{ \alpha_1 (\alpha_1 - 1) + \left[\left[(-\ln u_1)^{\alpha_{12}} + (-\ln u_2)^{\alpha_{12}} \right]^{\frac{\alpha_1}{\alpha_{12}}} + (-\ln u_3)^{\alpha_1} \right]^{\frac{1}{\alpha_1}} \right\}$$

$$+ \ e^{-\left\{ \left[(-\ln u_1)^{\alpha_{12}} + (-\ln u_2)^{\alpha_{12}} \right]^{\frac{\alpha_1}{\alpha_{12}}} + (-\ln u_3)^{\alpha_1} \right\}^{\frac{1}{\alpha_1}}} . \left[(-\ln u_1)^{\alpha_{12}} + (-\ln u_2)^{\alpha_{12}} \right]^{\frac{2\alpha_1}{\alpha_{12}} - 2}$$

$$\cdot \left\{ \left[(-\ln u_1)^{\alpha_{12}} + (-\ln u_2)^{\alpha_{12}} \right]^{\frac{\alpha_1}{\alpha_{12}}} + (-\ln u_3)^{\alpha_1} \right\}^{\frac{2}{\alpha_1} - 3} . (-\ln u_1)^{\alpha_{12} - 1} . (-\ln u_2)^{\alpha_{12} - 1} . (-\ln u_3)^{\alpha_1 - 1} . \frac{1}{u_1 \ u_2 \ u_3}$$

$$. \left\{ \alpha_1 (\alpha_1 - 1) + \left[\left[(-\ln u_1)^{\alpha_{12}} + (-\ln u_2)^{\alpha_{12}} \right]^{\frac{\alpha_1}{\alpha_{12}}} + (-\ln u_3)^{\alpha_1} \right]^{\frac{1}{\alpha_1}} \right\} \quad (39)$$

Again for the hierarchical copula, the Gumbel Copula based joint density function is

$$h(x, y, z) = \frac{\partial^3 C(u_1, u_2, u_3)}{\partial u_1 \, \partial u_2 \, \partial u_3} f_1(x) \, f_2(y) \, f_3(z) \quad (40)$$

3. Numerical Results

In this section, we present the numerical results based on 1000 simulation runs. The samples were generated at random by using the multivariate normal distributions associated with the mean vectors and covariance matrices as indicated below. The misclassification error rate P12 which is classifying Population 1 as Population 2, and the misclassification error rate P21 which is classifying Population 2 as Population 1 were estimated empirically through this simulation. This simulation was done for the Vine structures based on the Gaussian, Clayton, and Gumbel Copula models, and the 'Hierarchy' models based on the Clayton and Gumbel Copulas.

We present the numerical results for dimension p = 3. The mean vectors and the covariance matrices are assumed to be known. As noted earlier, the mean vectors and the covariance matrices studied are listed below.

$$\mu_1^{(1)} = 2.2, \quad \mu_2^{(1)} = 2.4, \quad \mu_3^{(1)} = 1.9, \quad \mu_1^{(2)} = 1.7, \quad \mu_2^{(2)} = 1.1, \quad \mu_3^{(2)} = 1.4$$

Table 1. Misclassification Error rate Comparison

Σ_1	Σ_2	Vine Copula			Hierarchical Copula	
		Clayton P12(P21)	Gaussian P12(P21)	Gumbel P12(P21)	Clayton P12(P21)	Gumbel P12(P21)
$\begin{pmatrix}19 & 18 & 13\\ 18 & 26 & 10\\ 13 & 10 & 35\end{pmatrix}$	$\begin{pmatrix}4 & 1 & 2\\ 1 & 9 & -3\\ 2 & -3 & 25\end{pmatrix}$	0.358(0.407)	0.306(0.310)	0.495(0.485)	0.380(0.482)	0.300(0.490)
$\begin{pmatrix}30 & 16 & 6\\ 16 & 10 & 4\\ 6 & 4 & 2\end{pmatrix}$	$\begin{pmatrix}6 & 6 & 1\\ 6 & 8 & 2\\ 1 & 2 & 1\end{pmatrix}$	0.350(0.390)	0.302(0.315)	0.490(0.472)	0.405(0.464)	0.400(0.472)
$\begin{pmatrix}19 & 18 & 13\\ 18 & 26 & 10\\ 13 & 10 & 35\end{pmatrix}$	$\begin{pmatrix}6 & 6 & 1\\ 6 & 8 & 2\\ 1 & 2 & 1\end{pmatrix}$	0.320(0.310)	0.165(0.185)	0.487(0.464)	0.180(0.452)	0.285(0.460)
$\begin{pmatrix}6 & 1 & -2\\ 1 & 13 & 4\\ -2 & 4 & 4\end{pmatrix}$	$\begin{pmatrix}11 & -8 & 3\\ -8 & 9 & -3\\ 3 & -3 & 2\end{pmatrix}$	0.487(0.476)	0.430(0.525)	0.482(0.480)	0.565(0.496)	0.660(0.490)
$\begin{pmatrix}30 & 16 & 6\\ 16 & 10 & 4\\ 6 & 4 & 2\end{pmatrix}$	$\begin{pmatrix}11 & -8 & 3\\ -8 & 9 & -3\\ 3 & -3 & 2\end{pmatrix}$	0.352(0.316)	0.219(0.180)	0.486(0.481)	0.190(0.476)	0.170(0.494)
$\begin{pmatrix}6 & 6 & 1\\ 6 & 8 & 2\\ 1 & 2 & 1\end{pmatrix}$	$\begin{pmatrix}11 & -8 & 3\\ -8 & 9 & -3\\ 3 & -3 & 2\end{pmatrix}$	0.440(0.375)	0.267(0.325)	0.490(0.478)	0.245(0.488)	0.492(0.464)
$\begin{pmatrix}4 & 1 & 2\\ 1 & 9 & -3\\ 2 & -3 & 25\end{pmatrix}$	$\begin{pmatrix}11 & -8 & 3\\ -8 & 9 & -3\\ 3 & -3 & 2\end{pmatrix}$	0.456(0.477)	0.366(0.375)	0.475(0.473)	0.480(0.485)	0.550(0.496)

4. Discussion and Conclusion

As is seen from Table 1, Vine based Copula Gaussian Copula is doing better than Vine based Clayton and Gumbel Copulas, and 'Hierarchy' based Clayton and Gumbel Copulas. This conclusion is based on the average error rates P12 and P21. This is an interesting result. Usually, the dimensional reductions leads to loss of information and hence we expect to see higher error rates. However, this was not the case here. Maybe, it is possible that the Vine structures do not lose too much of information. Maybe, this is the reason that Vine Copulas are extensively used in many fields such as Actuarial Science, Economics, and Finance. Furthermore, the Vine Copulas help us to avoid the possibility for singularities in the context of higher dimensional covariance matrices. In this paper, we considered a three dimensional situation. For this situation all three types of Vine Copulas; R Vine, C Vine, and D Vine are the same. However, this is not the case for dimensions higher than 3. The authors plan to investigate the misclassification error rate of Vine based comparisons for the higher dimensions in the future.

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