Parameters Estimation for Wear-out Failure Period of Three-Parameter Weibull Distribution

Takatoshi Sugiyama[1] & Toru Ogura[2]

[1] Department of applied Mathematics, Science Graduate School of Science, Tokyo University of Science, Tokyo, Japan

[2] Clinical Research Support Center, Mie University Hospital, Mie, Japan

Correspondence: Takatoshi Sugiyama, Institute of Statistical Science, Tokyo Japan

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Abstract

The shape parameter estimation using the minimum-variance linear estimator with hyperparameter (MVLE-H) method is believed to be effective for a wear-out failure period in a small sample. In the process of the estimation, our method uses the hyperparameter and estimate shape parameters of the MVLE-H method. To obtain the optimal hyperparameter c, it takes a long time, even in the case of the small sample. The main purpose of this paper is to remove the restriction of small samples. We observed that if we set the shape parameters, for sample size n and c, we can use the regression equation to infer the optimal c from n. So we searched in five increments and complemented the hyperparameter for the remaining sample sizes with a linear regression line. We used Monte Carlo simulations (MCSs) to determine the optimal hyperparameter for various sample sizes and shape parameters of the MVLE-H method. Intrinsically, we showed that the MVLE-H method performs well by determining the hyperparameter. Further, we showed that the location and scale parameter estimations are improved using the shape parameter estimated by the MVLE-H method. We verified the validity of the MVLE-H method using MCSs and a numerical example.

Keywords: parameter estimation, prior on location, three-parameter Weibull distribution, Wear-out failure period

1. Introduction

The Weibull and exponential distributions are often used to analyse lifetime data (Huang, WT & Huang, HH, 2006; Ogura et al., 2020). An exponential distribution is applied when the probability of occurrence is high immediately after the start, and then decreases monotonically. However, the Weibull distribution is applied when the probability of occurrence is high, and when it is low immediately after the start, and then increases. This is called the wear-out failure period. We used the Weibull distribution because we were interested in a wear-out failure period. A three-parameter Weibull distribution that does not require location parameter constraint is widely used. The probability density and cumulative distribution functions of the three-parameter Weibull distribution are expressed as follows:

$$g(x; m, \eta, \gamma) = \frac{m}{\eta}\left(\frac{x-\gamma}{\eta}\right)^{m-1} \exp\left[-\left(\frac{x-\gamma}{\eta}\right)^{m}\right], \tag{1}$$

$$G(x; m, \eta, \gamma) = 1 - \exp\left[-\left(\frac{x-\gamma}{\eta}\right)^{m}\right], \tag{2}$$

where $\eta > 0$, $m > 0$, and $\gamma < x$ are the scale, shape, and location parameters, respectively. However, it is difficult to estimate the three-parameter Weibull distribution because it cannot be obtained using maximum likelihood estimation (MLE). Several scholars have researched on the parameter estimation method for a three-parameter Weibull distribution, e.g., (Smith, 1985; Lawless, 2011; Murthy et al., 2004; Cousineau, 2009a; Cousineau, 2009b; Ng et al., 2012; Nagatsuka et al., 2013; Datsiou & Overend M, 2018; Sugiyama et al., 2019; Yang et al., 2019; Ogura et al., 2020). Nagatsuka et al., (2013) proposed the location and scale parameters free maximum likelihood estimators (LSPF-MLE) method, which estimates the shape parameter using the independent statistics of the location and scale parameters. The shape parameter estimation method using the minimum-variance linear estimator with a hyperparameter (MVLE-H) was proposed by Ogura et al., (2020).

In this study, we extended the MVLE-H method removing the restriction of small samples. We use Monte Carlo simulations (MCSs) to determine for the optimal hyperparameter used for the MVLE-H method for every sample size and shape parameter. However, it takes a long time to determine the optimal hyperparameter for all sample sizes, thus, we searched in increments of five, and complement the hyperparameter for the remaining sample sizes with a linear regression line. We estimated a temporary shape parameter using an existing method, which is an idea used in the w-MLE method, because the population shape parameter is unknown. We show that the MVLE-H method can be modified by determining the hyperparameter using the temporary shape parameter as prior information. In this paper, we discuss up to sample size $n = 40$. When sample size n is 40, to decide the hyperparameter c takes a few weeks in our computer. If we use supercomputer to calculate it, we may get the result for the sample size larger than 40. Further, we show that the estimation of the location and scale parameters can be improved using the shape parameter estimated by the MVLE-H method.

The remainder of this research is organised as follows. In Sect. 2, we describe the w-MLE, BL, LSPF-MLE, and MVLE-H methods. In Sect. 3, we describe the modified w-MLE, BL, and LSPF-MLE methods. In Sect. 4, we validate the effectiveness of the MVLE-H method using MCSs. In Sect. 5, we present an attempt to estimate the three-parameter Weibull distribution using a numerical example. Finally, we conclude the research in Sect. 6.

2. Existing Methods

Let $X_1, \ldots, X_n$ be random samples from the three-parameter Weibull distribution, and let $X_{(i)}$ denote the i-th order statistic, $\gamma < X_{(1)} < \cdots < X_{(n)}$ ($X_{(i)} \neq X_{(j)}$ for some $i \neq j$), $i = 1, \ldots, n$ ($n \geq 2$). In this paper, first we decide the hyperparameter, and then we estimate the shape parameter. By using this shape parameter we estimate the location and scale parameter, through each w-MLE, BL and LSPF-MLE method. To get the estimators, we have to calculate digitally. For example, in the explanation of the simulation, we will touch on the difference in the number of repetitions. We describe four existing parameter estimation methods: w-MLE, BL, LSPF-MLE, and MVLE-H methods.

2.1 w-MLE Method

The w-MLE method extends the MLE technique by incorporating three weights into the solutions of the maximum likelihood equations. The third weight W_3 is not a constant but varies during the search as new values of the estimating shape parameter are explored. However, linear interpolation was used because of the lack of a fast computational algorithm. The w-MLE method used for comparison in this paper is based on Cousineau (2009b), but W_3 is calculated by the following five steps based on Nagatsuka et al. (2013). (i) A temporary shape parameter, $\hat{m}_0$, is estimated by MLE for the two-parameter Weibull distribution in $(X_{(i)} - X_{(1)})$, $i = 2, \ldots, n$. (ii) Uniform random samples, $(u_1, \ldots, u_n)$, are generated from the standard uniform distribution. (iii) Using $\hat{m}_0$ and $(u_1, \ldots, u_n)$, we calculate

$$Y_3 = W_1 \frac{\frac{1}{n}\sum_{i=1}^{n}\left(\log \frac{1}{1-u_i}\right)^{-1/\hat{m}_0}}{\frac{1}{n}\sum_{i=1}^{n}\left(\log \frac{1}{1-u_i}\right)^{1-1/\hat{m}_0}}, \tag{3}$$

where W_1 is the median of $\frac{1}{n}\sum_{i=1}^{n}\log\frac{1}{1-u_i}$. (iv) Steps (i)–(iii) are repeated independently 30,000 times. Some calculate 10,000 times, but we know that 30,000 times lead to better results. (v) We calculate W_3 as the median of Y_3. In the MCSs, Steps (i)–(v) are performed for every time random samples from the three-parameter Weibull distribution. The location and shape parameters are estimated by minimizing

$$\begin{aligned}
&\mathrm{wMLE}(m,\gamma)\\
&= \left[\frac{W_2}{m} + \frac{1}{n}\sum_{i=1}^{n}\log\left(X_{(i)}-\gamma\right) - \frac{\sum_{i=1}^{n}\log\left(X_{(i)}-\gamma\right)\left(X_{(i)}-\gamma\right)^m}{\sum_{i=1}^{n}\left(X_{(i)}-\gamma\right)^m}\right]^2\\
&+\left[\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{(i)}-\gamma}\times\frac{\sum_{i=1}^{n}\left(X_{(i)}-\gamma\right)^m}{\sum_{i=1}^{n}\left(X_{(i)}-\gamma\right)^{m-1}} - W_3\right]^2,
\end{aligned} \tag{4}$$

where W_2 is the median of

$$\frac{\sum_{i=1}^{n}\log\left(\frac{1}{1-u_i}\right)\log\left(\log\left(\frac{1}{1-u_i}\right)\right)}{\sum_{i=1}^{n}\log\left(\frac{1}{1-u_i}\right)}.$$

Using $m = \hat{m}$ and $\eta = \hat{\eta}$ by manimising wMLE(m,γ), the scale parameter is estimated by $\hat{\eta} = \sqrt[\hat{m}]{\frac{1}{nW_1}\sum_{i=1}^{n}(x_i - \hat{\gamma})^{\hat{m}}}$. Because W_1 and W_2 depend only on n and not on the Weibull distribution, Cousineau (2009b) calculated W_1 and W_2 for $n = 1, \ldots, 16$ and listed them. When used with $n \leq 17$, we determine W_1 and W_2 by the median of 100,000 MCSs.

2.2 BL Method

The BL method multiplies the MLE using an empirical prior. The location and shape parameters were estimated by maximising the following:

$$\mathrm{BL1}(m,\gamma) = \frac{X_{(1)}-\gamma}{X_{(2)}-\gamma}\frac{\Gamma(n)m^n \prod_{i=1}^{n}(X_{(i)}-\gamma)^{m-1}}{\left[\sum_{i=1}^{n}(X_{(i)}-\gamma)^m\right]^n}. \tag{5}$$

Using $m = \hat{m}$ and $\gamma = \hat{\gamma}$ by maximising $\mathrm{BL1}(m,\gamma)$, the scale parameter is estimated by maximising the following:

$$\mathrm{BL2}(\eta) = \frac{\left[\sum_{i=1}^{n}(X_i-\hat{\gamma})^{\hat{m}}\right]^n}{\Gamma(n)\eta^{n\hat{m}}}\exp\left[-\frac{\sum_{i=1}^{n}(X_i-\hat{\gamma})^{\hat{m}}}{\eta^{\hat{m}}}\right]. \tag{6}$$

2.3 LSPF-MLE Method

The LSPF-MLE method of the shape parameter is estimated by maximising the following:

$$\begin{aligned}\mathrm{LSPF}(m) &= n!m^n\int_0^\infty\int_0^\infty v^{n-2}\left[\prod_{i=1}^{n}(u+Z_{(i)}v)\right]^{m-1}\\ &\quad\times\exp\left\{-\left[\sum_{i=1}^{n}(u+Z_{(i)}v)^m\right]\right\}dudv,\end{aligned} \tag{7}$$

where $Z_{(i)} = \frac{X_{(i)}-X_{(1)}}{X_{(n)}-X_{(1)}}$. Using $m = \hat{m}$, the temporary location and scale parameters are estimated using $\hat{\gamma}_{\mathrm{int}} = X_{(1)}$ and $\hat{\eta}_{\mathrm{int}} = \left[\frac{1}{n}\sum_{i=1}^{n}\left(X_i-\hat{\gamma}_{\mathrm{int}}\right)^{\hat{m}}\right]^{1/\hat{m}}$. Then, using $m = \hat{m}$, $\hat{\gamma}_{\mathrm{int}}$, and $\hat{\eta}_{\mathrm{int}}$, the location and scale parameters are estimated by $\hat{\gamma} = X_{(1)} - n^{-1/\hat{m}}\hat{\eta}_{\mathrm{int}}\Gamma(1+1/\hat{m})$ and $\hat{\eta} = \left[\frac{1}{n}\sum_{i=1}^{n}(X_i-\hat{\gamma})^{\hat{m}}\right]^{1/\hat{m}}$, respectively.

2.4 MVLE-H Method

The MVLE-H method of the shape parameter is expressed as follows:

$$\hat{m} = 1'S^{-1}1\Big/1'S^{-1}\begin{pmatrix}T_{(2)}\\ \vdots\\ T_{(n)}\end{pmatrix}, \tag{8}$$

where $1' = (1 \ \cdots \ 1)$ is $(n-1)$-dimensional vector,

$$T_{(i)} = \int_0^1 \frac{\log\left[c(1-u)(X_{(i)}-X_{(1)})+u\right]-\log(u)}{k(i,n)-k(1,n)}du, \tag{9}$$

$$k(i,n) = \int_0^1 \log\left[-\log(1-u)\right]\frac{n!u^{i-1}(1-u)^{n-i}}{(i-1)!(n-i)!}du, \tag{10}$$

$$\begin{aligned}S &= \frac{1}{k(i,n)-k(1,n)}\frac{1}{k(j,n)-k(1,n)}[\mathrm{cov}\left(\log(X_{(i)}-\gamma),\log(X_{(j)}-\gamma)\right)\\ &\quad-\mathrm{cov}\left(\log(X_{(i)}-\gamma),\log(X_{(1)}-\gamma)\right)-\mathrm{cov}\left(\log(X_{(j)}-\gamma),\log(X_{(1)}-\gamma)\right)\\ &\quad+\mathrm{var}\left(\log(X_{(1)}-\gamma)\right)],\end{aligned} \tag{11}$$

$$\begin{aligned}&\mathrm{cov}\left(\log(X_{(i)}-\gamma),\log(X_{(j)}-\gamma)\right)\\ &= \int_u^1\int_0^1\left\{\frac{\log\left[-\log(1-u)\right]}{m}-\frac{k(i,n)}{m}\right\}\left\{\frac{\log\left[-\log(1-v)\right]}{m}-\frac{k(j,n)}{m}\right\}\\ &\times\frac{n!u^{i-1}(v-u)^{j-i-1}(1-v)^{n-j}}{(i-1)!(j-i-1)!(n-j)!}dudv,\end{aligned} \tag{12}$$

$$\begin{aligned}&\mathrm{var}\left(\log(X_{(i)}-\gamma)\right)\\ &= \int_0^1\left\{\frac{\log\left[-\log(1-u)\right]}{m}-\frac{k(i,n)}{m}\right\}^2\frac{n!u^{i-1}(1-u)^{n-i}}{(i-1)!(n-i)!}du,\end{aligned} \tag{13}$$

$c > 0$ is a hyperparameter, and $i, j = 2, \ldots, n$. The right-hand side of (8) might appear to depend on γ and m, but Ogura et al. (2020) showed that it does not, when actually calculated. Notably, the right-hand side of (12) and (13) is given by $\log[-\log(1-u)]$ or $\log[-\log(1-v)]$ on their paper, but we proved that it is $\log[-\log(1-u)]/m$ or $\log[-\log(1-v)]/m$ because their work is considered as a clerical error. However, substituting (12) and (13) into (8), they are cancelled by the numerator and denominator, and the estimator in (8) is the same with or without $1/m$.

The MVLE-H method can only estimate the shape parameter, but the location and scale parameters can be improved for the w-MLE, BL, and LSPF-MLE methods using the shape parameter estimated by the MVLE-H method. Detailed discussion is in Sect. 3.

2.5 Optimal Hyperparameter c on MVLE-H Method

The performance of the MVLE-H method is enhanced by optimally selecting the hyperparameter c in (4). The optimal c value depends on n and m. However, although n can be determined from the data, m is unknown. Therefore, Ogura et al. (2020) determined optimal c using only n. The performance of the MVLE-H method can be improved by determining the hyperparameter for every n and m.

Using MCSs, we searched for the optimal hyperparameters c for every n and m. We set 32 cases ($\gamma = 0$, $\eta = 1$, $m = 1.5, 2, 3, 4, 5$) for the sample size $n = 5, 10, 15, 20, 25, 30, 35, 40$. The replication size used in this study was 4,000. We used the software R version 4.0.3 R Core Team (2021) for the MCSs. Tables 1-5 list the MCS results of the bias and root-mean-square error (RMSE) (RMSE = $\sqrt{\text{variance} + \text{bias}^2}$) for every hyperparameter c. The optimal c value was selected near the minimum RMSE value. Table 6 lists the optimal c values for every n and m. It is believed that the optimal c and n are represented by linear regression lines and are expressed as follows:

$$\hat{c}(1.5) = 0.70n + 1.5\ (m = 1.5),\quad \hat{c}(2) = 0.20n + 2.4\ (m = 2),$$
$$\hat{c}(3) = 0.10n + 1.8\ (m = 3),\quad \hat{c}(4) = 0.08n + 0.9\ (m = 4),$$
$$\hat{c}(5) = 0.04n + 1.0\ (m = 5).$$

Figure 1 shows the plots of n and c, and the linear regression lines. We searched for the optimal c in five increments of n and complemented the hyperparameter for the remaining sample sizes with a linear regression line. The hyperparameterc discussed here is robust in the estimation of the shape parameter by The MVLE-H method, and the actual c used is sufficient to be a value near ϵ of the optimal c. The relationship expression above and Figure 1 are the most important parts of this paper. As the Weibull distribution estimator is regular for shapes larger than 2 Smith (1985), the search for the optimum c was performed at $m = 2, 3, 4, 5$, but we believed that the optimum c at $m = 2.5, 3.5, 4.5$ could be estimated by linear interpolation.

We used a temporary shape parameter, $\hat{m}_0$, is estimated by the MLE for the two-parameter Weibull distribution in $(X_{(i)} - X_{(1)})$ and $i = 2, \ldots, n$ because m is unknown. The optimal c is determined using $\hat{c}(2)$, $\hat{c}(3)$, $\hat{c}(4)$, and $\hat{c}(5)$ for $\hat{m}_0 < 2.5$, $2.5 \leq \hat{m}_0 < 3.5$, $3.5 \leq \hat{m}_0 < 4.5$, and $4.5 \leq \hat{m}_0$, respectively.

Table 1. Bias and RMSE of the shape parameter estimated by the MVLE-H method with various hyperparameter c values for $m = 1.5$

$n \setminus c$		4.8	4.9	5.0	5.1	5.2
5	Bias	-0.017	-0.027	-0.036	-0.045	-0.054
	RMSE	0.393	0.390	0.388	0.385	0.383
$n \setminus c$		8.3	8.4	8.5	8.6	8.7
10	Bias	0.007	0.003	-0.001	-0.004	-0.007
	RMSE	0.305	0.304	0.303	0.303	0.302
$n \setminus c$		11.8	11.9	12.0	12.1	12.2
15	Bias	0.015	0.013	0.011	0.010	0.008
	RMSE	0.263	0.263	0.262	0.263	0.262
$n \setminus c$		15.3	15.4	15.5	15.6	15.7
20	Bias	0.009	0.007	0.006	0.003	0.002
	RMSE	0.235	0.235	0.234	0.234	0.234
$n \setminus c$		18.8	18.9	19.0	19.1	19.2
25	Bias	0.015	0.013	0.012	0.011	0.010
	RMSE	0.224	0.223	0.223	0.223	0.222
$n \setminus c$		22.3	22.4	22.5	22.6	22.7
30	Bias	0.001	0.001	0.000	-0.001	-0.002
	RMSE	0.197	0.197	0.197	0.197	0.197
$n \setminus c$		25.8	25.9	26.0	26.1	26.2
35	Bias	0.003	0.002	0.002	0.001	0.000
	RMSE	0.186	0.186	0.186	0.186	0.186
$n \setminus c$		29.3	29.4	29.5	29.6	29.7
40	Bias	-0.002	-0.002	-0.003	-0.003	-0.004
	RMSE	0.174	0.173	0.173	0.173	0.173

Table 2. Bias and RMSE of the shape parameter estimated by the MVLE-H method with various hyperparameter c values for $m = 2$

$n \setminus c$		3.2	3.3	3.4	3.5	3.6
5	Bias	-0.089	-0.111	-0.132	-0.152	-0.170
	RMSE	0.515	0.512	0.509	0.507	0.508
$n \setminus c$		4.2	4.3	4.4	4.5	4.6
10	Bias	0.063	0.050	0.034	0.020	0.010
	RMSE	0.421	0.419	0.411	0.406	0.405
$n \setminus c$		5.2	5.3	5.4	5.5	5.6
15	Bias	0.093	0.081	0.072	0.066	0.053
	RMSE	0.379	0.373	0.369	0.368	0.364
$n \setminus c$		6.2	6.3	6.4	6.5	6.6
20	Bias	0.107	0.087	0.083	0.074	0.067
	RMSE	0.352	0.344	0.345	0.342	0.338
$n \setminus c$		7.2	7.3	7.4	7.5	7.6
25	Bias	0.097	0.091	0.085	0.082	0.075
	RMSE	0.319	0.317	0.315	0.313	0.313
$n \setminus c$		8.2	8.3	8.4	8.5	8.6
30	Bias	0.086	0.078	0.078	0.072	0.069
	RMSE	0.300	0.296	0.297	0.294	0.293
$n \setminus c$		9.2	9.3	9.4	9.5	9.6
35	Bias	0.080	0.076	0.074	0.070	0.066
	RMSE	0.281	0.281	0.279	0.279	0.276
$n \setminus c$		10.2	10.3	10.4	10.5	10.6
40	Bias	0.071	0.068	0.066	0.063	0.059
	RMSE	0.264	0.266	0.263	0.264	0.262

Table 3. Bias and RMSE of the shape parameter estimated by the MVLE-H method with various hyperparameter c values for $m = 3$

$n \setminus c$		2.1	2.2	2.3	2.4	2.5
5	Bias	-0.329	-0.395	-0.431	-0.483	-0.518
	RMSE	0.792	0.793	0.814	0.812	0.844
$n \setminus c$		2.6	2.7	2.8	2.9	3.0
10	Bias	-0.091	-0.133	-0.161	-0.213	-0.240
	RMSE	0.598	0.595	0.592	0.593	0.610
$n \setminus c$		3.1	3.2	3.3	3.4	3.5
15	Bias	-0.039	-0.065	-0.086	-0.119	-0.140
	RMSE	0.526	0.527	0.522	0.531	0.524
$n \setminus c$		3.6	3.7	3.8	3.9	4.0
20	Bias	-0.017	-0.044	-0.067	-0.083	-0.108
	RMSE	0.488	0.483	0.481	0.484	0.490
$n \setminus c$		4.1	4.2	4.3	4.4	4.5
25	Bias	-0.025	-0.043	-0.062	-0.079	-0.098
	RMSE	0.453	0.452	0.452	0.452	0.455
$n \setminus c$		4.6	4.7	4.8	4.9	5.0
30	Bias	-0.651	-0.654	-0.658	-0.664	-0.667
	RMSE	0.807	0.808	0.809	0.811	0.811
$n \setminus c$		5.1	5.2	5.3	5.4	5.5
35	Bias	-0.038	-0.050	-0.065	-0.077	-0.091
	RMSE	0.408	0.409	0.409	0.410	0.413
$n \setminus c$		5.6	5.7	5.8	5.9	6.0
40	Bias	-0.054	-0.064	-0.077	-0.086	-0.094
	RMSE	0.398	0.401	0.399	0.401	0.401

Table 4. Bias and RMSE of the shape parameter estimated by the MVLE-H method with various hyperparameter c values for $m = 4$

$n \setminus c$		1.1	1.2	1.3	1.4	1.5
5	Bias	0.127	-0.014	-0.189	-0.322	-0.454
	RMSE	1.179	1.159	1.111	1.101	1.069
$n \setminus c$		1.5	1.6	1.7	1.8	1.9
10	Bias	0.124	0.009	-0.094	-0.179	-0.270
	RMSE	0.828	0.804	0.771	0.786	0.796
$n \setminus c$		1.9	2.0	2.1	2.2	2.3
15	Bias	0.060	-0.007	-0.098	-0.181	-0.229
	RMSE	0.726	0.714	0.700	0.709	0.717
$n \setminus c$		2.3	2.4	2.5	2.6	2.7
20	Bias	-0.021	-0.085	-0.141	-0.181	-0.233
	RMSE	0.664	0.654	0.658	0.665	0.672
$n \setminus c$		2.7	2.8	2.9	3.0	3.1
25	Bias	-0.111	-0.157	-0.198	-0.236	-0.276
	RMSE	0.613	0.617	0.626	0.632	0.648
$n \setminus c$		3.1	3.2	3.3	3.4	3.5
30	Bias	-0.155	-0.198	-0.225	-0.261	-0.290
	RMSE	0.597	0.603	0.612	0.621	0.633
$n \setminus c$		3.5	3.6	3.7	3.8	3.9
35	Bias	-0.211	-0.241	-0.267	-0.295	-0.321
	RMSE	0.591	0.603	0.607	0.621	0.629
$n \setminus c$		3.9	4.0	4.1	4.2	4.3
40	Bias	-0.250	-0.274	-0.297	-0.320	-0.345
	RMSE	0.580	0.593	0.599	0.611	0.621

Table 5. Bias and RMSE of the shape parameter estimated by the MVLE-H method with various hyperparameter c values for $m = 5$

$n \setminus c$		1.0	1.1	1.2	1.3	1.4
5	Bias	-0.107	-0.340	-0.563	-0.773	-0.896
	RMSE	1.481	1.427	1.392	1.411	1.474
$n \setminus c$		1.2	1.3	1.4	1.5	1.6
10	Bias	0.095	-0.100	-0.270	-0.384	-0.517
	RMSE	1.029	0.984	0.972	0.996	1.026
$n \setminus c$		1.4	1.5	1.6	1.7	1.8
15	Bias	0.123	-0.014	-0.159	-0.291	-0.394
	RMSE	0.899	0.873	0.868	0.868	0.904
$n \setminus c$		1.6	1.7	1.8	1.9	2.0
20	Bias	0.083	-0.052	-0.141	-0.241	-0.345
	RMSE	0.843	0.810	0.814	0.826	0.837
$n \setminus c$		1.8	1.9	2.0	2.1	2.2
25	Bias	0.030	-0.070	-0.156	-0.242	-0.329
	RMSE	0.762	0.758	0.762	0.774	0.790
$n \setminus c$		2.0	2.1	2.2	2.3	2.4
30	Bias	-0.035	-0.124	-0.205	-0.277	-0.346
	RMSE	0.740	0.739	0.749	0.764	0.783
$n \setminus c$		2.2	2.3	2.4	2.5	2.6
35	Bias	-0.080	-0.155	-0.222	-0.286	-0.354
	RMSE	0.722	0.724	0.735	0.746	0.769
$n \setminus c$		2.4	2.5	2.6	2.7	2.8
40	Bias	-0.133	-0.200	-0.261	-0.318	-0.372
	RMSE	0.696	0.700	0.715	0.732	0.757

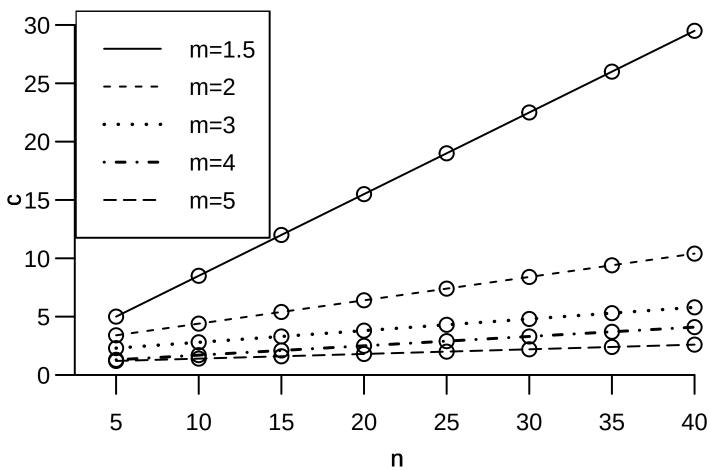

Figure 1. Plots of n and c, and linear regression line

3. Modified w-MLE, BL, and LSPF-MLE Methods

Using the shape parameter $\hat{m}$ in (8) estimated by the MVLE-H method, Ogura et al. (2020) improved the w-MLE, BL, and LSPF-MLE methods. These methods treat $\hat{m}$ as if it was known and estimate the location and scale parameters using the w-MLE, BL, and LSPF-MLE methods. These methods are called modified w-MLE, BL, and LSPF-MLE methods to distinguish them from the original three methods. We write these as M.w-MLE, M.BL and M.LSPE-MLE.

By considering $\hat{m}$ to be known, wMLE$(\hat{m}, \gamma)$ in (4) of the modified w-MLE method estimate only γ, BL1$(\hat{m}, \gamma)$ in (5) of the modified BL method estimate only γ, and LSPF(m) in (7) of the modified LSPF-MLE method is not used.

4. MCSs

The effectiveness of the estimated c in the MVLE-H method was verified using the MCSs. The sample size was set to, $n = 25, 30, 35, 40$, and $\gamma = 0$, $\eta = 1$, and $m = 1.5, 2, 2.5, 3, 3.5, 4, 4.5, 5$. Table 7 lists the bias and RMSE in the shape parameter obtained by the four methods (the MVLE-H, BL, w-MLE, and LSPF-MLE methods). Tables 8 and 9 list the bias and RMSE in the location and scale parameters obtained by the six methods (the modified w-MLE, w-MLE, modified BL, BL, modified LSPF-MLE and LSPF-MLE methods), respectively. Further, histograms of these MCSs results are shown in Figures SI1–SI16 in Supplementary Information. When the shape parameter estimated by the BL method was greater than 10, it was estimated to be 10 for the MCSs. The shape parameters estimated by the MVLE-H, w-MLE, and LSPF-MLE methods were not greater than 10. The RMSE of shape parameter estimated using the MVLE-H method was the smallest in all cases, while the RMSEs of location and scale parameters estimated by the modified w-MLE method were the smallest in all cases. We know that the estimation method w-MLE yields generally better and more stable estimates than the BL, LSPF-MLE methods. In the histogram, the modes of the shape parameter estimated by the MVLE-H method, and the location and scale parameters estimated by the modified w-MLE method were close to the population parameters.

Table 6. Optimal c for every n and m

$m\backslash n$	5	10	15	20	25	30	35	40
1.5	5.0	8.5	12.0	15.5	19.0	22.5	26.0	29.5
2	3.5	4.4	5.4	6.4	7.4	8.4	9.4	10.4
3	2.3	2.8	3.3	3.8	4.3	4.8	5.3	5.8
4	1.3	1.7	2.1	2.5	2.9	3.3	3.7	4.1
5	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6

Table 7. Comparison of bias and RMSE of the shape parameters estimated by the four methods

		MVLE-H		w-MLE		BL		LSPF-MLE	
n	m	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
25	1.5	0.005	0.215	0.212	0.880	-0.044	0.287	0.119	0.544
	2	0.089	0.312	-0.022	0.665	-0.177	0.550	0.231	0.968
	2.5	-0.029	0.372	-0.101	0.855	-0.176	1.059	0.301	1.347
	3	-0.060	0.451	-0.055	1.079	0.130	1.763	0.405	1.658
	3.5	-0.173	0.545	0.027	1.156	0.494	2.366	0.352	1.796
	4	-0.193	0.620	-0.028	1.081	0.841	2.834	0.290	1.887
	4.5	-0.248	0.709	-0.304	1.045	0.875	3.000	0.062	1.888
	5	-0.170	0.764	-0.647	1.130	0.883	3.093	-0.147	1.907
30	1.5	0.000	0.191	0.156	0.782	-0.057	0.243	0.075	0.414
	2	0.079	0.299	-0.029	0.601	-0.230	0.431	0.155	0.770
	2.5	-0.031	0.354	-0.097	0.776	-0.350	0.759	0.191	1.041
	3	-0.059	0.439	-0.068	0.992	-0.151	1.387	0.256	1.281
	3.5	-0.169	0.508	0.038	1.078	0.252	2.015	0.204	1.359
	4	-0.230	0.619	0.043	1.011	0.675	2.586	0.048	1.405
	4.5	-0.266	0.687	-0.187	0.913	0.874	2.862	-0.174	1.399
	5	-0.194	0.758	-0.552	1.000	0.836	2.948	-0.471	1.460
35	1.5	0.000	0.185	0.136	0.724	-0.057	0.230	0.063	0.356
	2	0.076	0.279	-0.030	0.528	-0.233	0.391	0.115	0.626
	2.5	-0.028	0.326	-0.086	0.691	-0.415	0.662	0.147	0.847
	3	-0.064	0.416	-0.074	0.933	-0.333	1.167	0.155	1.035
	3.5	-0.192	0.500	0.052	1.045	-0.002	1.695	0.050	1.073
	4	-0.266	0.606	0.094	0.983	0.524	2.405	-0.158	1.091
	4.5	-0.313	0.673	-0.100	0.832	0.747	2.673	-0.431	1.127
	5	-0.228	0.733	-0.460	0.896	0.789	2.853	-0.785	1.278
40	1.5	0.001	0.178	0.122	0.676	-0.056	0.216	0.057	0.325
	2	0.066	0.255	-0.036	0.462	-0.240	0.364	0.083	0.519
	2.5	-0.031	0.316	-0.076	0.659	-0.445	0.598	0.111	0.734
	3	-0.085	0.398	-0.089	0.889	-0.490	1.002	0.049	0.838
	3.5	-0.206	0.494	0.025	1.023	-0.131	1.552	-0.079	0.857
	4	-0.310	0.602	0.120	0.958	0.344	2.139	-0.342	0.887
	4.5	-0.360	0.688	-0.082	0.831	0.673	2.571	-0.682	1.025
	5	-0.252	0.706	-0.424	0.844	0.755	2.692	-1.035	1.246

Table 8. Comparison of bias and RMSE of the location parameter estimated by the six methods

		M. w-MLE		w-MLE		M.BL		BL		M. LSPF-MLE		LSPF-MLE	
n	m	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
25	1.5	0.024	0.083	-0.148	0.721	0.029	0.089	0.099	0.122	0.008	0.088	-0.011	0.118
	2	0.000	0.106	0.025	0.218	0.015	0.113	0.148	0.190	0.013	0.122	-0.002	0.168
	2.5	0.036	0.128	0.048	0.222	0.045	0.138	0.136	0.268	0.057	0.155	0.028	0.207
	3	0.038	0.139	0.020	0.276	0.054	0.150	0.049	0.379	0.090	0.182	0.059	0.233
	3.5	0.062	0.152	0.012	0.239	0.074	0.163	-0.046	0.475	0.130	0.211	0.101	0.253
	4	0.058	0.154	-0.077	0.368	0.077	0.167	-0.114	0.536	0.162	0.235	0.142	0.271
	4.5	0.065	0.159	0.069	0.220	0.081	0.170	-0.117	0.537	0.195	0.260	0.187	0.294
	5	0.044	0.153	-0.030	0.401	0.068	0.165	-0.106	0.518	0.220	0.279	0.228	0.315
30	1.5	0.024	0.075	-0.108	0.630	0.028	0.080	0.092	0.112	0.008	0.078	-0.004	0.096
	2	0.000	0.098	0.026	0.192	0.013	0.105	0.151	0.174	0.011	0.113	0.002	0.149
	2.5	0.031	0.117	0.042	0.204	0.039	0.125	0.175	0.224	0.048	0.143	0.026	0.184
	3	0.033	0.130	0.016	0.265	0.046	0.140	0.113	0.303	0.077	0.170	0.054	0.210
	3.5	0.057	0.142	0.009	0.217	0.066	0.151	0.005	0.401	0.113	0.195	0.089	0.227
	4	0.062	0.149	-0.102	0.385	0.076	0.160	-0.087	0.481	0.149	0.222	0.135	0.249
	4.5	0.070	0.154	0.050	0.194	0.082	0.164	-0.118	0.507	0.182	0.246	0.181	0.271
	5	0.045	0.148	-0.058	0.400	0.064	0.158	-0.106	0.489	0.204	0.264	0.221	0.296
35	1.5	0.020	0.067	-0.088	0.550	0.024	0.072	0.082	0.101	0.006	0.071	-0.004	0.086
	2	0.003	0.092	0.026	0.156	0.014	0.098	0.147	0.166	0.012	0.106	0.006	0.134
	2.5	0.028	0.110	0.038	0.185	0.034	0.117	0.187	0.215	0.040	0.133	0.022	0.166
	3	0.037	0.125	0.024	0.247	0.048	0.134	0.156	0.275	0.075	0.163	0.057	0.194
	3.5	0.060	0.138	0.003	0.210	0.066	0.146	0.057	0.338	0.107	0.186	0.090	0.210
	4	0.072	0.148	-0.112	0.376	0.084	0.159	-0.057	0.447	0.147	0.217	0.141	0.236
	4.5	0.072	0.150	0.028	0.174	0.082	0.159	-0.108	0.480	0.170	0.234	0.179	0.256
	5	0.055	0.146	-0.091	0.409	0.070	0.156	-0.101	0.482	0.199	0.257	0.230	0.290
40	1.5	0.018	0.061	-0.080	0.537	0.022	0.065	0.075	0.092	0.005	0.064	-0.004	0.076
	2	0.001	0.086	0.024	0.151	0.011	0.091	0.138	0.156	0.009	0.099	0.005	0.120
	2.5	0.029	0.105	0.035	0.174	0.034	0.111	0.189	0.208	0.038	0.127	0.023	0.153
	3	0.036	0.120	0.028	0.234	0.046	0.128	0.185	0.252	0.071	0.156	0.058	0.180
	3.5	0.063	0.135	0.010	0.206	0.068	0.142	0.086	0.308	0.103	0.180	0.094	0.197
	4	0.076	0.145	-0.127	0.382	0.085	0.154	-0.028	0.398	0.140	0.207	0.143	0.221
	4.5	0.082	0.151	0.026	0.172	0.090	0.159	-0.093	0.457	0.170	0.231	0.191	0.251
	5	0.054	0.143	-0.117	0.410	0.067	0.151	-0.104	0.460	0.185	0.245	0.228	0.278

5. Numerical Example

We analysed a dataset of 31 lifetimes of specific components of kitchen appliance products Liz et al. (2012) as follows:

3.84, 1.00, 4.14, 4.81, 5.72, 7.23, 8.08, 4.16, 4.17, 4.00, 4.42, 3.58, 3.92, 4.73, 5.42, 5.09, 5.59, 3.67, 5.76, 6.34, 6.07, 6.75, 4.07, 7.34, 6.00, 8.26, 8.01, 8.67, 4.24, 5.73, 5.50 (years).

We used MLE for the two-parameter Weibull distribution in $(X_{(i)} - X_{(1)})$, which was used in the w-MLE method to estimate a temporary shape parameter in determining the optimal c for the MVLE-H method. The temporary shape parameter was estimated to be $m = 3.267$, and the optimal c in the MVLE-H method was determined to be 4.9 by substituting $n = 31$ for $\hat{c}(3)$. Therefore, we recommended the shape parameter $\hat{m} = 2.872$ estimated by the MVLE-H method, the location parameter $\hat{\gamma} = 0.169$, and the scale parameter $\hat{\eta} = 5.676$ estimated by the modified w-MLE method.

We used the software R to estimate the shape parameters by the four methods (the MVLE-H, BL, w-MLE, and LSPF-MLE methods) and estimated the location and scale parameters by the six methods (the modified w-MLE, w-MLE, modified BL, BL, modified LSPF-MLE, and LSPF-MLE methods). The sample code is shown in Appendix. The results of the three-parameter estimation are summarised in Table 10.

The shape parameters estimated by the BL and the LSPF-MLE methods were 5.101 and 4.295, respectively. The reason for these large estimation results was that the RMSEs of the BL and the LSPF-MLE methods were large in the MCS results of the shape parameters at $n = 30$ and $m = 3$, in Table 7. The estimation results of the location and scale parameters between the modified w-MLE and the w-MLE methods showed a small difference. Meanwhile, the differences between

Table 9. Comparison of bias and RMSE of the scale parameter estimated by the six methods

		M. w-MLE		w-MLE		M.BL		BL		M. LSPF-MLE		LSPF-MLE	
n	m	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
25	1.5	-0.026	0.155	0.155	0.801	-0.040	0.161	-0.114	0.185	-0.020	0.162	0.004	0.211
	2	0.005	0.142	-0.038	0.262	-0.015	0.148	-0.165	0.217	-0.013	0.157	0.000	0.230
	2.5	-0.040	0.144	-0.063	0.258	-0.053	0.154	-0.154	0.281	-0.063	0.171	-0.032	0.252
	3	-0.041	0.147	-0.032	0.303	-0.060	0.159	-0.060	0.390	-0.092	0.188	-0.058	0.268
	3.5	-0.067	0.156	-0.020	0.260	-0.081	0.169	0.038	0.489	-0.131	0.212	-0.100	0.279
	4	-0.062	0.156	0.071	0.379	-0.082	0.170	0.108	0.549	-0.159	0.232	-0.139	0.290
	4.5	-0.069	0.160	-0.075	0.230	-0.087	0.172	0.110	0.550	-0.190	0.253	-0.185	0.307
	5	-0.047	0.152	0.025	0.407	-0.071	0.165	0.101	0.529	-0.211	0.269	-0.225	0.324
30	1.5	-0.025	0.142	0.113	0.697	-0.036	0.147	-0.105	0.172	-0.018	0.148	-0.002	0.181
	2	0.003	0.130	-0.039	0.234	-0.014	0.135	-0.172	0.208	-0.012	0.145	-0.008	0.202
	2.5	-0.034	0.134	-0.055	0.238	-0.045	0.143	-0.194	0.244	-0.053	0.159	-0.030	0.224
	3	-0.037	0.139	-0.027	0.290	-0.052	0.149	-0.126	0.315	-0.080	0.177	-0.055	0.239
	3.5	-0.061	0.149	-0.014	0.239	-0.072	0.159	-0.013	0.413	-0.114	0.198	-0.088	0.249
	4	-0.066	0.152	0.098	0.396	-0.082	0.164	0.081	0.494	-0.148	0.221	-0.134	0.263
	4.5	-0.072	0.155	-0.053	0.201	-0.086	0.166	0.114	0.518	-0.177	0.240	-0.179	0.280
	5	-0.048	0.148	0.053	0.405	-0.068	0.159	0.101	0.500	-0.196	0.255	-0.221	0.303
35	1.5	-0.020	0.130	0.093	0.606	-0.030	0.134	-0.093	0.157	-0.013	0.136	0.001	0.163
	2	0.002	0.122	-0.035	0.198	-0.013	0.127	-0.165	0.199	-0.011	0.136	-0.010	0.183
	2.5	-0.030	0.128	-0.047	0.217	-0.038	0.135	-0.205	0.237	-0.044	0.150	-0.024	0.203
	3	-0.039	0.133	-0.031	0.269	-0.052	0.142	-0.169	0.286	-0.076	0.169	-0.057	0.219
	3.5	-0.063	0.144	-0.007	0.230	-0.072	0.153	-0.065	0.348	-0.108	0.190	-0.090	0.229
	4	-0.076	0.151	0.111	0.387	-0.089	0.162	0.052	0.459	-0.145	0.215	-0.141	0.247
	4.5	-0.075	0.153	-0.031	0.181	-0.086	0.162	0.104	0.492	-0.166	0.231	-0.180	0.264
	5	-0.057	0.146	0.089	0.414	-0.073	0.156	0.098	0.492	-0.191	0.249	-0.232	0.295
40	1.5	-0.020	0.120	0.083	0.595	-0.029	0.124	-0.089	0.146	-0.014	0.125	-0.002	0.147
	2	0.003	0.116	-0.032	0.188	-0.010	0.120	-0.156	0.190	-0.008	0.128	-0.009	0.165
	2.5	-0.030	0.121	-0.044	0.203	-0.038	0.128	-0.208	0.233	-0.042	0.142	-0.025	0.186
	3	-0.040	0.130	-0.038	0.258	-0.051	0.138	-0.202	0.267	-0.073	0.163	-0.061	0.204
	3.5	-0.066	0.141	-0.014	0.225	-0.073	0.149	-0.094	0.318	-0.105	0.184	-0.095	0.211
	4	-0.080	0.149	0.126	0.393	-0.091	0.159	0.023	0.408	-0.140	0.208	-0.146	0.231
	4.5	-0.086	0.153	-0.029	0.179	-0.095	0.162	0.089	0.467	-0.167	0.228	-0.195	0.258
	5	-0.056	0.143	0.116	0.414	-0.070	0.152	0.101	0.469	-0.178	0.237	-0.234	0.283

the modified BL and the BL methods, and the modified LSPF-MLE and the LSPF-MLE methods were large, which was believed to be caused by the large shape parameters estimated by the BL and the LSPF methods. Moreover, the estimation results of the modified BL and the modified LSPF methods of location and scale parameters were reasonable because they used the shape parameter estimated by the MVLE-H method. Although the improvement of the modified w-MLE method was not as large as the improvement of the modified BL and the modified LSPF-MLE methods, it was considered that the estimation results of the location and scale parameters were improved by the w-MLE method. This was because the MCS results of the RMSE of the location and scale parameters at $n = 30$ and $m = 3$ in Tables 8 and 9 were the smallest.

6. Conclusions

We enhanced the performance of the MVLE-H method by using two ideas. First, we estimated the hyperparameter used in the MVLE-H method depending on the sample size and population shape parameter. We estimated a temporary shape parameter using an existing method and used it as prior information to determine the hyperparameter, because the population shape parameter was unknown. Second, we searched for the optimal hyperparameter in increments of five and complement the hyperparameter for the remaining n with a linear regression line. From the MCS results, the RMSE of the MVLE-H method was the smallest for the shape parameter estimated, and the RMSE of the modified w-MLE method was the smallest for the location and scale parameters estimated. Therefore, the performance of the estimation of wear failure time parameters for a three-parameter Weibull distribution is improved estimating the shape parameter on the modified MVLE-H and the location and the scale parameter on the w-MLE methods.

Table 10. Three-parameter Weibull distribution estimation by seven methods using numerical example

Method	Shape	Location	Scale
MVLE-H	2.872	-	-
M. w-MLE	-	0.371	5.490
w-MLE	3.772	-0.522	6.493
M. BL	-	0.190	5.635
BL	5.101	-2.362	8.392
M. LSPF-MLE	-	-0.320	6.110
LSPF-MLE	4.295	-1.134	7.130

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Conflict of interest

The authors declare that they have no conflict of interest.

References

Cousineau, D. (2009a). Nearly unbiased estimators for the three-parameter Weibull distribution with greater efficiency than the iterative likelihood method. *British Journal of Mathematical and Statistical Psychology, 62*(1), 167-191. https://doi.org/10.1348/000711007X270843

Cousineau, D. (2009b). Fitting the three-parameter Weibull distribution: Review and evaluation of existing and new methods. *IEEE Transactions on Dielectrics and Electrical Insulation, 16*(1), 281-288. https://doi.org/10.1109/TDEI.2009.4784578

Datsiou, K. C., & Overend, M, (2018). Weibull parameter estimation and goodness-of-fit for glass strength data. *Struct Saf., 73,* 29-41. https://doi.org/10.1016/j.strusafe.2018.02.002

Hall, P., & Wang, J. Z. (2005). Bayesian likelihood methods for estimating the end point of a distribution. *J Roy Stat. Soc. B, 67*(5), 717-729. https://doi.org/10.1111/j.1467-9868.2005.00523.x

Huang, W. T., & Huang, H. H. (2006). Empirical Bayes estimation of the guarantee lifetime in a two-parameter exponential distribution. *Stat. Probabil. Lett., 76*(16), 1821–1829. https://doi.org/10.1016/j.spl.2006.04.034

Lawless, J. F. (2011). *Statistical Models and Methods for Lifetime Data.* John Wiley & Sons, New Jersey.

Murthy, D. N. P., Xie, M., & Jiang, R. (2004). *Weibull Models*. John Wiley & Sons, New Jersey.

Nagatsuka, H., Kamakura, T., & Balakrishnan, N. (2013). A consistent method of estimation for the three-parameter Weibull distribution. *Comput Stat Data An, 58*(1), 210-226. https://doi.org/10.1016/j.csda.2012.09.005

Ng, H. K. T., Luo, L., Hu, Y., & Duan, F. (2012). Parameter estimation of three-parameter Weibull distribution based on progressively Type-II censored samples. *J Stat Comput Sim., 82*(11), 1661-1678. https://doi.org/10.1080/00949655.2011.591797

Ogura, T., Sugiyama, T., & Sugiura, N. (2020). Estimation of the Shape Parameter of a Wear-Out Failure Period for a Three-Parameter Weibull Distribution in a Small Sample. *Int. J. Stat. Probab., 9*(6), 39-48. https://doi.org/10.5539/ijsp.v9n6p39

R Core Team (2021). R: A Language and Environment for Statistical Computing. R Foundation for Statistical Computing, Vienna, Austria. Software available at https://www.R-project.org/

Smith, R. L. (1985). Maximum Likelihood Estimation in a Class of Nonregular Cases. *Biometrika, 72*(1), 67-90. https://doi.org/10.1093/biomet/72.1.67

Sugiyama, T., Ogura, T., & Sugiyama, T. (2019). Estimation of the threshold parameter of a wear-out failure period in the case of a three-parameter Weibull distribution. *Int J Knowl Eng Soft Data Paradig, 6*(2), 110-119. https://doi.org/10.1504/IJKESDP.2019.102818

Yang, F., Ren, H., & Hu, Z. (2019). Maximum likelihood estimation for three-parameter Weibull distribution using evolutionary strategy. Math Probl Eng. https://doi.org/10.1155/2019/6281781

Li, Z., Wang, L., & Suzuki, K. (2012). Lifetime estimation from field failure data with information on discarding. *J Reliab Eng Assoc. Jpn., 34*(8), 553-566. https://doi.org/10.11348/reajshinrai.34.8_553

Appendix

We show a sample code of the software R for estimating the three-parameter Weibull distribution using a numerical example in Sect. 5. Another practical example can be analyzed by replacing the vector X with a suitable one. When maximizing or minimizing a function, it is advisable to search several times with different initial values.

```
X<-sort(c(3.84,1.00,4.14,4.81,5.72,7.23,8.08,4.16,4.17,4.00,
4.42,3.58,3.92,4.73,5.42,5.09,5.59,3.67,5.76,6.34,6.07,6.75,
4.07,7.34,6.00,8.26,8.01,8.67,4.24,5.73,5.50));n<-length(X)
c2<-function(n){0.20*n+2.4};c3<-function(n){0.10*n+1.8}
c4<-function(n){0.08*n+0.9};c5<-function(n){0.04*n+1.0}
library(MASS);library(cubature)

#MVLE-H method
k<-function(i,n){f<-function(u){log(-log(1-u))*factorial(n)/
 factorial(i-1)/factorial(n-i)*u^(i-1)*(1-u)^(n-i)}
 integrate(f,0,1)$value}
cov_ij<-function(i,j,n,m){ki<-function(u){log(-log(1-u))*
 factorial(n)/factorial(i-1)/factorial(n-i)*u^(i-1)*(1-u)^(n-i
 )}; kj<-function(u){log(-log(1-u))*factorial(n)/factorial(j-1
 )/factorial(n-j)*u^(j-1)*(1-u)^(n-j)}
 kin<-integrate(ki,0,1)$value; kjn<-integrate(kj,0,1)$value
 g<-function(u){f<-function(v){(log(-log(1-u))-kin/m)*(log(-
 log(1-v))-kjn/m)*factorial(n)/factorial(i-1)/factorial(j-i-1)/
 factorial(n-j)*u^(i-1)*(v-u)^(j-i-1)*(1-v)^(n-j)}; integrate(
 f,u,1,rel.tol=1e-10)$value}; integrate(Vectorize(g),0,1,
 rel.tol=1e-10)$value}
var_i<-function(i,n,m){ki<-function(u){log(-log(1-u))*factorial(
 n)/factorial(i-1)/factorial(n-i)*u^(i-1)*(1-u)^(n-i)}; kin<-
 integrate(ki,0,1)$value; f<-function(u){(log(-log(1-u))-kin/m)^
 2*factorial(n)/factorial(i-1)/factorial(n-i)*u^(i-1)*(1-u)^(
 n-i)}; integrate(f,0,1)$value}
S<-function(n,m){S0<-matrix(NA,n-1,n-1); for(i in 2:n){for(j in
 2:n){if(i<j){S0[i-1,j-1]<-(cov_ij(i,j,n,m)-cov_ij(1,i,n,m)-
 cov_ij(1,j,n,m)+var_i(1,n,m))/(k(i,n)-k(1,n))/(k(j,n)-k(1,n))}
 if(j<i){S0[i-1,j-1]<-(cov_ij(j,i,n,m)-cov_ij(1,i,n,m)-cov_ij(
 1,j,n,m)+var_i(1,n,m))/(k(i,n)-k(1,n))/(k(j,n)-k(1,n))}
 if(i==j){S0[i-1,j-1]<-(var_i(i,n,m)-2*cov_ij(1,i,n,m)+var_i(
 1,n,m))/(k(i,n)-k(1,n))^2}}}; return(S0)}
MVLEH<-function(c,m0){T<-function(i){T0<-function(U){(log(c*(1-
 U)*(X[i]-X[1])+U)-log(U))/(k(i,n)-k(1,n))}; integrate(T0,0,1)$
 value}; vT<-NULL; for(i in 2:n){vT<-c(vT,T(i))}; rep(1,n-1)%*%
 solve(S(n,m0))%*%rep(1,n-1)/rep(1,n-1)%*%solve(S(n,m0))%*%vT}
MLE0<-fitdistr((X-X[1])[-1],densfun="weibull"); hat_m0<-MLE0$
 estimate[1]; if(hat_m0<2.5){opt_c<-c2(n)}; if(2.5<=hat_m0&
 hat_m0<3.5){opt_c<-c3(n)}; if(3.5<=hat_m0&hat_m0<4.5){
 opt_c<-c4(n)}; if(4.5<=hat_m0){opt_c<-c5(n)}
MVLEH_m<-c(MVLEH(opt_c,hat_m0))

#w-MLE method (This code is a simplified version of n=31.)
wbl3LogLikelihood<-function(data,params){sum(dweibull(
```

```
 data-params[3],shape=params[1],scale=params[2],log=TRUE))}
wbl3wMLE<-function(data,params=NULL,...){n<-length(data)
 if(!is.null(params)){gamma=params[1];alpha=params[3]}else{
 gamma=2; alpha=0.9*min(data)}; fctGamma<-function(x,n,gamma,
 alpha,W2){W2/gamma+sum(log(x-alpha))/n-sum((log(x-alpha)*(
 x-alpha)^gamma)/sum((x-alpha)^gamma))}; fctBeta<-function(x,
 n,gamma,alpha,W1){(sum((x-alpha)^gamma)/(n*W1))^(1/gamma)}
 fctAlpha<-function(x,n,gamma,alpha,W3){-W3+(sum(1/(x-alpha))*
 sum((x-alpha)^gamma)/sum((x-alpha)^(gamma-1)))/n}
J1<-AllJ1[AllJ1[1]==n,2]; J2<-AllJ2[AllJ2[1]==n,2]; sub<-AllJ3[
AllJ3[1]==n,,]; J3<-function(gamma){if(gamma<0.1){gamma=0.1}
if(gamma>5.0){gamma=5.0}; pos=floor(gamma/wMLEresolution)+1
low=sub[pos-1,]; hig=sub[pos,]; as.numeric(((gamma-low[2])/
 wMLEresolution*(hig[3]-low[3])+low[3]))}; theta<-c(gamma,alpha)
objective<-function(theta){(fctGamma(data,n,theta[1],theta[2],
 J2))^2+(fctAlpha(data,n,theta[1],theta[2],J3(theta[1])))^2}
res<-constrOptim(theta,f=objective,ui=diag(c(1,-1)),ci=c(0,
 -min(data)),method="Nelder-Mead",...); gamma<-res$par[1]
alpha<-res$par[2]; beta<-fctBeta(data,n,gamma,alpha,J1)
bestfitparams=c(gamma,beta,alpha); fit=wbl3LogLikelihood(data,
 bestfitparams); list(fit=fit,bestfitparams=bestfitparams)}
AllJ1<-data.frame(31,0.9888452); AllJ2<-data.frame(31,0.95217)
AllJ3<-data.frame(rep(31,40),seq(0.1,4,0.1),c(43.9840,42.3834,
38.4094,31.0778,22.0437,14.8054,10.1444,7.3710,5.6550,4.5772,
3.8565,3.3480,2.9805,2.6979,2.4819,2.3130,2.1754,2.0600,1.9653,
1.8852,1.8160,1.7584,1.7063,1.6606,1.6187,1.5843,1.5527,1.5232,
1.4971,1.4738,1.4523,1.4321,1.4146,1.3974,1.3819,1.3682,1.3543,
1.3424,1.3311,1.3204)); wMLEresolution<-AllJ3[2,2]-AllJ3[1,2]
wMLEres<-wbl3wMLE(X,c(0.25,Inf,min(X)-sd(X)/2))$bestfitparams
w_m<-wMLEres[1]; w_gamma<-wMLEres[3]; w_eta<-wMLEres[2]

#Modified w-MLE method
W1<-AllJ1[AllJ1[1]==n,2]; W2<-AllJ2[AllJ2[1]==n,2]
W3l<-AllJ3[AllJ3[1]==n&round(AllJ3[2],1)==floor(MVLEH_m*10)/10,
 3]; W3u<-AllJ3[AllJ3[1]==n&round(AllJ3[2],1)==ceiling(MVLEH_m*
 10)/10,3]; W3<-W3l+(W3u-W3l)*10*(MVLEH_m-floor(MVLEH_m*10)/10)
M.wMLE_f<-function(r){(W2/MVLEH_m+1/n*(sum(log(X-r)))-sum(log(
 X-r)*(X-r)^MVLEH_m)/sum((X-r)^MVLEH_m))^2+(1/n*sum(1/(X-r))*
 sum((X-r)^MVLEH_m)/sum((X-r)^(MVLEH_m-1))-W3)^2}
M.w_gamma<-optimize(M.wMLE_f,c(-10,X[1]))$minimum
M.w_eta<-(1/n/W1*sum((X-M.w_gamma)^MVLEH_m))^(1/MVLEH_m)

#BL method
b<-1; BL_f1<-function(a){(X[1]-a[2])/(X[2]-a[2])*gamma(n)*a[1]^
 n*prod((X-a[2])^(a[1]-1))/(sum((X-a[2])^a[1]))^n/b}
BL_f2<-function(eta){(sum((X-B_gamma)^B_m)^n/gamma(n)/(eta^(
 B_m*n))*exp(-sum((X-B_gamma)^B_m)/eta^B_m))}
b<-optim(c(0.5,X[1]),BL_f1,control=list(fnscale=-1),method=
 "L-BFGS-B",lower=c(0,-Inf),upper=c(2,X[1]-0.1*1e-10))$value
B_mg<-optim(c(5,X[1]),BL_f1,control=list(fnscale=-1),method=
 "L-BFGS-B",lower=c(0,-Inf),upper=c(10,X[1]-0.1*1e-10))
B_m<-B_mg$par[1];B_gamma<-B_mg$par[2];B_eta<-optimize(BL_f2,c(
 0,100),maximum=T)$maximum

#Modified BL method
M.BL_f1<-function(r){(X[1]-r)/(X[2]-r)*gamma(n)*MVLEH_m^n*prod(
 (X-r)^(MVLEH_m-1))/(sum((X-r)^MVLEH_m))^n}
```

```
M.BL_f2<-function(eta){(sum((X-M.B_gamma)^MVLEH_m)^n/gamma(n)/
 eta^(MVLEH_m*n)*exp(-sum((X-M.B_gamma)^MVLEH_m)/eta^MVLEH_m))}
M.B_gamma<-optimize(M.BL_f1,c(-10,X[1]-1e-10),maximum=T)$maximum
M.B_eta<-optimize(M.BL_f2,c(0,100),maximum=T)$maximum

#LSPF method
Z<-(X-X[1])/(X[n]-X[1]); LSPF_f<-function(m){g<-function(a){a[2
 ]^(n-2)*(prod(a[1]+Z*a[2]))^(m-1)*exp(-(sum((a[1]+Z*a[2])^m)))}
adaptIntegrate(g,c(0,0),c(Inf,Inf))$integral*m^n*factorial(n)}
L_m<-optimize(LSPF_f,c(0,6),maximum=T)$maximum
L_gamma0<-X[1]; L_eta0<-(1/n*sum((X-L_gamma0)^L_m))^(1/L_m)
L_gamma<-X[1]-n^(-1/L_m)*L_eta0*gamma(1+1/L_m); L_eta<-(1/n*
 sum((X-L_gamma)^L_m))^(1/L_m)

#Modified LSPF method
M.L_gamma0<-X[1]; M.L_eta0<-(1/n*sum((X-M.L_gamma0)^MVLEH_m))^(
 1/MVLEH_m); M.L_gamma<-X[1]-n^(-1/MVLEH_m)*M.L_eta0*gamma(1+1/
 MVLEH_m); M.L_eta<-(1/n*sum((X-M.L_gamma)^MVLEH_m))^(1/MVLEH_m)

#Result
res<-rbind(c(MVLEH_m,NA,NA),c(NA,M.w_gamma,M.w_eta),c(w_m,
 w_gamma,w_eta),c(NA,M.B_gamma,M.B_eta),c(B_m,B_gamma,B_eta),
 c(NA,M.L_gamma,M.L_eta),c(L_m,L_gamma,L_eta))
colnames(res)<-c("Shape","Location","Scale")
rownames(res)<-c("MVLE-H","M. w-MLE","w-MLE","M. BL","BL",
 "M. LSPF-MLE","LSPF-MLE"); res
```